MATH1400: Modelling with differential equations, 2011–12
Quiz 4: Friday 27th April – 20 minutes

Student's name: Tutor's name:

Quiz 4a

1. Solve the following second-order ODE:

$$y'' - 3y' + 2y = 4x, \quad \text{with } y(0) = 0 \text{ and } y'(0) = 0.$$

$$y'' - 4y' + 3y = \cancel{9}\,18\,x, \quad \text{with } y(0) = 0 \text{ and } y'(0) = 0.$$

[**10 marks**]

characteristic equation:

$\lambda^2 - 3\lambda + 2 = 0 = (\lambda-1)(\lambda-2)$

so $\lambda = 1, \lambda = 2$

$y_{CF} = C_1 e^{x} + C_2 e^{2x}$

RHS is 1st degree polynomial, so try $y_{PI} = Ax + B$

$y_{PI}' = A$

$y_{PI}'' = 0$

substitute into equation

$0 - 3(A) + 2(Ax+B) = 4x$

compare terms:

$x: \quad 2A = 4 \rightarrow A = 2$

$1: \quad 2B - 3A = 0 \rightarrow B = \frac{3A}{2} = 3$

so $y_{PI} = 2x + 3$

general solution:

$y = C_1 e^{x} + C_2 e^{2x} + 2x + 3$

$y(0) = C_1 + C_2 + 3 = 0$

$y'(0) = C_1 + 2C_2 + 2 = 0$

subtract: $0 - C_2 + 1 = 0$

$\rightarrow C_2 = 1, \; C_1 = -3 - C_2 = -4$

$y = -4e^{x} + e^{2x} + 2x + 3$

Quiz 4b

(omitting the words)

$\lambda^2 - 4\lambda + 3 = 0 = (\lambda-1)(\lambda-3)$

$\lambda = 1, \lambda = 3$

$y_{CF} = C_1 e^{x} + C_2 e^{3x}$

$y_{PI} = Ax + B$

$y_{PI}' = A$

$y_{PI}'' = 0$

$0 - \cancel{3}\,4(A) + 3(Ax + B) = \cancel{9}\,18x$

$x: \quad 3A = \cancel{9}\,18 \rightarrow A = \cancel{3}\,6$

$1: \quad -4A + 3B = 0 \rightarrow B = \frac{4A}{3} = \cancel{4}\,8$

$y_{PI} = \cancel{3x+4}\; 6x + 8$

$y = C_1 e^{x} + C_2 e^{3x} + \cancel{3}\,6x + \cancel{4}\,8$

$y(0) = C_1 + C_2 + \cancel{4}\,8 = 0$

$y'(0) = C_1 + 3C_2 + \cancel{3}\,6 = 0$

subtract $0 - 2C_2 + 2 = 0$

$\rightarrow C_2 = 1, \; C_1 = -8 - C_2 = -9$

$y = -9e^{x} + e^{3x} + 6x + 8$

2. Solve the following second-order ODE:

$$y'' + 2y' + 10y = 85\cos x.$$

[10 marks]

characteristic equation : $\lambda^2 + 2\lambda + 10 = 0$

$\rightarrow \quad \lambda = \frac{-2 \pm \sqrt{4 - 4\times 10}}{2} = \frac{-2 \pm \sqrt{-36}}{2}$

$\lambda = -1 \pm 3i$

so $y_{CF} = e^{-x}(C_1\cos 3x + C_2\sin 3x)$

the RHS is <u>not</u> in the CF, so take

$y_{PI} = A\cos x + B\sin x$

$y'_{PI} = -A\sin x + B\cos x$

$y''_{PI} = -A\cos x - B\sin x$

substitute into ODE

$(-A\cos x - B\sin x) + 2(-A\sin x + B\cos x) + 10(A\cos x + B\sin x) = 85\cos x$

$\cos x: \quad -A + 2B + 10A = 85$

$\sin x: \quad -B - 2A + 10B = 0$

or $\quad 9A + 2B = 85 \quad ①$

$-2A + 9B = 0 \quad ②$

$① \times 2 + ② \times 9$:

$0A + (4 + 81)B = 85 \times 2$

$\rightarrow B = 2, \quad A = \frac{9B}{2} = 9$

so ~~the~~ $y_{PI} = 9\cos x + 2\sin x$

and ~~the~~ general solution is

$y = e^{-x}(C_1\cos 3x + C_2\sin 3x) + 9\cos x + 2\sin x$

$$y'' + 2y' + 5y = 10\sin x.$$

$\lambda^2 + 2\lambda + 5 = 0$

$\lambda = \frac{-2 \pm \sqrt{4 - 4\times 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$

$\lambda = -1 \pm 2i$

$y_{CF} = e^{-x}(C_1\cos 2x + C_2\sin 2x)$

$y_{PI} = A\cos x + B\sin x$

$y'_{PI} = -A\sin x + B\cos x$

$y''_{PI} = -A\cos x + B\sin x$

$(-A\cos x - B\sin x) + 2(-A\sin x + B\cos x) + 5(A\cos x + B\sin x) = 10\sin x$

$\cos x: \quad -A + 2B + 5A = 0$

$\sin x: \quad -B - 2A + 5B = 10$

or $\quad 4A + 2B = 0 \quad ①$

$-2A + 4B = 10 \quad ②$

$① + 2\times ②$:

$0A + (2 + 8)B = 20$

$\rightarrow B = 2, \quad A = -\frac{B}{2} = -1$

$y_{PI} = -\cos x + 2\sin x$

$y = e^{-x}(C_1\cos 2x + C_2\sin 2x) - \cos x + 2\sin x$