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Chapter 1

Introduction

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1.1 Motivation: Why study differential equations?

Suppose we know how a certain quantity (for example, the temperature of coffee in a cup, the number of people infected with a virus, the concentration of carbon dioxide in the atmosphere) changes with time. The **rate of change** of this quantity is the **derivative**, so we can work out how quickly the temperature changes, how quickly the number of infected people changes, how quickly the concentration of carbon dioxide changes.

Suppose instead we know the value of the quantity **now** and we wish to predict its value in the **future**. To do this, we must know how quickly the quantity is changing. But the rate of change of a quantity will depend on the quantity itself: this gives rise to a **differential equation** – an equation relating the derivative of a quantity to its value.

As an example, suppose that $N(t)$ is the number of bacteria growing on a plate of nutrients. At the start of the experiment, suppose that there are 1000 bacteria, so $N(0) = 1000$. The rate of change of N will be proportional to N itself: if there are twice as many bacteria, then N will grow twice as rapidly. So we have:

$$\frac{dN}{dt} = \sigma N,$$

where σ is a constant, and dN/dt is the derivative (rate of change) of N with respect to time. We would have to do further experiments to find out the value of σ.

We can easily verify that

$$N(t) = 1000e^{\sigma t}$$

is a **solution** of this **differential equation** with the given **initial condition**. To do this, first calculate $N(0)$ and verify that it is the same as the number given:

$$N(0) = 1000e^0 = 1000.$$

Next, calculate dN/dt and verify that it satisfies the differential equation:

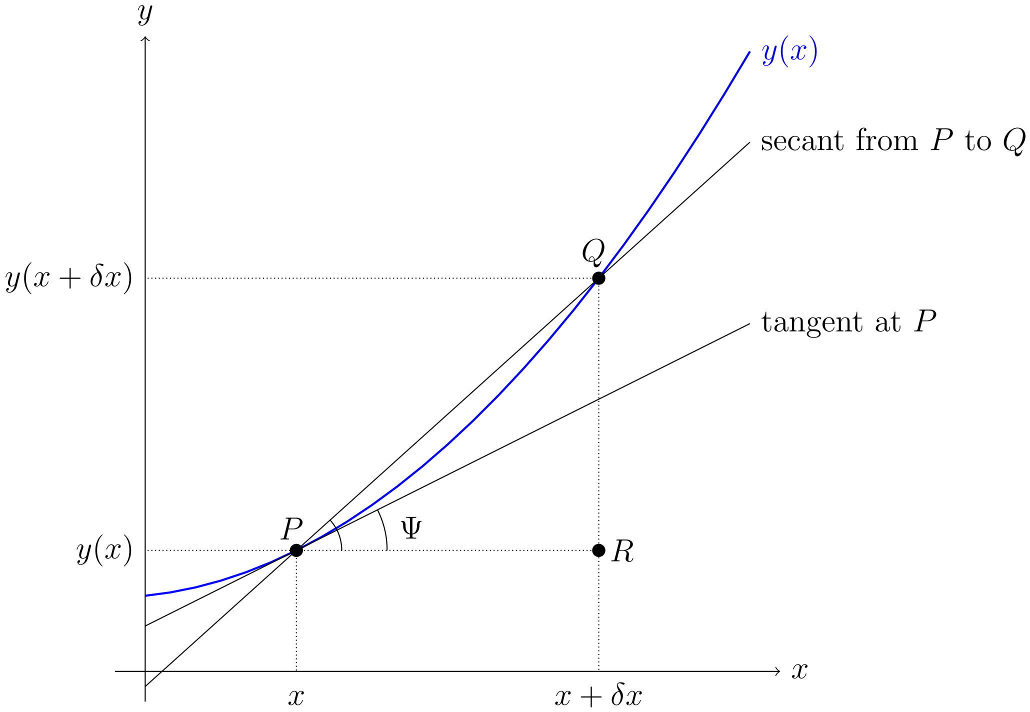
$$\frac{dN}{dt} = 1000\sigma e^{\sigma t} = \sigma\left(1000e^{\sigma t}\right) = \sigma N$$

as required.

1.2 Basics

1.2.1 The derivative

The derivative of a function $y(x)$ at a particular value of x is the slope of the tangent to the curve at the point P, or $(x, y(x))$.



Suppose $y(x)$ is a function; then the derivative dy/dx at a particular value of x is the slope of the curve at the point P (or, the slope of the line that is tangent to the curve at the point P):

$$\frac{dy}{dx} = \tan\Psi$$

If Q is a neighbouring point on the curve, then we can take the limit as Q tends to P:

$$\frac{dy}{dx} = \lim_{Q\to P}\frac{QR}{PR} = \lim_{\delta x\to 0}\frac{y(x+\delta x)-y(x)}{\delta x}$$

assuming that the limit exists.
For example, suppose $y(x) = x^3$. Then

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\delta x \to 0} \frac{(x+\delta x)^3 - (x)^3}{\delta x} \\
&= \lim_{\delta x \to 0} \left(\frac{x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 - x^3}{\delta x} \right) \\
&= \lim_{\delta x \to 0} \left(3x^2 + 3x(\delta x) + (\delta x)^2 \right) \\
&= 3x^2
\end{aligned}$$

so if P were the point $(1, 1)$, for example, the slope of the curve at P would be 3.

1.2.2 Dependent and independent variables

When considering differential equations, it is important to distinguish between the **dependent** variable(s) and the **independent** variable. For example, time t might be the independent variable, and the population $N(t)$ of bacteria on a plate could be the dependent variable: the population depends on time – time does not depend on the population! Thus we would naturally consider dN/dt, rather than dt/dN – both derivatives make sense mathematically, but dN/dt makes more sense in the context of this problem.

There can be more than one dependent variable, but there can only be one independent variable in an **ordinary differential equation (ODE)**. In situations where there is more than one independent variable, we obtain a partial differential equation (PDE). Throughout this course, when we write:

$$\frac{dy}{dx},$$

we are thinking of y as the dependent variable and x as the independent variable, but the names x and y aren't important: we could equally well think of

$$\frac{dy}{dt} \quad \text{or} \quad \frac{dx}{dt}$$

(and indeed we will do this later on in the course). In some situations, it is useful to change the role of the variables: write dx/dy instead of dy/dx.

1.3 Growth and decay

Suppose the concentration of a substance, or the population of an organism, or the temperature of coffee in a cup, changes with time t – call the interesting quantity $p(t)$. Suppose that we know the **initial value** of p at time $t = t_0$. At this time, the value of p is p_0, or:

$$p(t_0) = p_0. \tag{1.1}$$

Suppose that we want to predict the future values of $p(t)$ at any time t.

To find $p(t)$, we make use of the growth or decay rate of p. Let $k(p,t)\delta t$ be the amount by which p changes in the time interval $(t, t+\delta t)$. Suppose that we know $k(p,t)$. Then

$$p(t+\delta t) = p(t) + k(p,t)\delta t$$

Rearranging:

$$\frac{p(t+\delta t) - p(t)}{\delta t} = k(p,t)$$

In the limit of $\delta t \to 0$, we get the **differential equation**:

$$\frac{dp}{dt} = k(p,t). \tag{1.2}$$

For a complete solution to our problem of predicting the value of $p(t)$ at any time t, we must solve the differential equation (1.2) with the initial condition (1.1).

When **modelling** any physical/chemical/biological process in this way, we must make a list of assumptions about the situation (for example, what influences the rate of growth or decay $k(p,t)$ above?), we must obtain a differential equation, and solve this equation using the initial values. We then compare the solution of our model equation to the real situation, and ask: does our solution make sense? If not, we modify our assumptions and improve the model.

1.4 Introduction to Ordinary Differential Equations (ODEs)

An **ordinary differential equation (ODE)** is an equation linking a dependent variable $y(x)$ (or possibly more that one dependent variable), an independent variable x, and certain of the derivatives $y' = dy/dx$, $y'' = d^2y/dx^2$ etc.

Examples:

(a) $y' = 4x^7$

(b) $y'' - 7y' + 12y = 5\cos x$

(c) $y'' + 2(y')^2 = 75x^3$

(d) $yy''' + 1 = 0$

(e) $(y')^2 + y = 7x$

(f) $(y'')^2 + \sqrt{y} = 4$

The aim is to **solve** the differential equation, that is, to obtain a relationship between y and x that doesn't involve any derivatives. In some cases, it is possible to work out an **explicit solution**. In other cases, we may find a relationship between x and y that cannot be solved explicitly for y. For example, $x - y - y^3 = 0$ is enough to calculate y for any given value of x – this is called an **implicit** solution.

Example: Show that

$$y(x) = 2e^{3x} - 3e^{4x} + \frac{11}{34}\cos x - \frac{7}{34}\sin x$$

is an explicit solution of (b).

Solution:

$$\begin{aligned} y' &= \frac{dy}{dx} = 6e^{3x} - 12e^{4x} - \frac{11}{34}\sin x - \frac{7}{34}\cos x \\ y'' &= \frac{d^2y}{dx^2} = 18e^{3x} - 48e^{4x} - \frac{11}{34}\cos x + \frac{7}{34}\sin x \end{aligned}$$

and so

$$\begin{aligned} y'' - 7y' + 12y &= (18 - 42 + 24)e^{3x} + (-48 + 84 - 36)e^{4x} \\ &\quad + \left(-\frac{11}{34} + \frac{49}{34} + \frac{132}{34}\right)\cos x + \left(\frac{7}{34} + \frac{77}{34} - \frac{84}{34}\right)\sin x \\ &= 5\cos x \end{aligned}$$

as required.
The first part of the course will mainly be concerned with situations where it is possible to calculate explicit or implicit solutions of ODEs.
However, in many cases it is impossible to write down an explicit or implicit solution. Even in these cases, we can get useful information about **qualitative features** of solutions – this will come at the end of the course.

1.4.1 Classification

In order to know whether or not we are in a case where an explicit solution is possible, we need to be able to **classify** the ODE.

Order: The order of an ODE is the largest number of times we differentiate the dependent variable. So the orders of (a)–(f) are: 1, 2, 2, 3, 1, 2.

Linear: An ODE is linear if it contains no products or powers (other than one) of the dependent variable or its derivatives. A linear ODE cannot contain terms like y^2, yy', $\sqrt{y}$, $\cos y$ etc. Powers of the independent variable are allowed. So the examples (a) and (b) are linear and the others are nonlinear.

Autonomous: An ODE is autonomous if there is no explicit mention of the independent variable. So (d) and (f) are autonomous and the others are non-autonomous.

Example: The following first-order ODE is two-dimensional:

$$\begin{aligned} \frac{dx}{dt} &= 3x - 2y, \\ \frac{dy}{dt} &= x + y. \end{aligned}$$

Dimension: A first-order ODE is one-dimensional if there is only one dependent variable. In general, the dimension of an ODE with one dependent variable is equal to the order of the ODE, so the dimensions of (a)–(f) are: 1, 2, 2, 3, 1, 2. The dimension of a first-order ODE with several dependent variables is equal to the number of dependent variables. The dimension of second and higher order ODEs with several dependent variables can also be calculated.

1.4.2 Arbitrary constants

In general, in order to solve an nth order ODE, we will have to integrate n times, introducing n arbitrary constants C_1, C_2, $\ldots$, C_n, so the explicit solution $y(x)$ will depend on n arbitrary constants. We will write the solution as $y(x, C_1, C_2, \ldots, C_n)$.

Example: Show that

$$y(x) = C_1 e^{3x} + C_2 e^{4x} + \frac{11}{34}\cos x - \frac{7}{34}\sin x$$

is an explicit solution of (b), where C_1 and C_2 are unknown constants.

Solution: Do this at home.
The most **general solution** of an nth order ODE will contain n arbitrary constants. Conversely, given $y(x, C_1, C_2, \ldots, C_n)$, we can obtain an nth order ODE that does not contain the constants.

Example: Suppose that $n = 2$, and $y(x, C_1, C_2)$ is:

$$\begin{aligned} y &= C_1 e^x + C_2 e^{2x} \\ y' &= C_1 e^x + 2C_2 e^{2x} \\ y'' &= C_1 e^x + 4C_2 e^{2x} \end{aligned}$$

We can eliminate C_1 and C_2 by taking the right combination of y, y' and y''. To find the right combination, start by writing down a general linear combination of y, y' and y'':

$$y'' + Ay' + By = 0,$$

where A and B are constants whose values we will work out. The coefficient of y'' can be set equal to 1 without loss of generality. Next, substitute in the expressions for y, y' and y'':

$$C_1 e^x + 4C_2 e^{2x} + A\left(C_1 e^x + 2C_2 e^{2x}\right) + B\left(C_1 e^x + C_2 e^{2x}\right) = 0.$$

Collect terms together:

$$(1 + A + B)\, C_1 e^x + (4 + 2A + B)\, C_2 e^{2x} = 0,$$

This equality holds only when the coefficients of the two functions are both zero (since e^x and e^{2x} are linearly independent), so $1 + A + B = 0$ and $4 + 2A + B = 0$. The solution of this pair of linear equations is $A = -3$ and $B = 2$, so the required ODE is:

$$y'' - 3y' + 2y = 0.$$

In general, if we are given additional information about y and its derivatives, we can (usually) find the values of these constants.

Initial value problem (IVP): We are given the values of y, y', y'', …, $y^{(n-1)}$ at a single value of x.

Boundary value problem (BVP): We are given the values of y at n values of x, not all the same.

Example: Solve

$$y'' - 3y' + 2y = 0$$

with boundary conditions $y(0) = 0$ and $y(1) = e - e^2$.

Solution: Here $n = 2$ (a second-order ODE) and we are given the value of y at two different values of x – so this is a boundary value problem. We already have a solution with two arbitrary constants above:

$$y = C_1 e^x + C_2 e^{2x}$$

Substitute $x = 0$ and $x = 1$ into this explicit solution to find:

$$\begin{aligned} y(0) &= C_1 + C_2 = 0 \\ y(1) &= C_1 e + C_2 e^2 = e - e^2 \end{aligned}$$

The first equation gives us $C_2 = -C_1$, and the second equation gives us $C_1 = 1$, so the solution to our boundary value problem is:

$$y(x) = e^x - e^{2x}.$$

Verification: In many of the examples we will examine, it is possible to **verify** that we have the correct solution by checking that our answer indeed satisfies the differential equation and has the right initial or boundary values. Here we get:

$$\begin{aligned} y' &= e^x - 2e^{2x} \\ y'' &= e^x - 4e^{2x} \\ y'' - 3y' + 2y &= (1 - 3 + 2)e^x + (-4 + 6 - 2)e^{2x} = 0 \qquad \textbf{OK} \\ y(0) &= e^0 - e^0 = 0 \qquad \textbf{OK} \\ y(1) &= e - e^2 \qquad \textbf{OK} \end{aligned}$$

so our solution is correct.

Chapter 2

Solution of first-order ODEs

Contents

2.1 Introduction

A first-order ODE involves the dependent variable y and its first derivative $y' = dy/dx$. There will be a single initial value: $y = y_0$ when $x = x_0$ (often $x_0 = 0$, but not always). Then the general problem takes the form:

$$y' = f(y, x) \qquad \text{with} \quad y(x_0) = y_0, \tag{2.1}$$

where $f(y, x)$ is a given function of y and x.
Some ODEs are easy to solve, for example, if the function $f(x, y)$ depends only on x:

$$y' = f(x),$$

which is solved by $y = \int f(x)\,dx + C$, where C is an arbitrary constant of integration. However, just integrating like this almost never works.
We will treat three important special cases of first-order ODEs: **separable**, **linear** and **exact**, and show in each case how to find explicit solutions. We will also consider some examples that are not separable, exact or linear, but that can be transformed into one of these forms.

2.2 Separable first-order ODEs

A first-order ODE is **separable** if it can be written in the form:

$$y' = a(x)b(y) \qquad \text{with} \quad y(x_0) = y_0, \tag{2.2}$$

where $a(x)$ and $b(y)$ are functions of x and y respectively.
To solve this equation, we write:

$$\frac{dy}{dx} = a(x)b(y)$$

1. **Bring** the $b(y)$ over to the other side:

$$\frac{1}{b(y)}\frac{dy}{dx} = a(x)$$

2. Now **integrate** with respect to x, not forgetting the **constant of integration**:

$$\int \frac{1}{b(y)}\frac{dy}{dx}\,dx = \int a(x)\,dx + C$$

3. Next, **change variables** in the integral for y:

$$\int \frac{1}{b(y)}\,dy = \int a(x)\,dx + C$$

4. If we can **evaluate** these indefinite integrals (depends on the functions a and b), and then re-arrange the answer, we can write down the **general solution** $y(x)$ explicitly.
5. Finally, we can then use the **initial condition** $y(x_0) = y_0$ to evaluate the constant of integration C, and so find the **particular solution**. If you don't need the general solution, it is sometimes easier to do step 5 after doing the integrals but before solving for the explicit solution.
6. And last of all, we can **verify** that the answer satisfies the ODE and the initial value.

Example: Consider

$$\frac{dy}{dx} = \frac{y}{1+x} \qquad \text{with} \quad y(1) = -\frac{1}{2}.$$

Find the explicit solution, and verify that the answer is correct.

Solution: Here,

$$a(x) = \frac{1}{1+x} \qquad \text{and} \qquad b(y) = y$$

Proceed as above:
1. Bring the $b(y)$ over to the other side:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{1+x}$$

2. and 3. Now integrate with respect to x, not forgetting the constant of integration, and change variables in the integral for y:

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{y}\,dy = \int \frac{1}{1+x}\,dx + C$$

4. We can evaluate these indefinite integrals:

$$\log|y| = \log|1+x| + C$$

and we can re-arrange to find y explicitly by taking exponentials of both sides:

$$\exp(\log|y|) = |y| = \exp\left(\log|1+x| + C\right) = e^C|1+x|.$$

If we define $K = e^C$ (a positive constant), we can remove the absolute value signs if we introduce a ±:

$$y = \pm K(1+x)$$

Since K is arbitrary, we can now remove the ± if we allow K to be negative:

$$y = K(1+x)$$

This is the general solution.
5. We use $y(1) = -\frac{1}{2}$ to evaluate K:

$$-\frac{1}{2} = K(1+1) \qquad \text{so} \quad K = -\frac{1}{4}$$

and so we write the particular solution:

$$y = -\frac{1}{4}(1+x)$$

6. We verify that this is correct. First we check $y(1) = -\frac{1}{4} \times (2) = -\frac{1}{2}$, which is OK, and then we differentiate and substitute into the ODE:

$$\begin{aligned} y' &= -\frac{1}{4} \\ \frac{y}{1+x} &= \frac{-\frac{1}{4}(1+x)}{1+x} = -\frac{1}{4} = y' \end{aligned}$$

so this is OK too.

