Math 1400 Mock Exam 2008

1a

$$xy(1+x^2)\,y' - y^2 = 1$$ separable or Bernoulli.

$$\frac{y}{1+y^2}\,y' = \frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} = \frac{A+Ax^2+Bx^2+Cx}{x(1+x^2)}$$

$$A=1,\ C=0,\ B=-A$$

$$= \frac{1}{x} - \frac{x}{1+x^2}$$

$$\int \frac{2y}{1+y^2}\,dy = \int \frac{2}{x} - \frac{2x}{1+x^2}\,dx =$$

$$\log(1+y^2) = \log x^2 - \log(1+x^2) + C = \log \frac{Kx^2}{1+x^2}$$

$$1+y^2 = \frac{Kx^2}{1+x^2}\ ; \quad y = \pm\sqrt{\frac{Kx^2}{1+x^2}-1}$$

$$y(1)=1 \rightarrow 1 = \pm\sqrt{\frac{K}{2}-1} \quad K=4 \text{ and } +: \quad y=\sqrt{\frac{4x^2}{1+x^2}-1}$$

1b $$y' = \frac{y}{x} - \frac{y^2}{x^2}$$ homogeneous or Bernoulli

$$y = xv\ ; \quad y' = v + xv' = v - v^2$$

$$xv' = -v^2 \qquad \frac{v'}{v^2} = -x \qquad \int \frac{dv}{v^2} = \frac{-x^2}{2} + \frac{C}{2} = \frac{-1}{v}$$

$$v = \frac{2}{x^2-C}\ ; \quad y = \frac{2x}{x^2-C}$$

1c $$y' + \frac{3}{x}y = \frac{e^{2x}}{x^3}$$ linear

$$IF = \exp\int\frac{3}{x} = x^3 \qquad (x^3y)' = e^{2x}$$

$$x^3y = \tfrac{1}{2}e^{2x} + C$$

$$y(1)=1 \rightarrow 1 = \tfrac{1}{2}e^2 + C\ ; \quad C = 1-\tfrac{1}{2}e^2$$

$$x^3y = \tfrac{1}{2}(e^{2x}-e^2)+1\ ; \quad y = \frac{\tfrac{1}{2}(e^{2x}-e^2)+1}{x^3}$$

Q2 a i) $\frac{dx}{dt} = -kx$

b) Separable: $x(t) = K e^{-kt}$

$x(0) = x_0 \Rightarrow x(t) = x_0 e^{-kt}$

ii) $x(T_h) = \frac{1}{2} x_0 \rightarrow \frac{1}{2} x_0 = x_0 e^{-kT_h}$

$T_h = \frac{1}{k} \log 2$; $k = \frac{1}{T_h} \log 2$

$x(T) = 0.8\, x_0 = x_0 e^{-\frac{T}{T_h} \log 2}$

$\log 0.8 = -\frac{T}{T_h} \log 2$; $T = \frac{\log 1.25}{\log 2} T_h$.

Q2b $\frac{d\theta}{dt} = -k(\theta - A)$

$\frac{d\theta}{dt} + k\theta = kA$

$IF = e^{kt}$

$\rightarrow \frac{d}{dt}(e^{kt}\theta) = kAe^{kt}$

$e^{kt}\theta = Ae^{kt} + C$

~~$\theta_0 = A + C$~~

$\theta(t_0) = \theta_0 \rightarrow C = e^{kt_0} \cdot \theta_0 - Ae^{kt_0}$

$\theta = A + (\theta_0 - A) e^{-k(t - t_0)}$

Points to mention:

A = Ambient temperature

$\theta - A$ = temp diff.

rate of cooling $\propto$ temp diff, k = const of prop.

and as $t \to \infty$, $\theta \to A$.

Ideally, the answer would be written in complete sentences!

Q3 $y'' - y' - 2y = e^{x} - e^{-x}$

$\lambda^2 - \lambda - 2 = 0 \rightarrow \lambda = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = 2, -1$

$y_{CF} = C_1 e^{-x} + C_2 e^{2x}$

$y_{PI} = Ae^{x} + Bx\, e^{-x}$ (xe^{-x} since e^{-x} is a CF)

$y_{PI}' = Ae^{x} + Be^{-x} - Bxe^{-x}$

$y_{PI}'' = Ae^{x} - 2Be^{-x} + Bxe^{-x}$

$y_{PI}'' - y_{PI}' - 2y_{PI} = Ae^{x} - 2Be^{-x} + \underline{Bxe^{-x}}$

$- Ae^{x} - Be^{-x} + \underline{Bxe^{-x}}$

$-2Ae^{x} \underline{- 2xBe^{-x}}$

$= -2Ae^{x} - 3Be^{-x} = e^{x} - e^{-x}$

$A = -\frac{1}{2},\ B = \frac{1}{3} \rightarrow y = C_1 e^{-x} + C_2 e^{2x} - \frac{1}{2}e^{x} + \frac{1}{3}xe^{-x}$

b $y'' + 4y = \sin x \rightarrow y_{CF} = C_1 \cos 2x + C_2 \sin 2x$

$y_{PI} = A\cos x + B\sin x$; $3A\cos x + 3B\sin x = \sin x$

$A = 0,\ B = \frac{1}{3}$

$\rightarrow y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{3}\sin x$

$y(0) = C_1 = 0$ $y\left(\frac{\pi}{4}\right) = C_2 + \frac{1}{3}\frac{1}{\sqrt{2}} = 0$