Example: Find (and verify) the solution of

$$\frac{dy}{dx} + 4 = y^2 \qquad \text{with} \quad y(0) = 4.$$

Solution:

$$y(x) = 2\left(\frac{3+e^{4x}}{3-e^{4x}}\right)$$

2.3 Linear first-order ODEs and the Integrating Factor (IF) method

Linear first-order ODEs are an important class of ODEs of the form:

$$y' + P(x)y = Q(x) \qquad \text{with} \quad y(x_0) = y_0, \tag{2.3}$$

where $P(x)$ and $Q(x)$ are given functions of x. The Integrating Factor method involves multiplying both sides of the ODE by an unknown function $R(x)$ and then choosing $R(x)$ so that the ODE is easier to solve.
Taking the ODE above and multiplying by $R(x)$ yields:

$$R(x)y' + P(x)R(x)y = Q(x)R(x)$$

Suppose that we could choose $R(x)$ so that the left-hand side (LHS) of this was the derivative of $R(x)y(x)$:

$$\frac{d}{dx}\left(R(x)y(x)\right) = R(x)y' + R'(x)y = Q(x)R(x)$$

so we see that this can be achieved if

$$\frac{dR}{dx} = RP(x).$$

Recall that $P(x)$ is given to us, so this equation is separable, and can be solved:

$$\begin{aligned} \frac{1}{R}\frac{dR}{dx} &= P \\ \int \frac{1}{R}\frac{dR}{dx}\,dx = \int \frac{1}{R}\,dR &= \int P(x)\,dx \\ \log|R| &= \int P(x)\,dx + C \\ R &= \pm e^C \exp\left(\int P(x)\,dx\right). \end{aligned}$$

Since multiplying R by a constant won't change it's ability to convert $y' + P(x)y$ into $(Ry)'$, we can choose the + sign and set e^C to 1, resulting in an expression for R:

$$R(x) = \exp\left(\int P(x)\,dx\right). \tag{2.4}$$

$R(x)$ is called the **integrating factor** of the first-order linear ODE (2.3), and $R(x)$ satisfies $R' = RP$.
Now we have a method for solving first-order linear ODEs:
1. Arrange the ODE into the form of (2.3):

$$y' + P(x)y = Q(x)$$

and calculate the **integrating factor** $R(x)$:

$$R(x) = \exp\left(\int P(x)\,dx\right),$$

so $R' = RP$.
2. Multiply both sides of the ODE by $R(x)$ and write the LHS as a **perfect derivative**:

$$Ry' + RPy = Ry' + R'y = \frac{d}{dx}\left(R(x)y\right) = R(x)Q(x)$$

3. **Integrate** both sides with respect to x, not forgetting the **constant of integration**:

$$\int \frac{d}{dx}\left(R(x)y\right)\,dx = R(x)y = \int R(x)Q(x)\,dx + C$$

In principle, we can evaluate the integral on the RHS.
4. **Divide** by $R(x)$ to get the **general solution**:

$$y = \frac{1}{R(x)}\int R(x)Q(x)\,dx + \frac{C}{R(x)}.$$

Note that the x's inside and outside the integral have different meaning.
5. Finally, we can then use the **initial condition** $y(x_0) = y_0$ to evaluate the constant of integration C, and so find the **particular solution**.
6. Last of all, we can **verify** that the answer satisfies the ODE and the initial value.

Example: Consider

$$\frac{dy}{dx} + xy = x \qquad \text{with} \quad y(0) = 5.$$

Find the explicit solution, and verify that the answer is correct.

Solution: Here,

$$P(x) = x \qquad \text{and} \qquad Q(x) = x$$

Proceed as above:
1. Calculate the integrating factor $R(x)$:

$$R(x) = \exp\left(\int x\,dx\right) = \exp\left(\frac{x^2}{2}\right) = e^{\frac{x^2}{2}}$$

2. and 3. Multiply both sides by $R(x)$ and integrate:

$$\int \frac{d}{dx}\left(R(x)y\right)\,dx = R(x)y = \int xe^{\frac{x^2}{2}}\,dx + C = e^{\frac{x^2}{2}} + C$$

using the substitution $u = \frac{x^2}{2}$.
4. Divide by $R(x)$ to get y:

$$y = 1 + Ce^{-\frac{x^2}{2}}$$

This is the general solution.
5. Use the initial condition to evaluate C:

$$y(0) = 5 = 1 + Ce^0 = 1 + C, \qquad\qquad C = 4$$

so the particular solution is

$$y = 1 + 4e^{-\frac{x^2}{2}}$$

6. Verify the initial condition: $y(0)$ is indeed 5, and verify that the ODE is satisfied:

$$\begin{aligned} y' &= -4xe^{-\frac{x^2}{2}} \\ y' + xy &= -4xe^{-\frac{x^2}{2}} + x + 4xe^{-\frac{x^2}{2}} = x \qquad \text{OK} \end{aligned}$$

Example: Find (and verify) the general solution of

$$(x+1)y' + y = x(x+1)$$

Solution:

$$y(x) = \frac{\frac{1}{3}x^3 + \frac{1}{2}x^2 + C}{x+1}$$

2.4 ODEs that can be transformed into Separable or Linear first-order

There are many different kinds of first-order ODEs that can be transformed into one of the forms discussed above, and then solved – far more than we will do here.

2.4.1 Homogeneous ODEs

Homogeneous ODEs are properly called equations of homogeneous degree. They are ODEs that can be arranged into the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \tag{2.5}$$

that is, the RHS is a function of the combination $\frac{y}{x}$.
This can be solved by writing

$$y(x) = xv(x),$$

where $v(x)$ is a new dependent variable. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(xv) = v + x\frac{dv}{dx} = f\left(\frac{y}{x}\right) = f(v) \\ \frac{dv}{dx} &= \frac{f(v) - v}{x}. \end{aligned}$$

This is a separable equation for v and can be solved by the method of section 2.2. Remember to return to the original dependent variable $y(x)$ at the end of the calculation.

Example: Consider

$$x\frac{dy}{dx} = x + 3y \qquad \text{with} \quad y(1) = 2.$$

Find the explicit solution, and verify that the answer is correct.

Solution: Divide by x to get $y' = 1 + 3\frac{y}{x}$, and then proceed as above. Here

$$y' = f\left(\frac{y}{x}\right) = 1 + \frac{3y}{x} \qquad \text{so} \qquad f(v) = 1 + 3v.$$

Writing $y = xv$, we have

$$y' = v + xv' = f(v) = 1 + 3v$$

or

$$v' = \frac{1 + 2v}{x}$$

This is separable, so we use the method of section 2.2:
1. Bring the function of the dependent variable to the LHS:

$$\frac{1}{1 + 2v}\frac{dv}{dx} = \frac{1}{x}$$

2. and 3. Now integrate with respect to x, not forgetting the constant of integration, and change variables in the integral for v:

$$\int \frac{1}{1+2v}\frac{dv}{dx}\,dx = \int \frac{1}{1+2v}\,dv = \int \frac{1}{x}\,dx + C$$

4. We can evaluate these indefinite integrals:

$$\frac{1}{2}\log|1+2v| = \log|x| + C$$

and we can re-arrange to find v explicitly by multiplying by 2 and taking exponentials of both sides:

$$\exp(\log|1+2v|) = |1+2v| = \exp\left(2\log|x| + 2C\right) = x^2 e^{2C}.$$

Remove the absolute value signs:

$$1 + 2v = \pm e^{2C}\,x^2$$

If we define $K = \pm e^{2C}$ (a new constant), we can solve for v:

$$v = \frac{1}{2}\left(Kx^2 - 1\right)$$

Recall that $v = y/x$, so we multiply both sides by x to get the general solution:

$$y = \frac{x}{2}\left(Kx^2 - 1\right)$$

5. We use $y(1) = 2$ to evaluate K:

$$2 = \frac{1}{2}(K-1) \qquad \text{so} \quad K = 5$$

and so we write the particular solution:

$$y = \frac{x}{2}\left(5x^2 - 1\right)$$

6. We verify that this is correct. First we check $y(1) = \frac{1}{2}(5-1) = 2$, which is OK, and then we differentiate and substitute into the ODE:

$$\begin{aligned} xy' &= \frac{x}{2}\left(15x^2 - 1\right) \\ x + 3y &= \frac{x}{2}\left(15x^2 - 1\right) = xy' \end{aligned}$$

so this is OK too.

Example: Find (and verify) the general solution of

$$x^2\frac{dy}{dx} = y^2 + 3xy + x^2$$

Solution:

$$y(x) = -x\left(1 + \frac{1}{C + \log|x|}\right)$$

2.4.2 Bernoulli's equation

Jacob Bernoulli (1654–1705) was one of the many prominent mathematicians in the Bernoulli family.
Bernoulli's equation is an equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{2.6}$$

where $n \neq 1$. (If $n = 1$, write $(P(x) - Q(x))y$, and we get a first-order linear ODE.)
This can be solved by writing

$$z(x) = y^{1-n},$$

where $z(x)$ is a new dependent variable. Then

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

Now divide (2.6) by y^n to get

$$\begin{aligned} y^{-n}\frac{dy}{dx} + P(x)y^{1-n} &= Q(x) \\ \frac{1}{1-n}\frac{dz}{dx} + P(x)z &= Q(x) \\ \frac{dz}{dx} + (1-n)P(x)z &= (1-n)Q(x) \end{aligned}$$

This is a first-order linear ODE for z and can be solved by the method of section 2.3. Remember to return to the original dependent variable $y(x)$ at the end of the calculation.

Example: Consider

$$\frac{dy}{dx} + 4y = e^x y^3$$

Find the general solution, and verify that the answer is correct.

Solution: Here $n = 3$, $P(x) = 4$ and $Q(x) = e^x$. Write $z(x) = y^{1-n} = y^{-2}$, and

$$\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$$

Divide the ODE by y^3 to get

$$\begin{aligned} y^{-3}\frac{dy}{dx} + 4y^{-2} &= e^x \\ -\frac{1}{2}\frac{dz}{dx} + 4z &= e^x \\ \frac{dz}{dx} - 8z &= -2e^x \end{aligned}$$

This is a first-order linear ODE for z, so we use the method of section 2.3.
1. Here, we've already used the names $P(x)$ and $Q(x)$, but we can calculate the integrating factor $R(x)$:

$$R(x) = \exp\left(\int -8\,dx\right) = e^{-8x}$$

2. and 3. Multiply both sides by $R(x)$ and integrate:

$$\int \frac{d}{dx}\left(R(x)z\right)\,dx = R(x)z = \int -2e^{x}e^{-8x}\,dx + C = \int -2e^{-7x}\,dx + C = \frac{2}{7}e^{-7x} + C$$

4. Divide by $R(x)$ to get z:

$$z = \frac{2}{7}e^{x} + Ce^{8x}$$

Remember that $z = y^{-2}$, so $y = \pm z^{-1/2}$:

$$y(x) = \frac{\pm 1}{\sqrt{\frac{2}{7}e^{x} + Ce^{8x}}}$$

This is the general solution.
6. Verify that the ODE is satisfied (taking the + case only for ease of writing):

$$\begin{aligned} y' &= -\frac{\frac{2}{7}e^{x} + 8Ce^{8x}}{2\left(\frac{2}{7}e^{x} + Ce^{8x}\right)^{3/2}} \\ y' + 4y &= -\frac{\frac{2}{7}e^{x} + 8Ce^{8x}}{2\left(\frac{2}{7}e^{x} + Ce^{8x}\right)^{3/2}} + \frac{4}{\left(\frac{2}{7}e^{x} + Ce^{8x}\right)^{1/2}} \\ &= \frac{-\frac{1}{7}e^{x} - 4Ce^{8x} + \frac{8}{7}e^{x} + 4Ce^{8x}}{\left(\frac{2}{7}e^{x} + Ce^{8x}\right)^{3/2}} \\ &= \frac{e^{x}}{\left(\frac{2}{7}e^{x} + Ce^{8x}\right)^{3/2}} = e^{x}y^{3} \qquad \text{OK} \end{aligned}$$

Example: Find (and verify) the general solution of

$$\frac{dy}{dx} + \frac{y}{x} = 2x^{3}y^{4}$$

Solution:

$$y(x) = \left(-6x^4 + Cx^3\right)^{-1/3}$$

2.5 Exact first-order ODEs

Suppose the general solution of some first-order ODE can be written in the form

$$H(x,y)=C,$$

where H is a function of two variables whose partial derivatives with respect to x and y are themselves differentiable, and C is a constant. For each value of C, this equation defines a curve in the (x,y) plane. If we think of y as a function of x along these curves, we get:

$$H(x,y(x))=C.$$

Differentiate with respect to x using the chain rule:

$$\frac{d}{dx}H(x,y(x))=\frac{\partial H}{\partial x}+\frac{\partial H}{\partial y}\frac{dy}{dx}=0.$$

Recall that if $\frac{\partial H}{\partial x}$ and $\frac{\partial H}{\partial y}$ are differentiable, then

$$\frac{\partial^2 H}{\partial x\partial y}=\frac{\partial^2 H}{\partial y\partial x}.$$

Now, suppose that we have an ODE of the form:

$$M(x,y)+N(x,y)\frac{dy}{dx}=0, \tag{2.7}$$

where the partial derivatives of M and N exist.

There are two questions. First, under what circumstances can we reverse the chain rule and obtain $H(x,y)=C$ as the solution of the ODE? If this can be done, then the function H should satisfy:

$$\frac{\partial H}{\partial x}=M(x,y) \qquad \text{and} \qquad \frac{\partial H}{\partial y}=N(x,y).$$

If such an H exists, then $\frac{\partial^2 H}{\partial x\partial y}=\frac{\partial^2 H}{\partial y\partial x}$, so M and N must satisfy

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}.$$

In this case, the ODE (2.7) is called **exact**.

The second question is, given that the ODE is exact ($\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$), how can the function H be found? In general H can be hard (or impossible) to find, but a process of **"anti-partial differentiation"** can sometimes work:

$$\begin{aligned}
\frac{\partial H}{\partial x} &= M(x,y) \qquad \text{so} \qquad H(x,y)=\int M(x,y)\,dx+C_1(y),\\
\frac{\partial H}{\partial y} &= N(x,y) \qquad \text{so} \qquad H(x,y)=\int N(x,y)\,dy+C_2(x),
\end{aligned}$$

where $C_1(y)$ and $C_2(x)$ are arbitrary functions. The aim is to find $C_1(y)$ and $C_2(x)$ such that the two candidate solutions above are equal.

Example: Find (and verify) the solution of

$$2xy\frac{dy}{dx} - 2x + y^2 = 0 \quad \text{with} \quad y(1) = 2.$$

Solution: Here $M(x,y) = -2x + y^2$ and $N(x,y) = 2xy$. We can verify that:

$$\frac{\partial M}{\partial y} = 2y \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 2y,$$

so the ODE is exact. We attempt the anti-partial differentiation:

$$\int M(x,y)\,dx = -x^2 + xy^2 + C_1(y), \qquad \text{and} \qquad \int N(x,y)\,dy = xy^2 + C_2(x).$$

These are equal if we choose $C_1(y) = 0$ and $C_2(x) = -x^2$, so the general solution is:

$$H(x,y) = -x^2 + xy^2 = C.$$

Putting in the initial value $y(1) = 2$ gives $C = 3$, so the implicit solution is

$$-x^2 + xy^2 = 3.$$

In this case, this can be written explicitly as

$$y = \sqrt{\frac{x^2+3}{x}},$$

where we have chosen the positive square root to satisfy $y(1) = 2$. To verify that this is correct, we check $y(1) = \sqrt{4} = 2$, then we differentiate and substitute into the ODE:

$$\begin{aligned} y' &= \frac{1}{2}\left(\frac{x^2+3}{x}\right)^{-\frac{1}{2}}\left(\frac{(2x)x-(x^2+3)}{x^2}\right) = \frac{1}{2y}\left(\frac{x^2-3}{x^2}\right) \\ 2xy\,y' &= \frac{x^2-3}{x} \\ y^2 - 2x &= \frac{x^2+3}{x} - 2x = \frac{3-x^2}{x} \end{aligned}$$

so $2xy\,y' - 2x + y^2 = 0$ and the ODE is satisfied.

Notes:

1. If an ODE is not exact, sometimes multiplying it by a function of x or y (or of both) can make it exact. Such functions are called integrating factors. For example, dividing the example above by y yields an equation (which happens to be a Bernoulli equation) whose integrating factor is y.

2. A separable ODE $y' = a(x)b(y)$ (2.2) written in the form

$$\frac{1}{b(y)}\frac{dy}{dx} = a(x)$$

is exact, since $M = -a(x)$ is not a function of y and $N = \frac{1}{b(y)}$ is not a function of x.

3. Similarly, a linear first-order ODE $y' + P(x)y = Q(x)$ (2.3) is exact, once it has been multiplied by its integrating factor $R(x)$ (with $R' = RP$)

$$Ry' + RPy = RQ,$$

since $M(x,y) = R(x)P(x)y - R(x)Q(x)$ and $N(x,y) = R(x)$ satisfy $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = R'$.

Chapter 3

Applications of first-order ODEs

Contents

3.1 Radioactivity and Carbon dating

There are three isotopes of Carbon: ^{12}C, ^{13}C and ^{14}C. Almost all Carbon is made up of the first two (^{12}C and ^{13}C) because ^{14}C is radioactive: it decays with a half-life of 5730 years to form ^{14}N. Although it decays quite quickly, it is constantly being produced in the upper atmosphere by the action of cosmic rays. The equilibrium level of ^{14}C is about 1 part per trillion (10^{12}).

When an organism dies, it ceases to absorb Carbon from the environment, so the amount of ^{14}C it contains will decrease as this decays radioactively. The time since the death of the organism can be estimated by measuring how much ^{14}C there is left.

Let $x(t)$ be the proportion of ^{14}C at time t. In a short interval δt, the concentration will decrease by an amount proportional to how much is left, and to the length of the time interval:

$$x(t + \delta t) = x(t) - kx(t)\delta t$$

where k is a positive constant. Rearranging:

$$\frac{x(t+\delta t) - x(t)}{\delta t} = -kx$$

and taking the limit of small δt (see chapter 1):

$$\frac{dx}{dt} = -kx \qquad \text{with } x(t_0) = x_0. \tag{3.1}$$

We can solve this separable first-order ODE:

$$\begin{aligned}\int \frac{1}{x}\frac{dx}{dt}\,dt = \int \frac{1}{x}\,dx &= -\int k\,dt + C\\ \log|x| &= -kt + C\\ |x| &= e^C e^{-kt}\end{aligned}$$

Use the fact that $x > 0$ and set $A = e^C$:

$$x = Ae^{-kt}$$

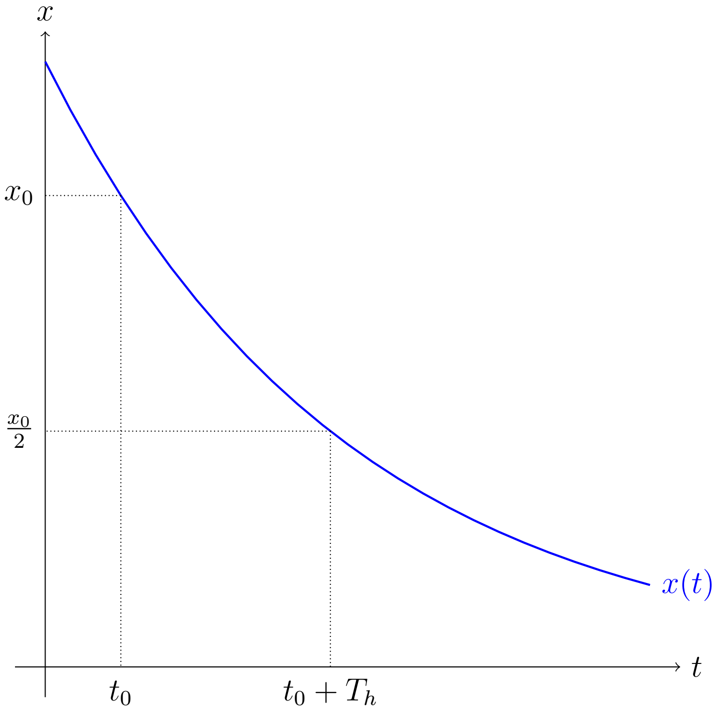
We can now use the initial condition $x(t_0) = x_0$ to find A:

$$x_0 = Ae^{-kt_0}, \qquad A = x_0 e^{kt_0}$$

which results in the solution for $x(t)$:

$$x = x_0 e^{-k(t-t_0)}. \tag{3.2}$$

Graphically:



The **half-life** T_h is the time after which half of the ^{14}C has decayed. So, from (3.2), at time $t_0 + T_h$:

$$x(t_0 + T_h) = \frac{x_0}{2} = x_0 e^{-k(t_0+T_h-t_0)} = x_0 e^{-kT_h}.$$

or

$$T_h = \frac{1}{k}\log 2.$$

For ^{14}C, $T_h = 5730\,\text{years}$, so $k = \frac{1}{T_h}\log 2 = 0.000121/\text{year}$.

Example: A fossilised bone is found to contain 0.1% of its original ^{14}C. Find the age of the fossil.

Solution: Let t be today's date, and suppose the fossil is T years old, so it was fossilised at time $t_0 = t - T$. At this time it had x_0 ^{14}C, and now it has $0.001x_0$.

$$x(t) = 0.001x_0 = x_0 e^{-k(t-t_0)} = x_0 e^{-kT},$$

which we can solve for T:

$$T = -\frac{1}{k}\log 0.001 = \frac{1}{k}\log 1000 = 57100\,\text{years}.$$

Example: A nuclear breeder reactor produces waste that contains (amongst other things) the isotope ^{239}Pu (Plutonium-239). After 15 years, the initial concentration of ^{239}Pu in the waste has decreased by 0.043%. Find the half-life of the isotope.

Solution:

Example: A sample of thread contains 10^{10} atoms of ^{14}C. How many disintegrations per second will there be?

Solution: $\frac{dx}{dt} = -kx$, so there are 0.000121×10^{10} disintegrations per year, or 0.04 disintegrations per second.

3.2 Population growth models

Let $p(t)$ be the population of a country at time t. From chapter 1:

$$\frac{dp}{dt} = k(p,t) = B(p,t) - D(p,t) + M(p,t),$$

where
$B(p,t)$ represents input (births etc.)
$D(p,t)$ represents output (deaths etc.)
$M(p,t)$ represents net migration into the country.
The total rate of change of population is $k(p,t) = B(p,t) - D(p,t) + M(p,t)$. For now, we will assume that $M(p,t) = 0$. To progress further, we need to model the birth and death rate.

3.2.1 The Malthusian model

Thomas Robert Malthus FRS: 1766–1834, English clergyman, political economist and demographer.

Malthus (1798) suggested a model for the birth and death rates: these are proportional to the population:

$$B(p,t) = bp(t), \qquad \text{and} \qquad D(p,t) = dp(t),$$

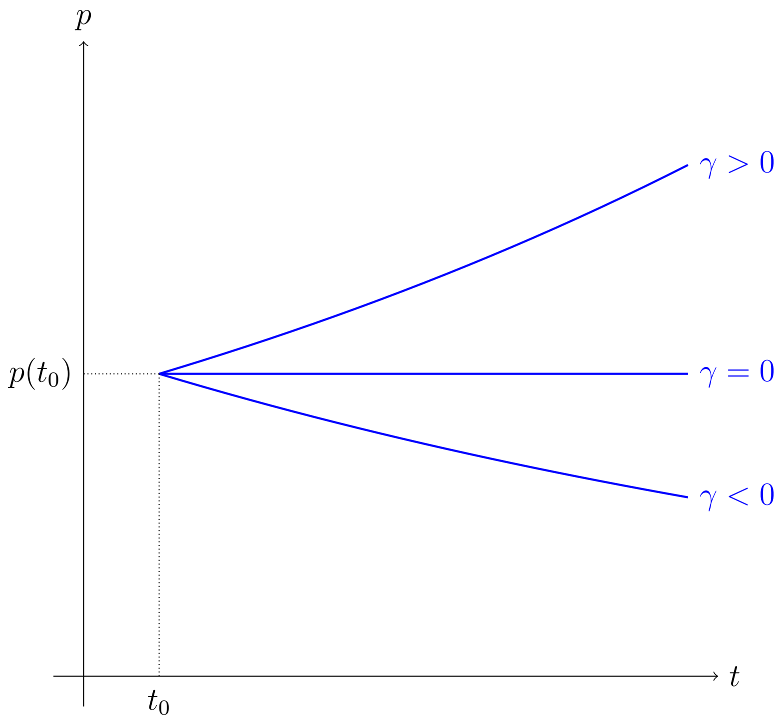
where b and d are constants, so

$$\frac{dp}{dt} = (b-d)p = \gamma p \tag{3.3}$$

where $\gamma = b - d$ is a constant: γ is called the **growth rate**. We can solve this equation:

$$p(t) = p(t_0)e^{\gamma(t-t_0)}$$

This model is quite limited, and it predicts that the population with increase without bound if $\gamma > 0$.



Example: In 1770, the population of Great Britain was estimated to be 6.4 million. By 1790, the population had grown to 8 million. Estimate γ, and predict the population in the year 2010.

Solution: Take $t_0 = 1770$, and $p(t_0) = 6.4 \times 10^6$. So

$$p(1790) = 8 \times 10^6 = p(t_0)e^{(1790-1770)\gamma} = 6.4 \times 10^6 \times e^{20\gamma}.$$

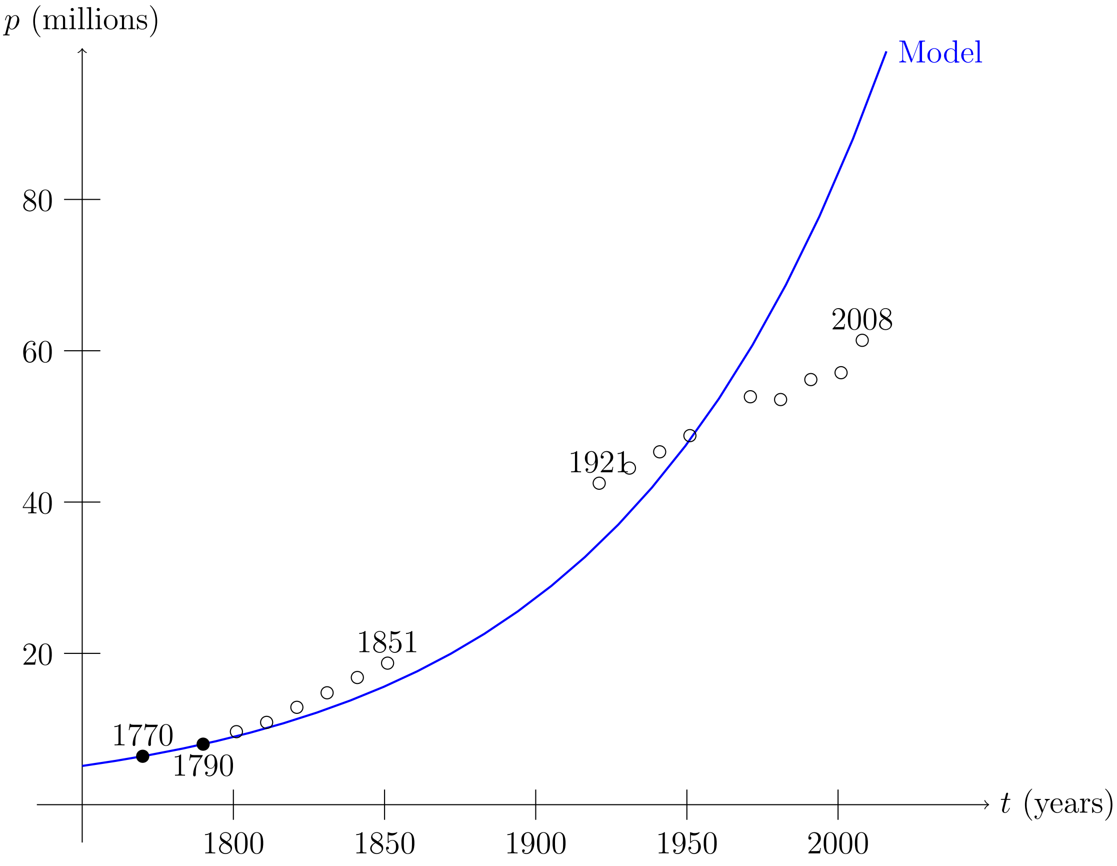
which results in

$$\gamma = \frac{1}{20}\log\left(\frac{8 \times 10^6}{6.4 \times 10^6}\right) = 0.0112/\text{year}$$

Now we can get $p(2010)$:

$$p(2008) = p(t_0)e^{(2010-1770)\gamma} = 6.4 \times 10^6 \times e^{240\times 0.0112} = 93 \times 10^6$$

In fact, the current population of Great Britain is about 61×10^6 – not bad agreement considering how simple the model is. However, looking in more detail suggests that the agreement is not that good – not surprising, since we have not included effects like immigration, birth and death rates that change with time, changes in agriculture that allow more food production etc.



3.2.2 The Logistic model

Pierre François Verhulst: 1804–1849, Belgian mathematician.
One problem with the Malthusian model is that it predicts either the population grows without bound ($\gamma > 0$) or that it decays to extinction ($\gamma < 0$). Of course, populations cannot grow without bound – there can be competition for food, resources or space – and this effect can be modelled by supposing that the growth rate γ depends on p. Suppose that there is a maximum population $p_\infty > 0$, for which the growth rate γ is zero, and γ is positive if $p < p_\infty$. The simplest model of this would have γ depending linearly on p:

$$\gamma = \mu \left(1 - \frac{p}{p_\infty} \right)$$

where μ is the growth rate in the limit of very small population ($p \to 0$).
Putting this together results in the **logistic equation**:

$$\frac{dp}{dt} = \mu p \left(1 - \frac{p}{p_\infty} \right) \qquad \text{with } p(t_0) = p_0 \tag{3.4}$$

This population model was first written down by Verhulst (1838) and is a successful model of yeast, bacteria or fruit flies (in a controlled environment), but still too simple for more realistic situations.

Nonetheless, we can solve the separable ODE (3.4) (it is also a Bernoulli equation):

$$\begin{aligned}
\int \frac{1}{p\left(1-\frac{p}{p_\infty}\right)}\frac{dp}{dt}\,dt = \int \frac{p_\infty}{p(p_\infty - p)}\,dp &= \int \mu\,dt + C \\
\int \left(\frac{1}{p} + \frac{1}{p_\infty - p}\right) dp &= \mu t + C \\
\log|p| - \log|p_\infty - p| &= \mu t + C \\
\frac{p}{p_\infty - p} &= Ae^{\mu t}
\end{aligned}$$

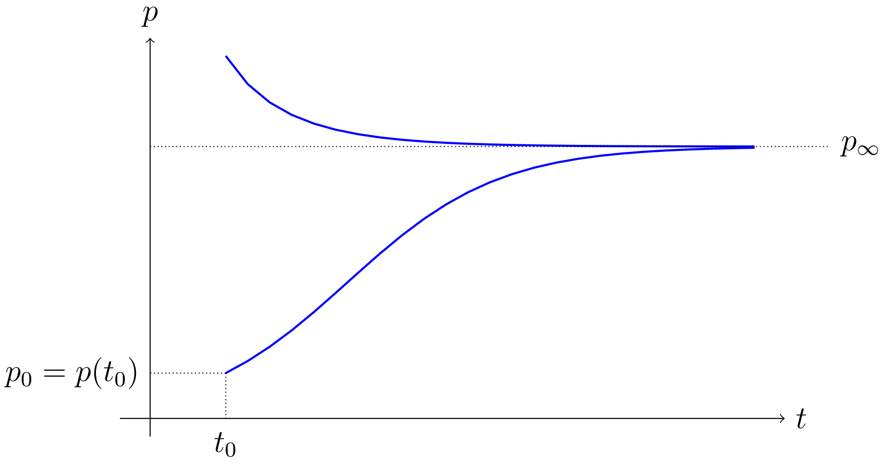
where $A = \pm e^C$ and we have used partial fractions to do the integral. We can now use the initial condition $p(t_0) = p_0$ to find A:

$$A = \frac{p_0}{p_\infty - p_0}e^{-\mu t_0} \qquad \text{so} \qquad \frac{p}{p_\infty - p} = \frac{p_0}{p_\infty - p_0}e^{\mu(t-t_0)}$$

Now rearrange to find $p(t)$:

$$\begin{aligned}
p &= \frac{p_0}{p_\infty - p_0}(p_\infty - p)e^{\mu(t-t_0)} \\
p\left(1 + \frac{p_0}{p_\infty - p_0}e^{\mu(t-t_0)}\right) &= p_\infty \frac{p_0}{p_\infty - p_0}e^{\mu(t-t_0)} \\
p(t) &= \frac{p_\infty p_0 e^{\mu(t-t_0)}}{p_\infty - p_0 + p_0 e^{\mu(t-t_0)}} \\
&= \frac{p_\infty p_0}{(p_\infty - p_0)e^{-\mu(t-t_0)} + p_0} \\
&= \frac{p_\infty}{1 + \left(\frac{p_\infty}{p_0} - 1\right)e^{-\mu(t-t_0)}}
\end{aligned}$$

Exercise: Verify that this expression has $p(t_0) = p_0$ and that it satisfies the logistic equation (3.4). Verify also that as $t \to \infty$, we have $e^{-\mu(t-t_0)} \to 0$, so $p(t) \to p_\infty$.



The graph illustrates that if $0 < p_0 < p_\infty$, the population grows, and saturates at p_∞, but if $p_0 > p_\infty$, the population decays down to p_∞ (assuming that $p_\infty > 0$). What happens if $p_0 = 0$? What happens if $p_0 < 0$? (This is trickier than it looks!)

3.3 Newton's law of cooling

Sir Isaac Newton FRS: 1643–1727, English physicist, mathematician, astronomer, natural philosopher, alchemist and theologian.
Newton's law of cooling states that if an object is hotter than the ambient temperature, then the rate of cooling of the object is proportional to the temperature difference:

$$\frac{d\Theta}{dt} = -k(\Theta - A), \qquad \text{with } \Theta(t_0) = \Theta_0 \tag{3.5}$$

where $\Theta(t)$ is the object's temperature, A is the ambient temperature (a constant), and k is a positive constant. This is a first-order linear ODE:

$$\frac{d\Theta}{dt} + k\Theta = kA$$

so the integrating factor is $R(t) = \exp\left(\int k\,dt\right) = \exp(kt)$. If we multiply (3.5) by this integrating factor, we get:

$$\frac{d}{dt}\left(\Theta e^{kt}\right) = kAe^{kt}$$

Integrating this, using the initial condition and rearranging results in:

$$\Theta(t) = A + (\Theta_0 - A)e^{-k(t-t_0)}$$

Exercise: Verify that this expression is correct.
Note that as $t \to \infty$, we have $\Theta(t) \to A$.

3.4 Mixing problems

Suppose we have a container full of a salt solution of a certain concentration. If we pump in a solution with a different concentration at a certain rate, mix well and extract the mixture at the same rate, how does the concentration change with time? To solve this, we consider the law of conservation of salt:

$$\begin{pmatrix}\text{Rate of change of}\\ \text{salt in the container}\end{pmatrix} = \begin{pmatrix}\text{Rate of inflow of}\\ \text{salt into the container}\end{pmatrix} - \begin{pmatrix}\text{Rate of outflow of}\\ \text{salt from the container}\end{pmatrix}$$

Example: A $1000\,\ell$ tank of water initially contains $10\,\mathrm{kg}$ of dissolved salt. A pipe brings a salt solution (concentration $0.005\,\mathrm{kg}\ell^{-1}$) into the tank at a rate of $2\,\ell\mathrm{s}^{-1}$, and a second pipe carries away the excess solution. Calculate $C(t)$, the concentration of salt, assuming that the tank is well mixed.

Solution: Let $V = 1000\,\ell$ be the volume of water in the tank (this is constant) and let $m(t)$ be the mass of salt dissolved in the water, so $m(0) = 10\,\mathrm{kg}$. Let $C(t) = m/V$ be the concentration of salt in the water, assuming that the salt is well mixed, so $C(0) = 0.01\,\mathrm{kg}\ell^{-1}$.

The rate of inflow of salt (in kg/s) is $C_{in}r_{in}$, where $C_{in} = 0.005\,\mathrm{kg}\ell^{-1}$ is the concentration of salt in the inflow, and $r_{in} = 2\,\ell\mathrm{s}^{-1}$ is the rate of inflow.
The rate of outflow of salt (in kg/s) is $C_{out}r_{out}$, where $C_{out} = C(t)$ is the concentration of salt in the outflow, equal to the concentration of salt in the water in the tank in this case, and $r_{out} = r_{in} = 2\,\ell\mathrm{s}^{-1}$ is the rate of outflow, equal to the rate of inflow in this case.
From the law of concentration of salt, we get

$$\frac{dm}{dt} = C_{in}r_{in} - C_{out}r_{out}$$

Now make use of the fact that $m(t) = C(t)V$, $r_{out} = r_{in}$ and $C_{out} = C(t)$ to get

$$V\frac{dC}{dt} = (C_{in} - C)r_{in} \quad \text{or} \quad \frac{dC}{dt} + \frac{r_{in}}{V}C = \frac{r_{in}}{V}C_{in}$$

This is a first-order linear ODE; its integrating factor is $\exp\left(\frac{r_{in}}{V}t\right)$ and the solution in this case is

$$C(t) = C_{in} + K\exp\left(\frac{-r_{in}}{V}t\right),$$

where K is a constant. Putting in the initial condition and the numbers results in:

$$C(t) = 0.005 + 0.005e^{-0.002t}$$

where C is in $\mathrm{kg}\ell^{-1}$ and t is in seconds. (Exercise: work through the details and verify this yourself.)

Example: A $1000\,\ell$ tank of water is initially half-full of fresh water. A pipe brings a salt solution (concentration $0.005\,\mathrm{kg}\ell^{-1}$) into the tank at a rate of $5\,\ell\mathrm{s}^{-1}$, and a second pipe carries away the overflow. Calculate $C(t)$, the concentration of salt, assuming that the tank is well mixed.

Solution:

$$C(t) = \begin{cases} \dfrac{0.025t}{500+5t} & 0 \le t \le 100 \\ 0.005 - 0.0025\exp(-0.005(t-100)) & t \ge 100 \end{cases}$$

3.5 First-order model of supply and demand

Let $P(t)$ be the price of a product, and let Q_S and Q_D be the supply of and demand for the product. Suppose that the supply of a product increases as the price goes up, that the demand for a product decreases as the price goes up, and that the rate of change of price of the product is positive (the price goes up) when demand exceeds supply. This situation can be described by a first-order differential equation.