$y = -\frac{1}{3\sqrt{2}}\sin 2x + \frac{1}{3}\sin x$

c. $y'' + 6y' + 13y = 13x^2 - x + 22$

$\lambda^2 + 6\lambda + 13 = 0$ $\lambda = \frac{-6 \pm \sqrt{36 - 52}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$

$y_{CF} = (C_1 \cos 2x + C_2 \sin 2x)e^{-3x}$

$y_{PI} = Ax^2 + Bx + C$

$2A + 6(2Ax + B) + 13(Ax^2 + Bx + C)$

$= 13Ax^2 + (13B + 12A)x + 13C + 6B + 2 = 13x^2 - x + 22$

$A = 1,\ B = -1$ $13C - 6 + 2 = 22$ $C = 2$

$y = (C_1 \cos 2x + C_2 \sin 2x)e^{-3x} + x^2 - x + 2$

Q4a $y'' + 4y' + 4y = 4x + e^{2x}$

$\lambda^2 + 4\lambda + 4 = (\lambda+2)^2 \quad \lambda = -2$

$y_{CF} = C_1 e^{-2x} + C_2 x e^{-2x}$

$y_{PI} = Ax + B + Ce^{2x}$

$4Ce^{2x} + 4A + 8Ce^{2x} + 4Ax + 4B + 4Ce^{2x} = 4x + e^{2x}$

$A = 1, \quad B = -1, \quad C = \frac{1}{16}$.

$y = c_1 e^{-2x} + C_2 x e^{-2x} + x - 1 + \frac{1}{16} e^{2x}$

b $y = u e^x$ <u>Reduction of order</u>

$y' = u' e^x + u e^x \qquad y'' = u'' e^x + 2u' e^x + u e^x$

$(1+x)(u'' + 2u' + u) e^x - (2x+3)(u' + u) e^x + (x+2) u e^x$

$= \Big((1+x) u'' + \big(2(1+x) - (2x+3)\big) u' \Big) e^x = e^x (1+x)^2$

$(1+x) u'' + - u' = (1+x)^2$

write $u' = v$:

$v' - \frac{1}{1+x} v = 1 + x$

$IF = \exp \int -\frac{1}{1+x} = \exp(-\log(1+x)) = \frac{1}{1+x}$

$\left(\frac{1}{1+x} v\right)' = 1$

$\frac{1}{1+x} v = x + C$

$v = x(1+x) + c_1(1+x) = x^2 + (1+c_1)x + c_1$

$u = \int v = \frac{x^3}{3} + (1+c_1)\frac{x^2}{2} + c_1 x + c_2$

$y = u e^x$

$= \left(\frac{x^3}{3} + (1+c_1)\frac{x^2}{2} + c_1 x + c_2 \right) e^x$

Q5 a

(i) $dF = (3x^2 - 3y^2)\,dx + -6xy\,dy$

(ii) $dF = \left(y^2 e^x + \cos(x+y)\right) dx$

$\qquad + \left(2y e^x + \cos(x+y)\right) dy$

(iii) $dF = \left(\frac{2x}{x^2+y^2} - 2y\right) dx + \left(\frac{2y}{x^2+y^2} - 2x\right) dy$

b. stat pt: $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$

max: $F_{xx} \cdot F_{yy} - (F_{xy})^2 > 0$ and $F_{xx} + F_{yy} < 0$

min $\qquad > 0 \qquad > 0$

saddle $\qquad < 0$

c. $F = y(x^2 + y^2 - 1) = x^2 y + y^3 - y$

$F_x = 2xy = 0 \rightarrow x = 0$ or $y = 0$

$F_y = x^2 + 3y^2 - 1 = 0$

$x = 0 \rightarrow 3y^2 - 1 = 0, \quad y = \pm\frac{1}{\sqrt{3}} \quad (0, \frac{1}{\sqrt{3}}), (0, -\frac{1}{\sqrt{3}})$

$y = 0 \rightarrow x^2 - 1 = 0 \quad x = \pm 1. \quad (1, 0), (-1, 0)$

$F_{xx} = 2y \qquad F_{xy} = 2x \qquad F_{yy} = 6y$

x	y	F_{xx}	F_{yy}	F_{xy}	$F_{xx} \cdot F_{yy} - (F_{xy})^2$	$F_{xx} + F_{yy}$
1	0	0	0	2	-4	saddle
-1	0	0	0	-2	-4	saddle
0	$\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	$\frac{6}{\sqrt{3}}$	0	4	+ min
0	$-\frac{1}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}}$	$-\frac{6}{\sqrt{3}}$	0	4	− max