Example: Suppose that

$$Q_D = A - BP \qquad \text{and} \qquad Q_S = C + DP,$$

where A, B, C and D are postive constants. Suppose also that

$$\frac{dP}{dt} = E(Q_D - Q_S), \qquad \text{with} \qquad P(0) = P_0.$$

Set up, solve and interpret the ODE for $P(t)$.

Solution:

Chapter 4

Second-order linear ODEs

Contents

4.1 Introduction

Second-order linear ODEs are of the form:

$$e(x)\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(x)y = h(x), \tag{4.1}$$

where $e(x)$, $f(x)$, $g(x)$ and $h(x)$ are given functions of x. The equation is second-order because the highest derivative is $\frac{d^2y}{dx^2}$, and it is linear because there are no products of y with itself or its derivatives.

There are two important classifications:

The equation is **homogeneous** if $h(x) \equiv 0$, otherwise it is **inhomogeneous**. Note that the word "homogeneous" is used in a sense different from that of section 2.4.1

The equation is **constant coefficient** if $e(x)$, $f(x)$ and $g(x)$ are all constants (rather than functions), otherwise it is **non-constant coefficient**.

The homogeneous equation corresponding to (4.1) is:

$$e(x)\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(x)y = 0 \tag{4.2}$$

Note that $y = 0$ is always a solution of this equation.

The general solution of (4.1) will contain two arbitrary constants (since the equation is second-order). The particular solution can be specified in two ways:

Initial value problem (IVP): We are given the values of y and y' at a single value of x.

Boundary value problem (BVP): We are given the values of y at two different values of x.
In the case of initial value problems, there is a useful theorem that tells us there is a unique solution of the ODE (4.1).

Theorem (for IVPs): Let $e(x)$, $f(x)$, $g(x)$ and $h(x)$ be continuous on an interval I of the x-axis, and let $e(x) \neq 0$ for all $x \in I$. If the initial condition is specified at a point in I, then the solution of equation (4.1) **exists** and is **unique**.

Example: Show that

$$y = C_1 \sin x + C_2 \cos x,$$

where C_1 and C_2 are arbitrary constants, satisfies the ODE

$$y'' + y = 0.$$

Hence solve the IVP:

$$y'' + y = 0, \qquad \text{with } y(0) = 0 \text{ and } y'(0) = 1,$$

and the BVP:

$$y'' + y = 0, \qquad \text{with } y(0) = 0 \text{ and } y(\pi) = 1.$$

Solution: Note that $e = 1$, $f = 0$ and $g = 1$ are all continuous functions of x, and $e \neq 0$, so the existence theorem for IVPs applies. To show that the given function satisfies the ODE, we need some derivatives:

$$y' = C_1 \cos x - C_2 \sin x, \qquad y'' = -C_1 \sin x - C_2 \cos x = -y,$$

so $y'' + y = 0$. For both the IVP and the BVP, we have $y(0) = 0$. Substitute $x = 0$ into the expression for $y(x)$:

$$y(0) = C_1 \sin 0 + C_2 \cos 0 = C_2 = 0$$

so $C_2 = 0$ and $y(x) = C_1 \sin x$.
For the IVP, we have $y'(0) = C_1 = 1$, so $C_1 = 1$ and $y(x) = \sin x$: the solution exists and is unique, as expected.
For the BVP, we have $y(\pi) = C_1 \sin \pi = 0 \neq 1$, so we cannot have $y(\pi) = 1$: there is no solution. This is an example of an **ill-posed problem**, and is one reason why the existence theorem is only for Initial value problems.

4.2 Superposition principle for linear ODEs

Suppose that $y_1(x)$ and $y_2(x)$ are two solutions of the homogeneous linear ODE (4.2). Then $C_1y_1 + C_2y_2$ is also a solution of (4.2), where C_1 and C_2 are any constants. This combination $C_1y_1 + C_2y_2$ is called a linear superposition (or linear combination) of the two functions $y_1(x)$ and $y_2(x)$.

Proof: First we calculate the derivatives of the linear combination:

$$\begin{aligned}
y &= C_1y_1 + C_2y_2 \\
y' &= C_1y_1' + C_2y_2' \\
y'' &= C_1y_1'' + C_2y_2'' \\
ey'' + fy' + gy &= e\,(C_1y_1'' + C_2y_2'') + f\,(C_1y_1' + C_2y_2') + g\,(C_1y_1 + C_2y_2) \\
&= C_1\,(ey_1'' + fy_1' + gy_1) + C_2\,(ey_2'' + fy_2' + gy_2) \\
&= 0
\end{aligned}$$

So y satisfies (4.2).
Now suppose that $y_1(x)$ is a solution of the inhomogeneous ODE(4.1), and that $y_2(x)$ is a solution of the homogeneous (4.2). Then $y_1(x) + C_2y_2(x)$ is also a solution of the inhomogeneous ODE(4.1). The proof is similar.
Note that this can be extended to higher order. Note also that this principle does not hold for nonlinear ODEs.

4.3 Reduction of order for linear ODEs

In general, (4.1) can be hard to solve. However, suppose we have managed to find one solution of the homogeneous ODE (4.2) (perhaps by a lucky guess). We can use this solution to turn the second-order linear ODE (4.1) into a first-order linear ODE, which we can then solve using the methods of Chapter 2.3. This technique is called **reduction of order**.
1. Let $y_1(x)$ be the solution of (4.2) that we know, so

$$ey_1'' + fy_1' + gy_1 = 0$$

Now let

$$y(x) = u(x)y_1(x),$$

where $u(x)$ is an unknown function of x. We will derive a first-order ODE for u'.
2. Calculate some derivatives:

$$\begin{aligned}
y &= uy_1 \\
y' &= u'y_1 + uy_1' \\
y'' &= u''y_1 + 2u'y_1' + uy_1''
\end{aligned}$$

3. Substitute y into (4.1), noting that the u term should cancel exactly:

$$\begin{aligned} ey'' + fy' + gy &= e\left(u''y_1 + 2u'y_1' + uy_1''\right) + f\left(u'y_1 + uy_1'\right) + g\left(uy_1\right) \\ &= \left(ey_1\right)u'' + \left(2ey_1' + fy_1\right)u' + \left(ey_1'' + fy_1' + gy_1\right)u \\ &= \left(ey_1\right)u'' + \left(2ey_1' + fy_1\right)u' \\ &= h(x) \end{aligned}$$

4. Note that u itself does not appear in this equation. We can therefore define a new dependent variable $v = u'$ and write an ODE for v:

$$\left(ey_1\right)v' + \left(2ey_1' + fy_1\right)v = h$$

5. This is a first-order linear ODE, which can be solved by the methods of section 2.3 (using the Integrating Factor method). Verify that the solution $v(x)$ is correct.
6. Once we have the general solution $v(x)$ (with one arbitrary constant C_1), we integrate once more to find $u(x) = \int v\,dx + C_2$, remembering to add the second arbitrary constant C_2. Now we have $u(x)$.
7. Hence we can find the general solution $y(x)$ of (4.1) by calculating $y = uy_1$.
8. We can use the initial values or boundary values to calculate the values of the arbitrary constants, and hence find the particular solution.
9. And last of all, we can **verify** that the answer satisfies the ODE and the initial or boundary conditions.

Example: Solve

$$xy'' + 2(x+1)y' + (x+2)y = x^2e^{-x}, \qquad \text{with } y(1) = y(2) = 0$$

given that $y_1 = e^{-x}$ is a solution of the corresponding homogeneous problem.

Solution:
1. Verify that $y_1(x) = e^{-x}$ is a solution of the corresponding homogeneous problem:

$$\begin{aligned} y_1 &= e^{-x}, \qquad y_1' = -e^{-x}, \qquad y_1'' = e^{-x} \\ xy_1'' + 2(x+1)y_1' + (x+2)y_1 &= xe^{-x} - 2(x+1)e^{-x} + (x+2)e^{-x} \\ &= 0 \end{aligned}$$

Now let

$$y(x) = u(x)y_1(x) = ue^{-x}$$

where $u(x)$ is an unknown function of x. We will derive a first-order ODE for u'.
2. Calculate some derivatives:

$$\begin{aligned} y &= ue^{-x} \\ y' &= u'e^{-x} - ue^{-x} \\ y'' &= u''e^{-x} - 2u'e^{-x} + ue^{-x} \end{aligned}$$

3. Substitute y into the ODE, noting that the u term should cancel exactly:

$$\begin{aligned} xy'' + 2(x+1)y' + y &= x\left(u''e^{-x} - 2u'e^{-x} + ue^{-x}\right) + 2(x+1)\left(u'e^{-x} - ue^{-x}\right) + (x+2)ue^{-x} \\ &= xu''e^{-x} + \left(-2x + 2(x+1)\right)u'e^{-x} + \\ &\quad + \left(x - 2(x+1) + (x+2)\right)ue^{-x} \\ &= xu''e^{-x} + 2u'e^{-x} \\ &= x^2e^{-x} \end{aligned}$$

since x^2e^{-x} is the RHS of the original ODE.

4. Note that u itself does not appear in this equation. We can therefore define a new dependent variable $v = u'$ and write an ODE for v:

$$xv' + 2v = x^2$$

or, dividing by x:

$$v' + \frac{2}{x}v = x$$

5. This is a first-order linear ODE, which can be solved by the methods of section 2.3 (using the Integrating Factor method). The Integrating Factor is:

$$R(x) = \exp\left(\int \frac{2}{x}\,dx\right) = \exp(2\log|x|) = x^2$$

Multiply the ODE by this and integrate:

$$\begin{aligned} x^2v' + 2xv = (x^2v)' &= x^3 \\ x^2v &= \frac{x^4}{4} + C_1 \\ v &= \frac{x^2}{4} + \frac{C_1}{x^2} \end{aligned}$$

Verify that the solution $v(x)$ is correct (exercise).
6. We integrate once more to find $u(x) = \int v\,dx + C_2$:

$$u = \int \left(\frac{x^2}{4} + \frac{C_1}{x^2}\right)dx = \frac{x^3}{12} - \frac{C_1}{x} + C_2$$

remembering to add the second arbitrary constant C_2.
7. The general solution $y(x)$ of the ODE is:

$$y = uy_1 = \left(\frac{x^3}{12} - \frac{C_1}{x} + C_2\right)e^{-x}$$

8. We can use the initial values or boundary values to calculate the values of the arbitrary constants, and hence find the particular solution:

$$\begin{aligned}
y(1) = 0 &= \left(\frac{1}{12} - C_1 + C_2\right) e^{-1} \\
y(2) = 0 &= \left(\frac{8}{12} - \frac{C_1}{2} + C_2\right) e^{-2} \\
0 &= 1 - 12C_1 + 12C_2 \\
0 &= 8 - 6C_1 + 12C_2 \\
0 &= 7 + 6C_1 \\
C_1 &= -\frac{7}{6}, \qquad C_2 = -\frac{5}{4} \\
y(x) &= \left(\frac{x^3}{12} + \frac{7}{6x} - \frac{5}{4}\right) e^{-x}
\end{aligned}$$

9. We can verify that the answer satisfies the ODE and the boundary conditions:

$$\begin{aligned}
y(1) &= \left(\frac{1}{12} + \frac{7}{6} - \frac{5}{4}\right) e^{-1} = 0 \\
y(2) &= \left(\frac{8}{12} + \frac{7}{12} - \frac{5}{4}\right) e^{-2} = 0 \\
y' &= \left(-\frac{x^3}{12} + \frac{x^2}{4} - \frac{7}{6x} - \frac{7}{6x^2} + \frac{5}{4}\right) e^{-x} \\
y'' &= \left(\frac{x^3}{12} - \frac{x^2}{2} + \frac{x}{2} - \frac{7}{6x} + \frac{7}{3x^2} + \frac{7}{3x^3} - \frac{5}{4}\right) e^{-x} \\
xy'' + 2(x+1)y' + (x+2)y &= \cdots = x^2 e^{-x} \qquad \text{(Exercise)}
\end{aligned}$$

4.4 Linear constant coefficient ODEs

Second-order linear ODEs with constant coefficients are of the form:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = h(x), \tag{4.3}$$

where a and b are constants, we have divided so that the coefficient of y'' is 1, and $h(x)$ is a given function of x.

Again, there is a homogeneous version of this equation:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0, \tag{4.4}$$

We will motivate the method for this type of ODE by examining the corresponding first-order problem.

4.4.1 First-order linear constant coefficient ODEs

Suppose we had the ODE:

$$\frac{dy}{dx} + ay = h(x), \tag{4.5}$$

where a is a constant. This is a first-order linear ODE, and can be solved by the integrating factor method. We write the integrating factor as:

$$R(x) = \exp\left(\int a\,dx\right) = e^{ax}$$

We multiply by this:

$$y'e^{ax} + aye^{ax} = (ye^{ax})' = he^{ax}$$

we integrate:

$$ye^{ax} = \int he^{ax}\,dx + C$$

and we divide by R:

$$y(x) = e^{-ax}\int he^{ax}\,dx + Ce^{-ax}$$

Note that the solution is made up of two parts:

The Complementary Function:

$$y_{CF} = Ce^{-ax}$$

The Particular Integral:

$$y_{PI} = e^{-ax}\int he^{ax}\,dx$$

The general solution is then the sum of the Complementary Function and the Particular Integral:

$$y(x) = y_{CF} + y_{PI}$$

We examine the two parts in turn.
The **Complementary Function** is a solution of the homogeneous version of the ODE:

$$y'_{CF} + ay_{CF} = -ae^{-ax} + ae^{-ax} = 0$$

An alternative way of finding the Complementary Function is to **postulate** that the solution of the homogeneous ODE is of the form:

$$y_{CF} = e^{\lambda x}$$

where λ is unknown. We substitute this form into the homogeneous ODE:

$$y'_{CF} + ay_{CF} = \lambda e^{\lambda x} + ae^{\lambda x} = (\lambda + a)e^{\lambda x} = 0$$

or, dividing by $e^{\lambda x}$:

$$\lambda + a = 0$$

This equation is called the **characteristic equation** of the ODE (4.5), and its solution is $\lambda = -a$, so the Complementary Function is $y_{CF} = e^{-ax}$. In general, we can multiply this by an arbitrary constant (recall section 4.2), so $y_{CF} = Ce^{-ax}$.
For the **Particular Integral**, we need to examine some examples (using integration by parts):

$$\begin{aligned}
h(x) &= 1, & y_{PI} = e^{-ax}\int e^{ax}\,dx = \frac{1}{a}\\
h(x) &= x, & y_{PI} = e^{-ax}\int xe^{ax}\,dx = \frac{ax-1}{a^2}\\
h(x) &= x^2, & y_{PI} = e^{-ax}\int x^2e^{ax}\,dx = \frac{a^2x^2-2ax+2}{a^3}\\
h(x) &= e^{kx}, & y_{PI} = e^{-ax}\int e^{kx}e^{ax}\,dx = \frac{1}{k+a}e^{kx} \quad \text{if } k+a \neq 0\\
h(x) &= e^{-ax}, & y_{PI} = e^{-ax}\int e^{-ax}e^{ax}\,dx = xe^{-ax}\\
h(x) &= xe^{-ax}, & y_{PI} = e^{-ax}\int xe^{-ax}e^{ax}\,dx = \frac{x^2}{2}e^{-ax}\\
h(x) &= \sin(kx), & y_{PI} = e^{-ax}\int \sin kxe^{ax}\,dx = \frac{a\sin kx - k\cos kx}{a^2+k^2}\\
h(x) &= \cos(kx), & y_{PI} = e^{-ax}\int \cos kxe^{ax}\,dx = \frac{a\cos kx + k\sin kx}{a^2+k^2}
\end{aligned}$$

Exercise (on the examples sheet): verify that these are correct.
From this we can notice some patterns:

1. If $h(x)$ is a constant, y_{PI} is a constant.
2. If $h(x)$ is a polynomial, y_{PI} is a polynomial of the same degree.
3. If $h(x)$ is an exponential different from the Complementary Function, y_{PI} is the same exponential times a constant.
4. If $h(x)$ is the same exponential as the Complementary Function, y_{PI} is x times the Complementary Function times a constant.
5. If $h(x)$ a polynomial times the Complementary Function, y_{PI} is a polynomial of one degree higher times the Complementary Function.
6. If $h(x)$ is a sine or a cosine, y_{PI} is a combination of sines and cosines.

From this observation, we could **guess** the form of the Particular Integral instead of evaluating it. This is called the **method of undetermined coefficients**.

1. If $h(x)$ is a constant, try $y_{PI} = A$.

2. If $h(x)$ is an nth-degree polynomial, try $y_{PI} = Ax^n + Bx^{n-1} + \dots$: a polynomial of the same degree.
3. If $h(x)$ is an exponential e^{kx}, different from the Complementary Function, try $y_{PI} = Ae^{kx}$: the same exponential times a constant.
4. If $h(x)$ is the same exponential as the Complementary Function (e^{-ax}), try $y_{PI} = Axe^{-ax}$.
5. If $h(x)$ an nth-degree polynomial times the Complementary Function, try $y_{PI} = (Ax^{n+1} + Bx^n + \dots)e^{-ax}$: a polynomial of one degree higher times the Complementary Function.
6. If $h(x)$ is a sine or a cosine ($\sin kx$ or $\cos kx$), try $y_{PI} = A\sin kx + B\cos kx$: a combination of sines and cosines.

In each case, we substitute the guessed form of the Particular Integral into the ODE, and work out the values of the constants A, B, …, that work. If $h(x)$ is more than one of these, add together the corresponding guesses.

Once we have found the Complementary Function and a Particular Integral, the general solution is the sum of these (see section 4.2):

$$y(x) = y_{CF} + y_{PI}$$

Example: Find the general solution of:

$$\frac{dy}{dx} + 2y = x + e^x$$

Solution:

Complementary Function: we substitute $y = e^{\lambda x}$ into the homogeneous version of the ODE:

$$\lambda e^{\lambda x} + 2e^{\lambda x} = 0$$

and divide by $e^{\lambda x}$ to get the characteristic equation:

$$\lambda + 2 = 0$$

The solution of this is $\lambda = -2$, so

$$y_{CF} = Ce^{-2x}$$

where C is an arbitrary constant.

Particular Integral: we substitute

$$y_{PI} = Ax + B + De^x$$

into the ODE – choosing $Ax + B$ as a general first-degree polynomial, and De^x as a constant times the exponential on the RHS. Note that A, B and D play a different

role from C: C is arbitrary, though its value can be specified by the initial condition, while A, B and D need to be chosen so that y_{PI} works for the given RHS.
We need a derivative:

$$y'_{PI} = A + De^x$$

and substitute into the ODE:

$$A + De^x + 2(Ax + B + De^x) = x + e^x$$

or

$$(A + 2B) + 2Ax + 3De^x = x + e^x$$

This is an identity: it is supposed to be true for all values of x. The only way this can happen is if the different parts of the LHS and the RHS have the same coefficients. There are three parts: the constant term, the x term, and the e^x term. If we look at the coefficients of these on the LHS and the RHS, we get:

$$\begin{array}{rl} \text{Constant:} & A + 2B = 0 \\ x \text{ term:} & 2A = 1 \\ e^x \text{ term:} & 3D = 1 \end{array}$$

which we can solve:

$$D = \frac{1}{3}, \qquad A = \frac{1}{2}, \qquad B = -\frac{1}{2}A = -\frac{1}{4}$$

so the full solution of the ODE is:

$$y = y_{CF} + y_{PI} = Ce^{-2x} - \frac{1}{4} + \frac{1}{2}x + \frac{1}{3}e^x$$

We can verify that this is correct:

$$\begin{aligned} y' &= -2Ce^{-2x} + \frac{1}{2} + \frac{1}{3}e^x \\ y' + 2y &= -2Ce^{-2x} + \frac{1}{2} + \frac{1}{3}e^x + 2Ce^{-2x} - \frac{1}{2} + x + \frac{2}{3}e^x \\ &= x + e^x \end{aligned}$$

Exercise: compute the solution using the method of Chapter 2.

4.4.2 Second-order constant coefficient linear ODEs

The method for second-order ODEs proceeds along exactly the same lines.

Example: Solve

$$y'' + 5y' + 4y = x^2$$

Solution:
Complementary Function: we substitute $y = e^{\lambda x}$ into the homogeneous version of the ODE:

$$\lambda^2 e^{\lambda x} + 5\lambda e^{\lambda x} + 4e^{\lambda x} = 0$$

we divide by $e^{\lambda x}$:

$$\lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4) = 0$$

we solve the **characteristic equation:**

$$\lambda = -1 \qquad \text{and} \qquad \lambda = -4$$

Either one of these roots will give us a Complementary Function (e^{-x} or e^{-4x}), but (recalling section 4.2), a superposition of these two functions is also a Complementary Function, so we write the Complementary Function as combination of e^{-x} and e^{-4x}:

$$y_{CF} = C_1 e^{-x} + C_2 e^{-4x}$$

Particular Integral: the RHS is a second-degree polynomial, so we substitute

$$y_{PI} = Ax^2 + Bx + D$$

into the ODE – choosing $Ax^2 + Bx + D$ as a general second-degree polynomial. For this, we need some derivatives:

$$\begin{aligned} y'_{PI} &= 2Ax + B \\ y''_{PI} &= 2A \\ y''_{PI} + 5y'_{PI} + 4y_{PI} &= 2A + 10Ax + 5B + 4Ax^2 + 4Bx + 4D \\ = 4Ax^2 + (10A + 4B)x + (2A + 5B + 4D) &= x^2 \end{aligned}$$

We equate the coefficients of x^2, x and the constant term on the LHS and the RHS:

$$\begin{array}{rll} x^2 \text{ term:} & 4A = 1, & A = \dfrac{1}{4} \\ x \text{ term:} & 10A + 4B = 0, & B = -\dfrac{5}{2}A = -\dfrac{5}{8} \\ \text{Constant:} & 2A + 5B + 4D = 0, & D = -\dfrac{1}{2}A - \dfrac{5}{4}B = \dfrac{21}{32} \end{array}$$

so our general solution is

$$y = y_{CF} + y_{PI} = C_1 e^{-x} + C_2 e^{-4x} + \frac{x^2}{4} - \frac{5x}{8} + \frac{21}{32}$$

Exercise: verify that this is correct.

4.4.3 The second-order homogeneous problem

One big difference between the first-order and the second-order problems is that the characteristic equation is a quadratic in the second-order case, and so it can have repeated or complex roots. How do we calculate the Complementary Function in these cases?

Recall the form of the homogeneous problem from (4.4)

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

If we substitute

$$y = e^{\lambda x}$$

we get

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0$$

Now divide by $e^{\lambda x}$ to get the **characteristic equation** for this ODE:

$$\lambda^2 + a\lambda + b = 0 \tag{4.6}$$

This equation is a quadratic for λ, so there are (in general) two roots, λ_1 and λ_2. Either one will give us a complementary function ($e^{\lambda_1 x}$ or $e^{\lambda_2 x}$), but (recalling section 4.2), a superposition of these two functions is also a complementary function: $C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$. The roots of the characteristic equation are (using the quadratic formula):

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

and so there are three important cases:

1. Real distinct roots: If $a^2 - 4b > 0$, there are two real distinct roots:

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

In this case, the Complementary Function is

$$y_{CF} = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$$

2. Real repeated roots: If $a^2 - 4b = 0$, there is one real root that is repeated:

$$\lambda = \frac{-a}{2}$$

In this case, we have one part of the Complementary Function:

$$y_1 = C_1e^{-ax/2}$$

and we can derive a second part using the method of section 4.3: write $y_2 = u(x)y_1$, where u is a new dependent variable. Then

$$\begin{aligned}
y_2 &= ue^{-ax/2} \\
y_2' &= u'e^{-ax/2} - \frac{a}{2}ue^{-ax/2} \\
y_2'' &= u''e^{-ax/2} - au'e^{-ax/2} + \frac{a^2}{4}ue^{-ax/2} \\
y_2'' + ay_2' + by_2 &= u''e^{-ax/2} - au'e^{-ax/2} + \frac{a^2}{4}ue^{-ax/2} + u'ae^{-ax/2} - \frac{a^2}{2}ue^{-ax/2} + bue^{-ax/2} \\
&= u''e^{-ax/2} \\
&= 0
\end{aligned}$$

where we have used $a^2 = 4b$ in this case. The equation we have to solve is

$$u'' = 0$$

so $u = C_1 + C_2x$, and the Complementary Function is $y = uy_1$:

$$y_{CF} = C_1e^{-ax/2} + C_2xe^{-ax/2}$$

Note the $e^{-ax/2}$ and the $xe^{-ax/2}$. We can verify this:

$$\begin{aligned}
y_{CF}' &= -\frac{C_1a}{2}e^{-ax/2} - \frac{C_2a}{2}xe^{-ax/2} + C_2e^{-ax/2} \\
y_{CF}'' &= \frac{C_1a^2}{4}e^{-ax/2} + \frac{C_2a^2}{4}xe^{-ax/2} - C_2ae^{-ax/2} \\
y_{CF}'' + ay_{CF}' + by_{CF} &= \frac{C_1a^2}{4}e^{-ax/2} + \frac{C_2a^2}{4}xe^{-ax/2} - C_2ae^{-ax/2} \\
&\quad - \frac{C_1a^2}{2}e^{-ax/2} - \frac{C_2a^2}{2}xe^{-ax/2} + C_2ae^{-ax/2} + C_1be^{-ax/2} + C_2bxe^{-ax/2} \\
&= C_1\left(-\frac{a^2}{4} + b\right)e^{-ax/2} + C_2\left(-\frac{a^2}{4} + b\right)xe^{-ax/2} \\
&= 0
\end{aligned}$$

since $a^2 = 4b$ in this case.

3. Complex roots: If $a^2 - 4b < 0$, there are two distinct complex roots. In this case, let

$$p = \frac{-a}{2} \qquad \text{and} \qquad q = \frac{\sqrt{4b - a^2}}{2}$$

so

$$\lambda_1 = p + iq, \qquad \text{and} \qquad \lambda_2 = p - iq,$$

As in the case of real distinct roots, we can write:

$$y_{CF} = Ae^{\lambda_1 x} + Be^{\lambda_2 x} = Ae^{(p+iq)x} + Be^{(p-iq)x}$$

In this expression, A and B will be in general complex numbers. If we want to be sure of a real solution, we can tidy this up:

$$\begin{aligned} y_{CF} &= e^{px}\left(Ae^{iqx}+Be^{-iqx}\right) \\ &= e^{px}\left(A(\cos(qx)+i\sin(qx))+B(\cos(qx)-i\sin(qx))\right) \\ &= e^{px}\left((A+B)\cos(qx)+i(A-B)\sin(qx)\right) \\ &= e^{px}\left(C_1\cos(qx)+C_2\sin(qx)\right) \end{aligned}$$

where $C_1 = A+B$ and $C_2 = i(A-B)$. In spite of appearances, C_1 and C_2 can be real. We can verify that this is correct (exercise).

Summary: The characteristic equation is

$$\lambda^2 + a\lambda + b = 0$$

If there are two real roots, λ_1 and λ_2, then the Complementary Function is

$$y_{CF} = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

If there is one repeated real root λ_1, then the Complementary Function is

$$y_{CF} = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$$

If there are two complex roots, $p \pm iq$, then the Complementary Function is

$$y_{CF} = e^{px}\left(C_1\cos(qx)+C_2\sin(qx)\right)$$

Example: (not covered in detail) Solve the Initial Value Problem:

$$y'' + 3y' + 2y = 0, \qquad \text{with } y(0) = 1 \text{ and } y'(0) = 2$$

Solution: We find the characteristic equation by substituting $y = e^{\lambda x}$ into the ODE, and dividing by $e^{\lambda x}$:

$$\begin{aligned} \lambda^2 e^{\lambda x} + 3\lambda e^{\lambda x} + 2e^{\lambda x} &= 0 \\ \lambda^2 + 3\lambda + 2 &= 0 \end{aligned}$$

Solve this equation for λ:

$$(\lambda+1)(\lambda+2) = 0, \qquad \text{so } \lambda = -1 \text{ or } \lambda = -2.$$

Note that these are real and distinct, so the Complementary Function is

$$y_{CF} = C_1 e^{-x} + C_2 e^{-2x}$$

There is no particular integral, since the RHS of the ODE is zero, and so the general solution is:

$$y = C_1 e^{-x} + C_2 e^{-2x}$$

For the initial values, we need a derivative:

$$y' = -C_1e^{-x} - 2C_2e^{-2x}$$

so the initial values give us:

$$\begin{aligned} y(0) = C_1 + C_2 &= 1 \\ y'(0) = -C_1 - 2C_2 &= 2 \end{aligned}$$

Adding:

$$-C_2 = 3$$

so

$$\begin{aligned} C_2 &= -3 \\ C_1 &= 1 - C_2 = 4 \\ y(x) &= 4e^{-x} - 3e^{-2x} \end{aligned}$$

Exercise: verify that this is correct.

Example: (not covered in detail) Solve the Boundary Value Problem:

$$y'' + 4y' + 4y = 0, \qquad \text{with } y(1) = 2 \text{ and } y(3) = -1$$

Solution: We find the characteristic equation by substituting $y = e^{\lambda x}$ into the ODE, and dividing by $e^{\lambda x}$:

$$\begin{aligned} \lambda^2e^{\lambda x} + 4\lambda e^{\lambda x} + 4e^{\lambda x} &= 0 \\ \lambda^2 + 4\lambda + 4 &= 0 \end{aligned}$$

Solve this equation for λ:

$$(\lambda + 2)^2 = 0, \qquad \text{so } \lambda = -2.$$

Note that this is a single real root, so the Complementary Function is

$$y_{CF} = (C_1 + C_2x)e^{-2x}$$

There is no particular integral, since the RHS of the ODE is zero, and so the general solution is:

$$y = (C_1 + C_2x)e^{-2x}$$

The boundary values give us:

$$\begin{aligned} y(1) = (C_1 + C_2)e^{-2} &= 2 \\ y(3) = (C_1 + 3C_2)e^{-6} &= -1 \\ C_1 + C_2 &= 2e^2 \\ C_1 + 3C_2 &= -e^6 \end{aligned}$$

Subtracting:

$$2C_2 = -e^6 - 2e^2$$

so

$$\begin{aligned}
C_2 &= -\frac{e^2}{2}\left(e^4+2\right) \\
C_1 &= 2e^2 - C_2 = 2e^2 + \frac{e^2}{2}\left(e^4+2\right) = \frac{e^2}{2}\left(e^4+6\right) \\
y(x) &= \frac{e^2}{2}\left((e^4+6)-(e^4+2)x\right)e^{-2x}
\end{aligned}$$

Exercise: verify that this is correct.

Example: Solve the Initial Value Problem:

$$4y'' + 4y' + 17y = 0, \qquad \text{with } y(0) = 0 \text{ and } y'(0) = 2$$

Solution: We find the characteristic equation by substituting $y = e^{\lambda x}$ into the ODE, and dividing by $e^{\lambda x}$:

$$\begin{aligned}
4\lambda^2 e^{\lambda x} + 4\lambda e^{\lambda x} + 17 e^{\lambda x} &= 0 \\
4\lambda^2 + 4\lambda + 17 &= 0
\end{aligned}$$

We could have divided by 4 to get this into standard form, but there is no need. Solve this equation for λ:

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4\times 4\times 17}}{2\times 4} = \frac{-4 \pm \sqrt{-256}}{8} = \frac{-4 \pm 16i}{8} = -\frac{1}{2} \pm 2i$$

Note that these are complex: the real part is $-\frac{1}{2}$ and the imaginary part is 2. The Complementary Function is

$$y_{CF} = \left(C_1 \cos 2x + C_2 \sin 2x\right) e^{-x/2}$$

There is no particular integral, since the RHS of the ODE is zero, and so the general solution is:

$$y = \left(C_1 \cos 2x + C_2 \sin 2x\right) e^{-x/2}$$

For the initial values, we need a derivative:

$$y' = \left(-2C_1 \sin 2x + 2C_2 \cos 2x\right) e^{-x/2} - \frac{1}{2}\left(C_1 \cos 2x + C_2 \sin 2x\right) e^{-x/2}$$

so the initial values give us:

$$\begin{aligned}
y(0) = C_1 &= 0 \\
y'(0) = -\frac{1}{2}C_1 + 2C_2 &= 2
\end{aligned}$$

so:

$$\begin{aligned}
C_1 &= 0 \\
C_2 &= 1 \\
y(x) &= \sin 2x e^{-x/2}
\end{aligned}$$

Exercise: verify that this is correct.

4.4.4 The second-order inhomogeneous problem

Now that we know how to find the Complementary Function, we return to the inhomogeneous problem (4.3):

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = h(x).$$

As with first order linear ODEs (sections 2.3 and 4.4.1), there are two methods. One involves evaluating integrals to find the correct y_{PI} and is beyond the scope of this course. The second is a generalisation of the **method of undetermined coefficients** introduced in section 4.4.1: we **guess** the form of the Particular Integral depending on what $h(x)$ is, and work out the numbers that make our guess the correct one. In making the guess, we must take in to account the form of the Complementary Function, which will contain functions like: $e^{\lambda x}$, $xe^{\lambda x}$, $e^{px}\cos qx$, $e^{px}\sin qx$ and a constant ($\lambda = 0$).
The basic suggestions are:

1. If $h(x)$ does not include anything that looks like the Complementary Function, try setting y_{PI} to a function that looks like $h(x)$, but with all coefficients undetermined.

2. If $h(x)$ does includes functions that looks like the Complementary Function, try setting y_{PI} to a function that looks like $xh(x)$, again with all coefficients undetermined. If the characteristic equation has repeated root, try a function that looks like $x^2h(x)$.

3. If $h(x)$ contains more than one function, try adding together the corresponding guesses.

In a little more detail:

1. If $h(x)$ is a constant, try $y_{PI} = A$. (However, if part of the Complementary Function is a constant, try $y_{PI} = Ax$.)

2. If $h(x)$ is an nth-degree polynomial, try $y_{PI} = Ax^n + Bx^{n-1} + \ldots$: a polynomial of the same degree. (However, if part of the Complementary Function is a constant, try $y_{PI} = Ax^{n+1} + Bx^n + \ldots$: a polynomial of one degree higher.)

3. If $h(x)$ is an exponential e^{kx}, different from the Complementary Function, try $y_{PI} = Ae^{kx}$: the same exponential times a constant.

4. If $h(x)$ is the same as the Complementary Function ($e^{\lambda x}$, $e^{px}\cos qx$ etc.), try $y_{PI} = Axe^{\lambda x}$, $Axe^{px}\cos qx$ etc. If the characteristic equation has repeated root, try $y_{PI} = Ax^2e^{\lambda x}$.

5. If $h(x)$ is an nth-degree polynomial times the Complementary Function, try $y_{PI} = (Ax^{n+1} + Bx^n + \ldots)e^{\lambda x}$: a polynomial of one degree higher times the Complementary Function. If the characteristic equation has repeated roots, try $y_{PI} = (Ax^{n+2} + Bx^{n+1} + \ldots)e^{\lambda x}$: a polynomial of two degrees higher times the Complementary Function.

6. Whenever $h(x)$ involves a sine or a cosine ($\sin kx$ or $\cos kx$), possibly times an exponential, both sine and cosine should be included in the guess: try a combination of sines and cosines of the form: $y_{PI} = A\sin kx + B\cos kx$. This applies whether $h(x)$ is the same or different from the Complementary Function, though in the former case, multiply the guess by x.

In each case, we substitute the guessed form of the Particular Integral into the ODE, and work out the values of the constants A, B, $\ldots$, that work, by equating the coefficients of the different parts of the LHS and the RHS.
So now we have a complete method for solving second-order linear constant coefficient ODEs:
1. Find the characteristic equation and the Complementary Function y_{CF}. This should solve the homogeneous problem and contain two arbitrary constants.
2. Find the Particular Integral y_{PI}. This shouldn't contain any arbitrary constants.
3. The general solution is $y = y_{CF} + y_{PI}$.
4. Use the initial values or the boundary values to find the values of the two arbitrary constants (if possible). Note that you should do this after adding $y_{CF} + y_{PI}$.
5. Verify your solution is correct.

Example: Solve the Initial Value Problem:

$$y'' + 5y' + 4y = 4x^2 + 10x + 2, \qquad \text{with } y(0) = 1 \text{ and } y'(0) = 2$$

Solution: We find the characteristic equation by substituting $y = e^{\lambda x}$ into the homogeneous version of the ODE, and dividing by $e^{\lambda x}$:

$$\begin{aligned} \lambda^2 e^{\lambda x} + 5\lambda e^{\lambda x} + 4e^{\lambda x} &= 0 \\ \lambda^2 + 5\lambda + 4 &= 0 \end{aligned}$$

Solve this equation for λ:

$$(\lambda + 1)(\lambda + 4) = 0, \qquad \text{so } \lambda = -1 \text{ or } \lambda = -4.$$

Note that these are real and distinct, so the Complementary Function is

$$y_{CF} = C_1 e^{-x} + C_2 e^{-4x}$$

The RHS of the ODE is a quadratic function of x, so for the particular integral, we try a general quadratic function:

$$y_{PI} = Ax^2 + Bx + D$$

We need some derivatives:

$$y'_{PI} = 2Ax + B, \qquad \text{and} \qquad y''_{PI} = 2A,$$

Substitute these into the ODE:

$$\begin{aligned} y''_{PI} + 5y'_{PI} + 4y_{PI} &= 2A + 5(2Ax + B) + 4(Ax^2 + Bx + D) \\ &= 4Ax^2 + (4B + 10A)x + (4D + 5B + 2A) \\ &= 4x^2 + 10x + 2 \end{aligned}$$

This is an identity: it is supposed to be true for all values of x. The only way this can happen is if the different parts of the LHS and the RHS have the same coefficients. There are three parts: the constant term, the x term, and the x^2 term. If we look at the coefficients of these on the LHS and the RHS, we get:

$$\begin{aligned} \text{Constant:} &\qquad 2A + 5B + 4D = 2 \\ x \text{ term:} &\qquad 10A + 4B = 10 \\ x^2 \text{ term:} &\qquad 4A = 4 \end{aligned}$$

which we can solve:

$$A = 1, \qquad B = \frac{1}{4}(10 - 10A) = 0, \qquad D = \frac{1}{4}(2 - 2A - 5B) = 0$$

so the particular integral is

$$y_{PI} = x^2$$

and the general solution is:

$$y = C_1e^{-x} + C_2e^{-4x} + x^2$$

For dealing with the initial values, we need a derivative:

$$y' = -C_1e^{-x} - 4C_2e^{-4x} + 2x$$

so the initial values give us:

$$\begin{aligned} y(0) = C_1 + C_2 &= 1 \\ y'(0) = -C_1 - 4C_2 &= 2 \end{aligned}$$

Adding:

$$-3C_2 = 3$$

so:

$$\begin{aligned} C_2 &= -1 \\ C_1 &= 1 - C_2 = 2 \\ y(x) &= 2e^{-x} - e^{-4x} + x^2 \end{aligned}$$

Exercise: verify that this is correct.

Example: Solve the Initial Value Problem:

$$y'' + 5y' + 4y = \sin x, \qquad \text{with } y(0) = 1 \text{ and } y'(0) = 2$$

Solution: We already have the Complementary Function:

$$y_{CF} = C_1 e^{-x} + C_2 e^{-4x}$$

The RHS of the ODE is a sine or cosine, so for the particular integral, we try a general combination of sines and cosines:

$$y_{PI} = A\cos x + B\sin x$$

We need some derivatives:

$$y'_{PI} = -A\sin x + B\cos x, \qquad \text{and} \qquad y''_{PI} = -A\cos x - B\sin x$$

Substitute these into the ODE:

$$\begin{aligned} y''_{PI} + 5y'_{PI} + 4y_{PI} &= -A\cos x - B\sin x + 5(-A\sin x + B\cos x) + 4(A\cos x + B\sin x) \\ &= (3A + 5B)\cos x + (3B - 5A)\sin x \\ &= \sin x \end{aligned}$$

This is an identity: it is supposed to be true for all values of x. The only way this can happen is if the different parts of the LHS and the RHS have the same coefficients. There are two parts: the sine term and the cosine term. If we look at the coefficients of these on the LHS and the RHS, we get:

$$\begin{aligned} \text{sine term:} &\qquad -5A + 3B = 1 \\ \text{cosine term:} &\qquad 3A + 5B = 0 \end{aligned}$$

which we can solve. Add 3 times the top to 5 times the bottom:

$$-15A + 9B + 15A + 25B = 34B = 3$$

$$B = \frac{3}{34}, \qquad A = -\frac{5}{3}B = -\frac{5}{34}$$

so the particular integral is

$$y_{PI} = \frac{3\sin x - 5\cos x}{34}$$

and the general solution is:

$$y = C_1 e^{-x} + C_2 e^{-4x} + \frac{3\sin x - 5\cos x}{34}$$

For dealing with the initial values, we need a derivative:

$$y' = -C_1 e^{-x} - 4C_2 e^{-4x} + \frac{3\cos x + 5\sin x}{34}$$

so the initial values give us:

$$\begin{aligned} y(0) = C_1 + C_2 - \frac{5}{34} &= 1 \\ y'(0) = -C_1 - 4C_2 + \frac{3}{34} &= 2 \end{aligned}$$

Adding:

$$-3C_2 - \frac{2}{34} = 3$$

so:

$$\begin{aligned}
C_2 &= -1 - \frac{2}{102} = -\frac{104}{102} = -\frac{52}{51} \\
C_1 &= 1 - C_2 + \frac{5}{34} = 1 + \frac{104}{102} + \frac{15}{102} = \frac{221}{102} = \frac{13}{6} \\
y(x) &= \frac{13}{6}e^{-x} - \frac{52}{51}e^{-4x} + \frac{3\cos x + 5\sin x}{34}
\end{aligned}$$

Exercise: verify that this is correct.

Example: Solve the Initial Value Problem:

$$y'' + 5y' + 4y = e^{-x}, \qquad \text{with } y(0) = 1 \text{ and } y'(0) = 2$$

Solution: We already have the Complementary Function:

$$y_{CF} = C_1e^{-x} + C_2e^{-4x}$$

The RHS of the ODE is an exponential – but it is the same as one of the Complementary Functions, e^{-x}. We could try $y_{PI} = Ae^{-x}$, but a quick calculation will show that this doesn't work (exercise). From the notes above, we try x times the Complementary Function:

$$y_{PI} = Axe^{-x}$$

We need some derivatives:

$$y'_{PI} = Ae^{-x} - Axe^{-x} = A(1-x)e^{-x} \qquad \text{and} \qquad y''_{PI} = -Ae^{-x} - A(1-x)e^{-x} = A(x-2)e^{-x}$$

Substitute these into the ODE:

$$\begin{aligned}
y''_{PI} + 5y'_{PI} + 4y_{PI} &= A(x-2)e^{-x} + 5A(1-x)e^{-x} + 4Axe^{-x} \\
&= 3Ae^{-x} \\
&= e^{-x}
\end{aligned}$$

This is an identity: it is supposed to be true for all values of x. The only way this can happen is if the different parts of the LHS and the RHS have the same coefficients. There is one part: the e^{-x} term. If we look at the coefficient of this on the LHS and the RHS, we get:

$$e^{-x} \text{ term:} \qquad 3A = 1$$

which we can solve:

$$A = \frac{1}{3}$$

so the particular integral is

$$y_{PI} = \frac{1}{3}xe^{-x}$$

and the general solution is:

$$y = C_1 e^{-x} + C_2 e^{-4x} + \frac{1}{3} x e^{-x}$$

For dealing with the initial values, we need a derivative:

$$y' = -C_1 e^{-x} - 4C_2 e^{-4x} + \frac{1}{3}(1-x)e^{-x}$$

so the initial values give us:

$$\begin{aligned} y(0) = C_1 + C_2 &= 1 \\ y'(0) = -C_1 - 4C_2 + \frac{1}{3} &= 2 \end{aligned}$$

Adding:

$$-3C_2 + \frac{1}{3} = 3$$

so:

$$\begin{aligned} C_2 &= -1 + \frac{1}{9} = -\frac{8}{9} \\ C_1 &= 1 - C_2 = 1 + \frac{8}{9} = \frac{17}{9} \\ y(x) &= \frac{17}{9} e^{-x} - \frac{8}{9} e^{-4x} + \frac{1}{3} x e^{-x} \end{aligned}$$

Exercise: verify that this is correct.

Example: (not covered in detail) Solve the Initial Value Problem:

$$y'' + 6y' + 9y = 4e^{-5x}, \qquad \text{with } y(0) = 2 \text{ and } y'(0) = -1$$

Solution: We find the characteristic equation by substituting $y = e^{\lambda x}$ into the homogeneous version of the ODE, and dividing by $e^{\lambda x}$:

$$\begin{aligned} \lambda^2 e^{\lambda x} + 6\lambda e^{\lambda x} + 9 e^{\lambda x} &= 0 \\ \lambda^2 + 6\lambda + 9 &= 0 \end{aligned}$$

Solve this equation for λ:

$$(\lambda + 3)^2 = 0, \qquad \text{so } \lambda = -3.$$

Note that this is a single real root, so the Complementary Function is

$$y_{CF} = (C_1 + C_2 x)e^{-3x}$$

The RHS of the ODE is an exponential function of x that is different from the Complementary Function, so for the particular integral, we try a general exponential function of the same type as the RHS:

$$y_{PI} = Ae^{-5x}$$

We need some derivatives:

$$y'_{PI} = -5Ae^{-5x} \qquad \text{and} \qquad y''_{PI} = 25Ae^{-5x}$$

Substitute these into the ODE:

$$\begin{aligned} y''_{PI} + 6y'_{PI} + 9y_{PI} &= 25Ae^{-5x} - 30Ae^{-5x} + 9Ae^{-5x} \\ &= 4Ae^{-5x} \\ &= 4e^{-5x} \end{aligned}$$

This is an identity: it is supposed to be true for all values of x. The only way this can happen is if the different parts of the LHS and the RHS have the same coefficients. There is one part: the e^{-5x} term. If we look at the coefficients of this on the LHS and the RHS, we get:

$$e^{-5x} \text{ term:} \qquad 4A = 4$$

which we can solve: $A = 1$, so the particular integral is

$$y_{PI} = e^{-5x}$$

and the general solution is:

$$y = (C_1 + C_2x)e^{-3x} + e^{-5x}$$

For dealing with the initial values, we need a derivative:

$$y' = C_2e^{-3x} - 3(C_1 + C_2x)e^{-3x} - 5e^{-5x}$$

so the initial values give us:

$$\begin{aligned} y(0) = C_1 + 1 &= 2 \\ y'(0) = -3C_1 + C_2 - 5 &= -1 \end{aligned}$$

so:

$$\begin{aligned} C_1 &= 1 \\ C_2 &= -1 + 5 + 3C_1 = 7 \\ y(x) &= (1 + 7x)e^{-3x} + e^{-5x} \end{aligned}$$

Exercise: verify that this is correct.

Example: Solve the Initial Value Problem:

$$y'' + 6y' + 9y = 2e^{-3x}, \qquad \text{with } y(0) = 2 \text{ and } y'(0) = -1$$

Solution: We already have the Complementary Function:

$$y_{CF} = (C_1 + C_2 x)e^{-3x}$$

The RHS of the ODE is an exponential – but it is the same as one of the Complementary Functions, e^{-3x}. We could try $y_{PI} = Ae^{-3x}$, but a quick calculation will show that this doesn't work (exercise). Similarly, $y_{PI} = Axe^{-3x}$ doesn't work (exercise). From the notes above, we try x^2 times the Complementary Function:

$$y_{PI} = Ax^2 e^{-3x}$$

We need some derivatives:

$$y'_{PI} = A(2x - 3x^2)e^{-3x} \qquad \text{and} \qquad y''_{PI} = A(2 - 12x + 9x^2)e^{-3x}$$

Substitute these into the ODE:

$$\begin{aligned} y''_{PI} + 6y'_{PI} + 9y_{PI} &= A(2 - 12x + 9x^2)e^{-3x} + 6A(2x - 3x^2)e^{-3x} + 9Ax^2 e^{-3x} \\ &= 2Ae^{-3x} \\ &= 2e^{-3x} \end{aligned}$$

This is an identity: it is supposed to be true for all values of x. The only way this can happen is if the different parts of the LHS and the RHS have the same coefficients. There is one part: the e^{-3x} term. If we look at the coefficients of this on the LHS and the RHS, we get:

$$e^{-3x} \text{ term:} \qquad 2A = 2$$

which we can solve: $A = 1$, so the particular integral is

$$y_{PI} = x^2 e^{-3x}$$

and the general solution is:

$$y = (C_1 + C_2 x + x^2)e^{-3x}$$

For dealing with the initial values, we need a derivative:

$$y' = (C_2 + 2x)e^{-3x} - 3(C_1 + C_2 x + x^2)e^{-3x}$$

so the initial values give us:

$$\begin{aligned} y(0) = C_1 &= 2 \\ y'(0) = -3C_1 + C_2 &= -1 \end{aligned}$$

so:

$$\begin{aligned} C_1 &= 2 \\ C_2 &= -1 + 3C_1 = 5 \\ y(x) &= (2 + 5x + x^2)e^{-3x} \end{aligned}$$

Exercise: verify that this is correct.

Example: (not covered in detail) Solve the Initial Value Problem:

$$4y'' + 4y' + 17y = 17x^2 - 26x, \qquad \text{with } y(0) = 0 \text{ and } y'(0) = 2$$

Solution: We have already found the Complementary Function for this ODE:

$$y_{CF} = (C_1 \cos 2x + C_2 \sin 2x)\, e^{-x/2}$$

The RHS of the ODE is a quadratic function of x, so for the particular integral, we try a general quadratic function:

$$y_{PI} = Ax^2 + Bx + D$$

We need some derivatives:

$$y'_{PI} = 2Ax + B, \qquad \text{and} \qquad y''_{PI} = 2A,$$

Substitute these into the ODE:

$$\begin{aligned} 4y''_{PI} + 4y'_{PI} + 17y_{PI} &= 8A + 4(2Ax + B) + 17(Ax^2 + Bx + D) \\ &= 17Ax^2 + (17B + 8A)x + (17D + 4B + 8A) \\ &= 17x^2 - 26x \end{aligned}$$

This is an identity: it is supposed to be true for all values of x. The only way this can happen is if the different parts of the LHS and the RHS have the same coefficients. There are three parts: the constant term, the x term, and the x^2 term. If we look at the coefficients of these on the LHS and the RHS, we get:

$$\begin{aligned} \text{Constant:} \quad & 8A + 4B + 17D = 0 \\ x \text{ term:} \quad & 8A + 17B = -26 \\ x^2 \text{ term:} \quad & 17A = 17 \end{aligned}$$

which we can solve:

$$A = 1, \qquad B = \frac{1}{17}(-26 - 8A) = -2, \qquad D = \frac{1}{17}(0 - 8A - 4B) = 0$$

so the particular integral is

$$y_{PI} = x^2 - 2x$$

and the general solution is:

$$y = (C_1 \cos 2x + C_2 \sin 2x)\, e^{-x/2} + x^2 - 2x$$

For dealing with the initial values, we need a derivative:

$$y' = (-2C_1 \sin 2x + 2C_2 \cos 2x)\, e^{-x/2} - \frac{1}{2}\,(C_1 \cos 2x + C_2 \sin 2x)\, e^{-x/2} + 2x - 2$$

so the initial values give us:

$$\begin{aligned} y(0) = C_1 &= 0 \\ y'(0) = -\frac{1}{2}C_1 + 2C_2 - 2 &= 2 \end{aligned}$$

so:

$$\begin{aligned} C_1 &= 0 \\ C_2 &= 2 \\ y(x) &= 2\sin 2x e^{-x/2} + x^2 - 2x \end{aligned}$$

Exercise: verify that this is correct.

Example: (not covered in detail) Solve the Initial Value Problem:

$$4y'' + 4y' + 17y = \cos 2x - 8\sin 2x, \qquad \text{with } y(0) = 0 \text{ and } y'(0) = 2$$

Solution: We have already found the Complementary Function for this ODE:

$$y_{CF} = (C_1 \cos 2x + C_2 \sin 2x)\, e^{-x/2}$$

The RHS of the ODE is a trigonometric function of x, not equal to the Complementary Function (in spite of appearances) so for the particular integral, we try a general combination of sines and cosines:

$$y_{PI} = A\cos 2x + B\sin 2x$$

Aside: if the RHS had been (for example) $e^{-x/2}\sin 2x$, then we would have chosen $y_{PI} = (A\cos 2x + B\sin 2x)xe^{-x/2}$.
We need some derivatives:

$$y'_{PI} = -2A\sin 2x + 2B\cos 2x, \qquad \text{and} \qquad y''_{PI} = -4A\cos 2x - 4B\sin 2x$$

Substitute these into the ODE:

$$\begin{aligned} 4y''_{PI} + 4y'_{PI} + 17y_{PI} &= 4(-4A\cos 2x - 4B\sin 2x) + 4(-2A\sin 2x + 2B\cos 2x) \\ &\qquad + 17(A\cos 2x + B\sin 2x) \\ &= (A + 8B)\cos 2x + (B - 8A)\sin 2x \\ &= \cos 2x - 8\sin 2x \end{aligned}$$

This is an identity: it is supposed to be true for all values of x. The only way this can happen is if the different parts of the LHS and the RHS have the same coefficients. There are two parts: the $\cos 2x$ term and the $\sin 2x$ term. If we look at the coefficients of these on the LHS and the RHS, we get:

$$\begin{aligned} \cos 2x \text{ term:} &\qquad A + 8B = 1 \\ \sin 2x \text{ term:} &\qquad B - 8A = -8 \end{aligned}$$

which we can solve. Multiply the first equation by 8 and add:

$$\begin{aligned} 8A + 64B + B - 8A = 65B &= 8 - 8 = 0 \\ B &= 0, \qquad \text{and} \qquad A = 1 - 8B = 1 \end{aligned}$$

so the particular integral is

$$y_{PI} = \cos 2x$$

and the general solution is:

$$y = (C_1 \cos 2x + C_2 \sin 2x)\, e^{-x/2} + \cos 2x$$

For dealing with the initial values, we need a derivative:

$$y' = (-2C_1 \sin 2x + 2C_2 \cos 2x)\, e^{-x/2} - \frac{1}{2}(C_1 \cos 2x + C_2 \sin 2x)\, e^{-x/2} - 2 \sin 2x$$

so the initial values give us:

$$\begin{aligned} y(0) = C_1 + 1 &= 0 \\ y'(0) = -\frac{1}{2}C_1 + 2C_2 &= 2 \end{aligned}$$

so:

$$\begin{aligned} C_1 &= -1 \\ C_2 &= \frac{5}{4} \\ y(x) &= \left(-\cos 2x + \frac{5}{4}\sin 2x\right) e^{-x/2} + \cos 2x \end{aligned}$$

Exercise: verify that this is correct.

Example: Solve the Initial Value Problem:

$$y'' + 4y = 4 \sin 2x, \qquad \text{with } y(0) = 0 \text{ and } y'(0) = 2$$

Solution: We find the characteristic equation by substituting $y = e^{\lambda x}$ into the ODE, and dividing by $e^{\lambda x}$:

$$\begin{aligned} \lambda^2 e^{\lambda x} + 4e^{\lambda x} &= 0 \\ \lambda^2 + 4 &= 0 \end{aligned}$$

Solve this equation for λ:

$$\lambda^2 = -4, \qquad \text{so } \lambda = \pm 2i.$$

Note that these are complex: the real part is 0 and the imaginary part is 2. The Complementary Function is

$$y_{CF} = C_1 \cos 2x + C_2 \sin 2x$$

Aside: if the ODE were $y'' + k^2 y = 0$, then we would have $y_{CF} = C_1 \cos kx + C_2 \sin kx$. The RHS of the ODE is a trigonometric function of x, and in this case it is equal to a part of the Complementary Function, so for the particular integral, we try a general combination of x times sines and cosines:

$$y_{PI} = Ax\cos 2x + Bx\sin 2x$$

We need some derivatives:

$$y'_{PI} = (B-2Ax)\sin 2x + (A+2Bx)\cos 2x, \qquad \text{and} \qquad y''_{PI} = 4(B-Ax)\cos 2x - 4(A+Bx)\sin 2x$$

Substitute these into the ODE:

$$\begin{aligned} y''_{PI} + 4y_{PI} &= 4(B-Ax)\cos 2x - 4(A+Bx)\sin 2x + 4(Ax\cos 2x + Bx\sin 2x) \\ &= 4B\cos 2x - 4A\sin 2x \\ &= 4\sin 2x \end{aligned}$$

This is an identity: it is supposed to be true for all values of x. The only way this can happen is if the different parts of the LHS and the RHS have the same coefficients. There are two parts: the $\cos 2x$ term and the $\sin 2x$ term. If we look at the coefficients of these on the LHS and the RHS, we get:

$$\begin{aligned} \cos 2x \text{ term:} &\qquad 4B = 0 \\ \sin 2x \text{ term:} &\qquad -4A = 4 \end{aligned}$$

which we can solve:

$$B = 0, \qquad \text{and} \qquad A = -1$$

so the particular integral is

$$y_{PI} = -x\cos 2x$$

and the general solution is:

$$y = C_1\cos 2x + C_2\sin 2x - x\cos 2x$$

For dealing with the initial values, we need a derivative:

$$y' = -2C_1\sin 2x + 2C_2\cos 2x - \cos 2x + 2x\sin 2x$$

so the initial values give us:

$$\begin{aligned} y(0) = C_1 &= 0 \\ y'(0) = 2C_2 - 1 &= 2 \end{aligned}$$

so:

$$\begin{aligned} C_1 &= 0 \\ C_2 &= \frac{3}{2} \\ y(x) &= \frac{3}{2}\sin 2x - x\cos 2x \end{aligned}$$

Exercise: verify that this is correct.

4.5 Two-dimensional ODEs: phase plane analysis

It is convenient at this point to make a slight change of notation and rewrite the second-order ODE (4.3) with x as the dependent variable and t as the independent variable:

$$\ddot{x} + a\dot{x} + bx = \frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = h(t),$$

where a and b are constants, we have divided so that the coefficient of $\ddot{x}$ is 1, and $h(t)$ is a given function of t. Note the use of dots: $\dot{x} = \frac{dx}{dt}$ and $\ddot{x} = \frac{d^2x}{dt^2}$.
Further, we define a new variable y by:

$$y = \dot{x}, \qquad \text{or} \qquad \dot{x} = y,$$

so we can write:

$$\dot{y} = \ddot{x} = -ay - bx + h(t).$$

This pair of equations can be combined to form a **two-dimensional first-order** ODE (with two dependent variables x and y and one independent variable t):

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ h(t) \end{pmatrix}. \tag{4.7}$$

This two-dimensional first-order equation is essentially identical to the one-dimensional second-order ODE (4.3) – but it can be readily generalised in two ways. First, the matrix $\left[\begin{smallmatrix} 0 & 1 \\ -b & -a \end{smallmatrix}\right]$ can be replaced by a general 2×2 matrix $\mathbf{A}$, and the vector $\left(\begin{smallmatrix} 0 \\ h(t) \end{smallmatrix}\right)$ can be generalised, to get:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}. \tag{4.8}$$

Second, the ODE could be nonlinear:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x,y,t) \\ g(x,y,t) \end{pmatrix}, \tag{4.9}$$

where $f(x, y, t)$ and $g(x, y, t)$ are arbitrary functions. Writing the ODE in this way gets us into the realm of Nonlinear Differential Equations, Chaos and Dynamical Systems. In this section, we will only consider linear two-dimensional ODEs.
In addition, we will not consider the non-autonomous ODE (4.8), only the autonomous constant-coefficient ODE:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \tag{4.10}$$

where H_1 and H_2 are constants. We will also need the homogeneous ($H_1 = H_2 = 0$) version of this and of (4.7):

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.11}$$

Finally, we need the initial condition $(x(0), y(0)) = (x_0, y_0)$.

4.5.1 The phase plane

The general approach with two-dimensional autonomous ODEs such as (4.10) or (4.11) is to interpret the independent variable t as time, and to consider the time-dependent point $(x(t), y(t))$ in the (x, y)-plane (the phase plane). The point $(x(t), y(t))$ traces a curve in the phase plane called the **trajectory** of the initial condition (x_0, y_0). Different choices of initial condition result in different trajectories, and a diagram showing one or more trajectories is called a **phase portrait** of the ODE. Phase portraits will also show other features of the ODE, such as **equilibrium points**.

Example: Solve the ODE

$$\ddot{x} + x = 0.$$

Convert the ODE into a two-dimensional ODE and use the known solutions to sketch its phase portrait.

Solution: We find the characteristic equation by substituting $x = e^{\lambda t}$ into the ODE, and dividing by $e^{\lambda t}$:

$$\lambda^2 + 1 = 0$$

so the roots are $\lambda = \pm i$. Note that these are complex: the real part is 0 and the imaginary part is 1. The Complementary Function (and the general solution) is

$$x(t) = C_1 \cos t + C_2 \sin t.$$

To write this as a two-dimensional ODE, define $y = \dot{x}$ and we get:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

To construct trajectories through an initial condtion (x_0, y_0), we also need

$$y(t) = \dot{x} = -C_1 \sin t + C_2 \cos t.$$

We use the initial condition $(x(0), y(0)) = (x_0, y_0)$ to get

$$x(0) = x_0 = C_1 \qquad \text{and} \qquad y(0) = y_0 = C_2,$$

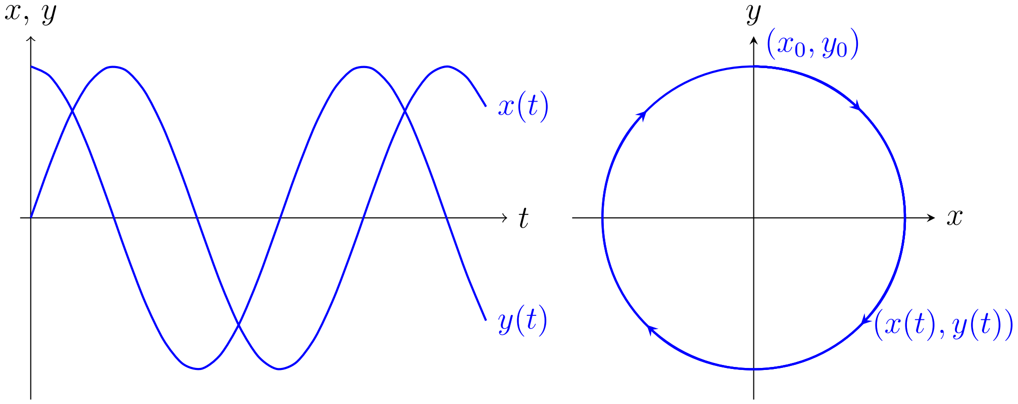
so

$$x(t) = x_0 \cos t + y_0 \sin t \qquad \text{and} \qquad y(t) = -x_0 \sin t + y_0 \cos t.$$

We observe that $x^2 + y^2$ is constant:

$$x^2(t) + y^2(t) = \cdots = x_0^2 + y_0^2,$$

so the trajectory of an initial condition (x_0, y_0) is a **circle** going through that point. Since $\dot{x} = y$, we note that x is an *increasing* function of time when $y > 0$, so points move to the right in the upper half-plane (and to the left in the lower half-plane), so trajectories trace out circles in a clockwise direction.



If $(x_0, y_0) = (0, 1)$, then $(x(t), y(t)) = (\sin t, \cos t)$: the unit circle, starting at $(0, 1)$ and proceeding clockwise. Different initial conditions would yield trajectories on circles of different radii. The curves $x(t)$ and $y(t)$ are **periodic** functions of time.

It is sometimes useful to compute the slope of a trajectory in the phase plane. This can be done using:

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{g(x, y)}{f(x, y)}$$

in the case of an autonomous nonlinear ODE (the autonomous version of (4.9)), or, in the case of (4.11), we get an ODE of homogeneous degree:

$$\frac{dy}{dx} = \frac{-bx - ay}{y},$$

whose solution can be found using the methods of section 2.4.1.

4.5.2 Equilibrium points

If the initial condition for (4.11) were chosen to be $(x_0, y_0) = (0, 0)$, then we would have $(\dot{x}, \dot{y}) = (0, 0)$ as well, and the point $(x(t), y(t))$ will not move from the origin. A point (X_{eqm}, Y_{eqm}) with $(\dot{x}, \dot{y}) = (0, 0)$ is called an **equilibrium point**; these are an important feature of phase portraits. Linear homogeneous ODEs, such as (4.11), usually have only one equilibrium point (the origin), linear inhomogeneous ODEs, such as (4.10), can have one equilibrium point (not the origin), while nonlinear ODEs, such as (4.9), can have several equilibrium points.

An **ordinary point** is a point (x, y) that is not an equilibrium point (so most points on the phase plane are ordinary points). There are theorems that guarantee (under appropriate conditions) that for autonomous ODEs such as (4.10) and (4.11):

- There is only one trajectory through each ordinary point;
- Any trajectory starting at an ordinary point cannot reach an equilibrium point in a finite amount of time;
- A trajectory through an ordinary point cannot cross itself unless it is a closed curve (as in the circle example above), in which case the trajectory is **periodic**.

Example: Solve the ODE

$$\ddot{x} + 2\dot{x} + 2x = 2.$$

Convert the ODE into a two-dimensional ODE, find the equilibrium point of the ODE, and use the known solutions to sketch its phase portrait.

Solution: We find the characteristic equation by substituting $x = e^{\lambda t}$ into the homogeneous version of the ODE, and dividing by $e^{\lambda t}$:

$$\lambda^2 + 2\lambda + 2 = 0$$

so the roots are $\lambda = -1 \pm i$. Note that these are complex: the real part is -1 and the imaginary part is 1. The Complementary Function is

$$x_{CF}(t) = e^{-t}\left(C_1 \cos t + C_2 \sin t\right).$$

We find the Particular Integral by trying a solution of the form $x_{PI} = A$ and substituting into the ODE: $\dot{x}_{PI} = 0$ and $\ddot{x}_{PI} = 0$, so $A = 1$. The general solution is:

$$x(t) = 1 + e^{-t}\left(C_1 \cos t + C_2 \sin t\right).$$

If (for example) the initial conditions were $x(0) = 0$ and $\dot{x}(0) = 1$, we would find $C_1 = -1$ and $C_2 = 0$, and so

$$x(t) = 1 - e^{-t}\cos t.$$

To write this as a two-dimensional ODE, define $y = \dot{x}$ and we get:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} y \\ -2y - 2x + 2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

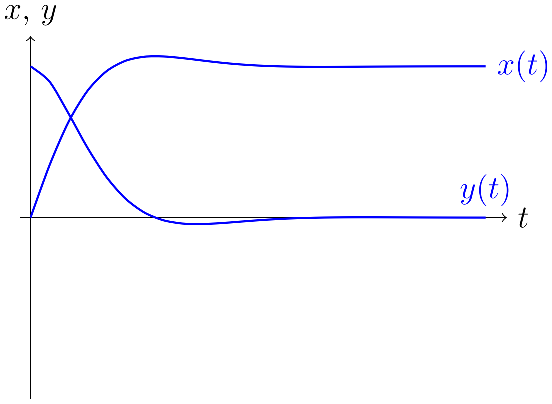
To find the equilibrium point, we solve the pair of equations $\dot{x} = 0$ and $\dot{y} = 0$:

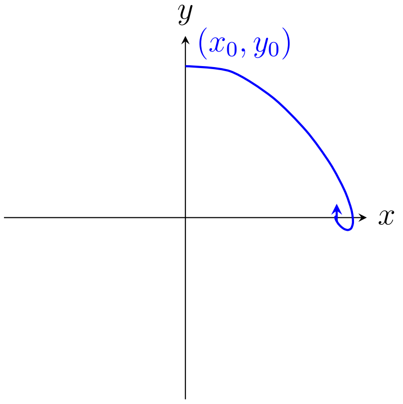
$$\dot{x} = y = 0, \qquad \text{so } Y_{eqm} = 0, \text{ and} \qquad \dot{y} = -2y - 2x + 2 = 0 \qquad \text{so } X_{eqm} = 1.$$

So the equilibrium point is $(X_{eqm}, Y_{eqm}) = (1, 0)$. We have already solved the equation (above) with initial condition $(x_0, y_0) = (0, 1)$:

$$x(t) = 1 - e^{-t}\cos t \qquad \text{and} \qquad y(t) = \dot{x} = e^{-t}\left(\cos t + \sin t\right).$$

We can plot $x(t)$ and $y(t)$ as well as the phase portrait $(x(t), y(t))$:





The trajectory appears to **spiral in** to the equilibrium point at $(1, 0)$.
Note that we can shift our coordinate system so that the equilibrium point is at the origin: define

$$\xi = x - X_{eqm} \qquad \text{and} \qquad \eta = y - Y_{eqm}.$$

In the example above, this results in:

$$\xi = x - 1 \qquad \text{and} \qquad \eta = y,$$

so

$$\dot{\xi} = \dot{x} = y = \eta \qquad \text{and} \qquad \dot{\eta} = \dot{y} = -2y - 2x + 2 = -2\eta - 2\xi$$

or

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

This is essentially just the homogeneous version of the original ODE.

4.5.3 Classification of equilibrium points

The phase portraits of the linear ODE (4.11) depends on the **eigenvalues** and **eigenvectors** of the matrix $\mathbf{A}$. To make this connection, consider the eigenvalues of $\left[\begin{smallmatrix} 0 & 1 \\ -b & -a \end{smallmatrix}\right]$, which are found by solving:

$$\begin{vmatrix} -\lambda & 1 \\ -b & -a - \lambda \end{vmatrix} = \lambda(a + \lambda) + b = \lambda^2 + a\lambda + b = 0.$$

Note that this is the same as the **characteristic equation** for the ODE $\ddot{x} + a\dot{x} + bx = 0$ (see section 4.4.2).
In the case of the general linear ODE:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where a, b, c and d are constants, the characteristic equation is found by solving

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - \mathrm{Tr}(\mathbf{A})\lambda + \mathrm{Det}(\mathbf{A}) = 0,$$

where $\mathrm{Tr}(\mathbf{A}) = a+d$ is the trace of the matrix, and $\mathrm{Det}(\mathbf{A}) = ad-bc$ is the determinant. The eigenvalues are:

$$\lambda = \frac{\mathrm{Tr}(\mathbf{A}) \pm \sqrt{(\mathrm{Tr}(\mathbf{A}))^2 - 4\mathrm{Det}(\mathbf{A})}}{2}$$

We know that the nature of the solution depends entirely on the roots of the characteristic equation – that is, the eigenvalues of $\mathbf{A}$. Recall that the sum of the eigenvalues of $\mathbf{A}$ is equal to $\mathrm{Tr}(\mathbf{A})$, that the product of the eigenvalues of $\mathbf{A}$ is equal to $\mathrm{Det}(\mathbf{A})$, and that when eigenvalues are complex, they come in complex conjugate pairs.

There are six main possibilities (excluding special cases):

1. Stable Node: eigenvalues are real, unequal and negative: $\lambda_2 < \lambda_1 < 0$. This requires $\mathrm{Tr}(\mathbf{A}) < 0$, $\mathrm{Det}(\mathbf{A}) > 0$ and $(\mathrm{Tr}(\mathbf{A}))^2 - 4\mathrm{Det}(\mathbf{A}) > 0$.
2. Stable Focus: eigenvalues are complex with negative real part: $\lambda = p \pm iq$ with $p < 0$ and $q > 0$. This requires $\mathrm{Tr}(\mathbf{A}) < 0$, $\mathrm{Det}(\mathbf{A}) > 0$ and $(\mathrm{Tr}(\mathbf{A}))^2 - 4\mathrm{Det}(\mathbf{A}) < 0$.
3. Centre: eigenvalues are complex with zero real part: $\lambda = p \pm iq$ with $p = 0$ and $q > 0$. This requires $\mathrm{Tr}(\mathbf{A}) = 0$ and $\mathrm{Det}(\mathbf{A}) > 0$.
4. Unstable Focus: eigenvalues are complex with positive real part: $\lambda = p \pm iq$ with $p > 0$ and $q > 0$. This requires $\mathrm{Tr}(\mathbf{A}) > 0$, $\mathrm{Det}(\mathbf{A}) > 0$ and $(\mathrm{Tr}(\mathbf{A}))^2 - 4\mathrm{Det}(\mathbf{A}) < 0$.
5. Unstable Node: eigenvalues are real, unequal and positive: $0 < \lambda_2 < \lambda_1$. This requires $\mathrm{Tr}(\mathbf{A}) > 0$, $\mathrm{Det}(\mathbf{A}) > 0$ and $(\mathrm{Tr}(\mathbf{A}))^2 - 4\mathrm{Det}(\mathbf{A}) > 0$.
6. Saddle: eigenvalues are real and have opposite sign: $\lambda_2 < 0 < \lambda_1$. This requires $\mathrm{Det}(\mathbf{A}) < 0$.

There are also special cases: when the eigenvalues are equal ($(\mathrm{Tr}(\mathbf{A}))^2 - 4\mathrm{Det}(\mathbf{A}) = 0$), or when one or both eigenvalues is zero ($\mathrm{Det}(\mathbf{A}) = 0$).
In the case of real unequal eigenvalues λ_1 and λ_2, with $\lambda_1 > \lambda_2$, it is possible to write down the solution of the ODE in terms of eigenvectors $\underline{v}_1$ and $\underline{v}_2$. Suppose that

$$\mathbf{A}\,\underline{v}_1 = \lambda_1\,\underline{v}_1 \qquad \text{and} \qquad \mathbf{A}\,\underline{v}_2 = \lambda_2\,\underline{v}_2,$$

and consider

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{\lambda_1 t}\,\underline{v}_1 + C_2 e^{\lambda_2 t}\,\underline{v}_2,$$

where C_1 and C_2 are arbitrary constants. Now compute

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = C_1\lambda_1 e^{\lambda_1 t}\,\underline{v}_1 + C_2\lambda_2 e^{\lambda_2 t}\,\underline{v}_2 = C_1 e^{\lambda_1 t}\,\mathbf{A}\,\underline{v}_1 + C_2 e^{\lambda_2 t}\,\mathbf{A}\,\underline{v}_2 = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix},$$

so the assumed form is a general solution of the ODE.
From this, we note two things: first, if $C_2 = 0$ (or $C_1 = 0$), then the trajectory lies along the eigenvector. Second, in the case $C_1 \neq 0$ and $C_2 \neq 0$, since, $\lambda_1 > \lambda_2$, in the limit of large positive t, we have $e^{\lambda_1 t} \gg e^{\lambda_2 t}$, so

$$\begin{pmatrix} x \\ y \end{pmatrix} \to C_1 e^{\lambda_1 t}\,\mathbf{A}\,\underline{v}_1.$$

Thus, the trajectory becomes aligned with the eigenvector $\underline{v}_1$, in the limit $t \to +\infty$. Conversely,

$$\begin{pmatrix} x \\ y \end{pmatrix} \to C_2 e^{\lambda_2 t}\,\mathbf{A}\,\underline{v}_2$$

in the limit $t \to -\infty$.
We can use this to understand the three cases with real eigenvalues:

1. Stable Node: eigenvalues are real and negative. Solve

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \mathbf{A}\begin{pmatrix}x\\ y\end{pmatrix}, \qquad \text{with} \qquad \mathbf{A} = \begin{bmatrix}-3 & -1\\ -1 & -3\end{bmatrix}$$

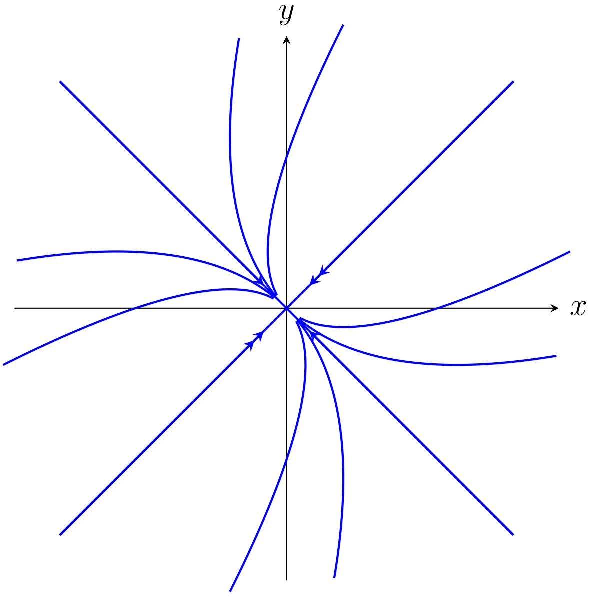
In this case, the characteristic equation is:

$$\begin{vmatrix}-3-\lambda & -1\\ -1 & -3-\lambda\end{vmatrix} = \lambda^2 + 6\lambda + 8 = 0.$$

The roots are $\lambda_1 = -2$ and $\lambda_2 = -4$. The eigenvectors are $\underline{v}_1 = \left(\begin{smallmatrix}1\\ -1\end{smallmatrix}\right)$ and $\underline{v}_2 = \left(\begin{smallmatrix}1\\ 1\end{smallmatrix}\right)$, so the general solution is:

$$\begin{pmatrix}x\\ y\end{pmatrix} = C_1 e^{-2t}\begin{pmatrix}1\\ -1\end{pmatrix} + C_2 e^{-4t}\begin{pmatrix}1\\ 1\end{pmatrix}$$

Choosing $C_1 = 1$ and $C_2 = 0$ results in the eigenspace $y = -x$, and choosing $C_1 = 0$ and $C_2 = 1$ results in the eigenspace $y = x$. For different choices of C_1 and C_2 we get different trajectories.



All trajectories go towards the equilibrium point at the origin. Stable nodes are also called sinks.

5. Unstable Node: eigenvalues are real and positive. Solve

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \mathbf{A}\begin{pmatrix}x\\ y\end{pmatrix}, \qquad \text{with} \qquad \mathbf{A} = \begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$$

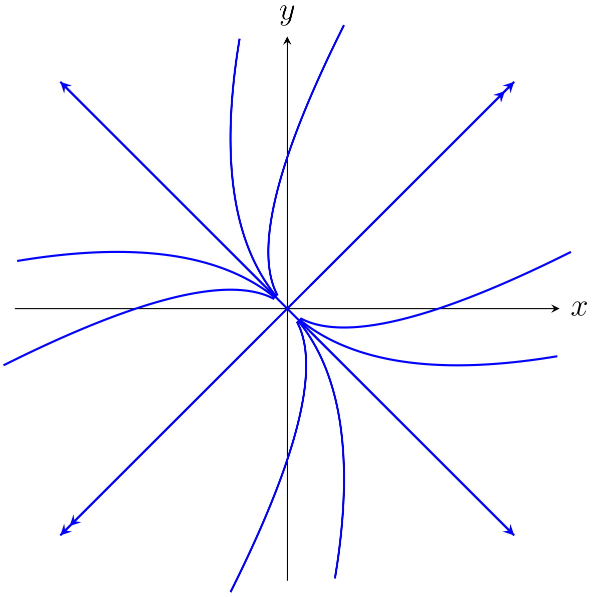
In this case, the characteristic equation is:

$$\begin{vmatrix}3-\lambda & 1\\ 1 & 3-\lambda\end{vmatrix} = \lambda^2 - 6\lambda + 8 = 0.$$

The roots are $\lambda_1 = 4$ and $\lambda_2 = 2$. The eigenvectors are $\underline{v}_1 = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ and $\underline{v}_2 = \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)$, so the general solution is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Choosing $C_1 = 1$ and $C_2 = 0$ results in the eigenspace $y = x$, and choosing $C_1 = 0$ and $C_2 = 1$ results in the eigenspace $y = -x$. For different choices of C_1 and C_2 we get different trajectories.



All trajectories go away from the equilibrium point at the origin. Unstable nodes are also called sources.

6. Saddle: eigenvalues are real and have opposite sign. Solve

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{with} \qquad \mathbf{A} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

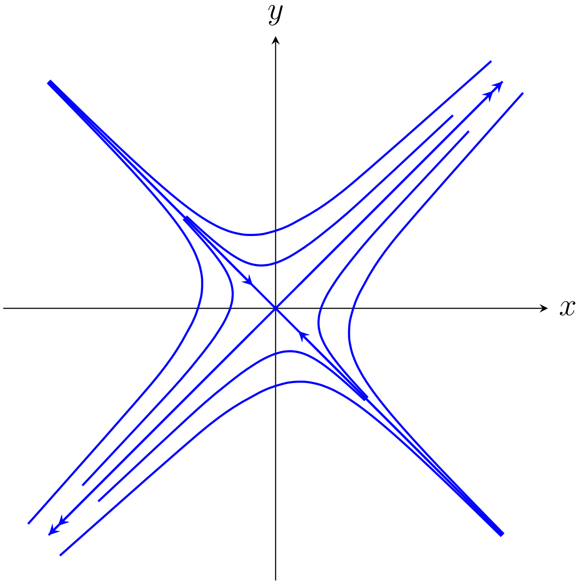
In this case, the characteristic equation is:

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 8 = 0.$$

The roots are $\lambda_1 = 4$ and $\lambda_2 = -2$. The eigenvectors are $\underline{v}_1 = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ and $\underline{v}_2 = \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)$, so the general solution is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Choosing $C_1 = 1$ and $C_2 = 0$ results in the eigenspace $y = x$, and choosing $C_1 = 0$ and $C_2 = 1$ results in the eigenspace $y = -x$. For different choices of C_1 and C_2 we get different trajectories.



All trajectories go towards the equilibrium point at the origin initially, but then bend around and leave the equilibrium point.
In the three cases with complex eigenvalues, a similar general solution can be written down in terms of the real and imaginary parts of the complex eigenvectors, but we will look qualitatively only. We have already seen an example of a centre: eigenvalues are complex with zero real part, and trajectories are closed loops (circles or ellipses). The effect of the real part of the eigenvalue is to make trajectories slowly spiral in to the origin (if the real part is negative) or slowly spiral away from the origin (if the real part is positive).

3. Centre: eigenvalues are complex with zero real part. Solve

$$\ddot{x} + x = 0.$$

This can be written in the form of a two-dimensional ODE:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{with} \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

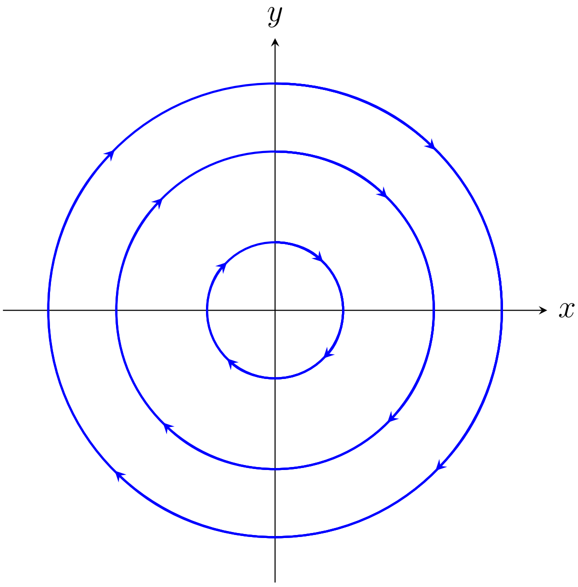
In this case, the characteristic equation is:

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0.$$

The roots are $\lambda_1 = \pm i$ (complex with zero real part). The eigenvectors are also complex, but we already have the solution from above:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \cos t + \begin{pmatrix} C_2 \\ -C_1 \end{pmatrix} \sin t$$

Choosing $C_1 = 1$ and $C_2 = 0$ (for example) results in the circle $x^2 + y^2 = 1$. For different choices of C_1 and C_2 we get different concentric circles.



For pure imaginary eigenvalues (not ±i), trajectories are concentric ellipses.

2. Stable Focus: eigenvalues are complex with negative real part. Solve

$$\ddot{x} + 2\dot{x} + 2x = 0.$$

This can be written in the form of a two-dimensional ODE:

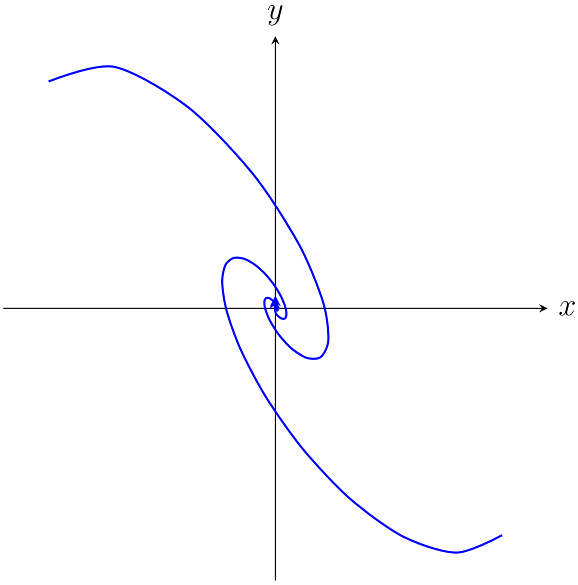
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{with} \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

In this case, the characteristic equation is:

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0.$$

The roots are $\lambda_1 = -1 \pm i$ (complex with negative real part). The eigenvectors are also complex, but we already have the solution from above:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 - C_1 \end{pmatrix} e^{-t} \cos t + \begin{pmatrix} C_2 \\ -C_1 - C_2 \end{pmatrix} e^{-t} \sin t$$



Trajectories spiral in towards the stable focus.

4. Unstable Focus: eigenvalues are complex with positive real part. Solve

$$\ddot{x} - 2\dot{x} + 2x = 0.$$

This can be written in the form of a two-dimensional ODE:

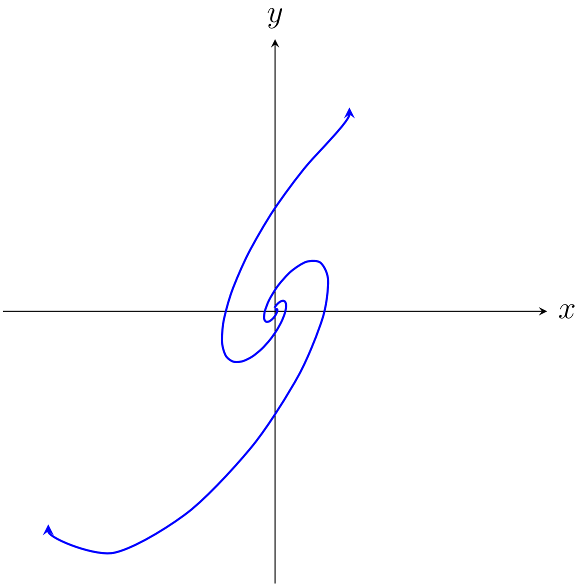
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{with} \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}$$

In this case, the characteristic equation is:

$$\begin{vmatrix} -\lambda & 1 \\ -2 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2 = 0.$$

The roots are $\lambda_1 = 1 \pm i$ (complex with positive real part). We can obtain the solution in a similar manner:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C_1 \\ C_1 + C_2 \end{pmatrix} e^t \cos t + \begin{pmatrix} C_2 \\ -C_1 + C_2 \end{pmatrix} e^t \sin t$$



Trajectories spiral away from the unstable focus.
In the special case of equal eigenvalues, the general solution involves the use of generalised eigenvectors.

Summary. To classify an equilibrium point for an ODE of the form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{with} \qquad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The equilibrium point (X_{eqm}, Y_{eqm}) is found by solving $\dot{x} = \dot{y} = 0$. The characteristic equation is:

$$\lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - \mathrm{Tr}(\mathbf{A})\lambda + \mathrm{Det}(\mathbf{A}) = 0.$$

Solve this to find roots λ_1 and λ_2:

Real roots: $(\mathrm{Tr}(\mathbf{A}))^2 - 4\mathrm{Det}(\mathbf{A}) > 0$

$\lambda_2 < \lambda_1 < 0$	Real negative roots: stable node ($\mathrm{Tr}(\mathbf{A}) < 0$, $\mathrm{Det}(\mathbf{A}) > 0$)
$0 < \lambda_2 < \lambda_1$	Real positive roots: unstable node ($\mathrm{Tr}(\mathbf{A}) > 0$, $\mathrm{Det}(\mathbf{A}) > 0$)
$\lambda_2 < 0 < \lambda_1$	Real roots of opposite sign: saddle ($\mathrm{Det}(\mathbf{A}) < 0$)

Complex roots $\lambda = p \pm iq$: $(\mathrm{Tr}(\mathbf{A}))^2 - 4\mathrm{Det}(\mathbf{A}) < 0$, which implies $\mathrm{Det}(\mathbf{A}) > 0$

$p < 0$	Complex roots with negative real part: stable focus ($\mathrm{Tr}(\mathbf{A}) < 0$)
$p = 0$	Complex roots with zero real part: centre ($\mathrm{Tr}(\mathbf{A}) = 0$)
$p > 0$	Complex roots with positive real part: unstable focus ($\mathrm{Tr}(\mathbf{A}) > 0$)

It is possible to use $\mathrm{Tr}(\mathbf{A})$ and $\mathrm{Det}(\mathbf{A})$ to determine the classification directly.
The classification can be used to sketch the phase protrait.

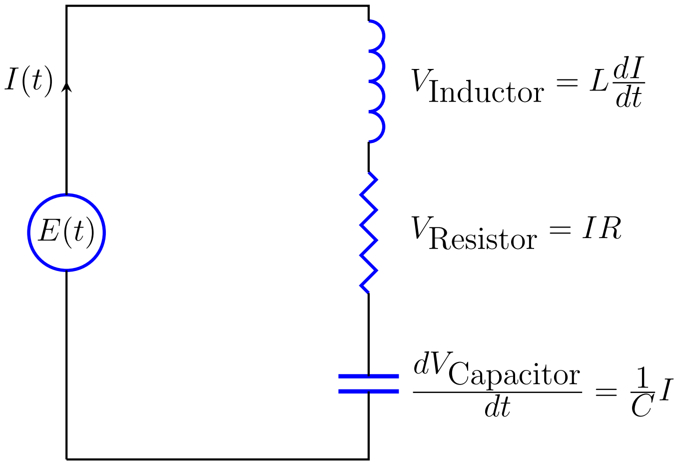
Chapter 5

Applications of second-order ODEs

Contents

5.1 Resonant electric circuits

A resonant electric circuit (or LRC circuit) consists of an imposed voltage $E(t)$, and three circuit elements: an **inductor**, a **resistor** and a **capacitor**:



Kirchhoff's Circuit Laws state that the current $I(t)$ is the same through each element, and that the sum of the voltages across each element is equal to the imposed voltage $E(t)$.

Voltages are measured in Volts, currents (I) in Amperes, charge (Q) in Coulombs, inductance (L) in Henrys, resistance (R) in Ohms, and capacitance (C) in Faradays.

Voltage drop across an inductance

$$V_{\text{Inductor}} = L\frac{dI}{dt}.$$

Voltage drop across a resistor

$$V_{\text{Resistor}} = IR.$$

Voltage drop across a capacitor

$$V_{\text{Capacitor}} = \frac{1}{C}Q,$$

where Q is the charge in the capacitor (related to the current by $I = dQ/dt$), so

$$\frac{dV_{\text{Capacitor}}}{dt} = \frac{1}{C}I.$$

The sum of the three voltages equals the imposed $E(t)$:

$$V_{\text{Inductor}} + V_{\text{Resistor}} + V_{\text{Capacitor}} = E(t).$$

Differentiate and substitute the three relations above:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}$$

We will solve this in the case of an imposed sinusoidal voltage of amplitude E_0 and frequency ω, that is, $E(t) = -E_0 \cos \omega t$:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \omega E_0 \sin \omega t.$$

First, we find the characterstic equation by substituting $I = e^{\lambda t}$ into the homogeneous equation, and dividing by $e^{\lambda t}$:

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0, \qquad \text{or} \qquad LC\lambda^2 + RC\lambda + 1 = 0,$$

The roots are:

$$\lambda = \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC}.$$

When LRC circuits are used as resonant electric circuits, the resistance R is small, so the roots are complex. Let the roots be

$$\lambda = -\alpha \pm i\sqrt{\omega_0^2 - \alpha^2},$$

where

$$\alpha = \frac{R}{2L}$$

is called the *attenuation factor* and

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

is the (undamped) *resonant frequency.* Thus the Complementary Function is

$$I_{CF} = e^{-\alpha t}\left(C_1\cos(\omega_d t) + C_2\sin(\omega_d t)\right).$$

with $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ being the damped resonant frequency. Note that since $\alpha > 0$, we have $I_{CF} \to 0$ as $t \to \infty$.
Now we look for a Particular Integral:

$$I_{PI} = A\cos(\omega t) + B\sin(\omega t).$$

Before proceeding, divide the ODE by L and use α and ω_0 to eliminate L and R: $LC = \omega_0^{-2}$ and $RC = 2\alpha\omega_0^{-2}$:

$$\frac{d^2I}{dt^2} + 2\alpha\frac{dI}{dt} + \omega_0^2 I = \omega\frac{E_0}{L}\sin\omega t.$$

Substitute the assumed form of the Particular Integral into the ODE:

$$\left(-A\omega^2 + 2B\alpha\omega + A\omega_0^2\right)\cos(\omega t) + \left(-B\omega^2 - 2A\alpha\omega + B\omega_0^2\right)\sin(\omega t) = \omega\frac{E_0}{L}\sin\omega t$$

Compare terms multiplying $\cos(\omega t)$ and $\sin(\omega t)$ to get a pair of equations for A and B:

$$-A\omega^2 + 2B\alpha\omega + A\omega_0^2 = 0 \qquad \text{and} \qquad -B\omega^2 - 2A\alpha\omega + B\omega_0^2 = \omega\frac{E_0}{L},$$

which can be solved:

$$A = -\frac{E_0}{L}\frac{2\alpha\omega^2}{(\omega^2-\omega_0^2)^2 + 4\alpha^2\omega^2} \qquad \text{and} \qquad B = \frac{E_0}{L}\frac{\omega(\omega^2-\omega_0^2)}{(\omega^2-\omega_0^2)^2 + 4\alpha^2\omega^2}.$$

The Particular Integral $I_{PI} = A\cos(\omega t) + B\sin(\omega t)$ can also be written in the form

$$I_{PI} = \sqrt{A^2+B^2}\sin(\omega t + \arctan(A/B))$$

(elementary trigonometry), so we call $\sqrt{A^2+B^2}$ the **amplitude** of I_{PI}:

$$\sqrt{A^2+B^2} = \frac{E_0}{L}\frac{\omega}{\sqrt{(\omega^2-\omega_0^2)^2 + 4\alpha^2\omega^2}}.$$

The general solution is $I_{CF} + I_{PI}$, but since $I_{CF} \to 0$ as $t \to \infty$, only I_{PI} remains after I_{CF} has decayed away. As a function of ω, this is maximum at $\omega = \omega_0$.
We can plot the amplitude of I_{PI} as a function of ω in the case (for example) $L = 1\,\mathrm{H}$, $R = 2\,\Omega$ and $C = 0.5\,\mathrm{F}$, with $E_0 = 1\,\mathrm{V}$, so the original ODE is

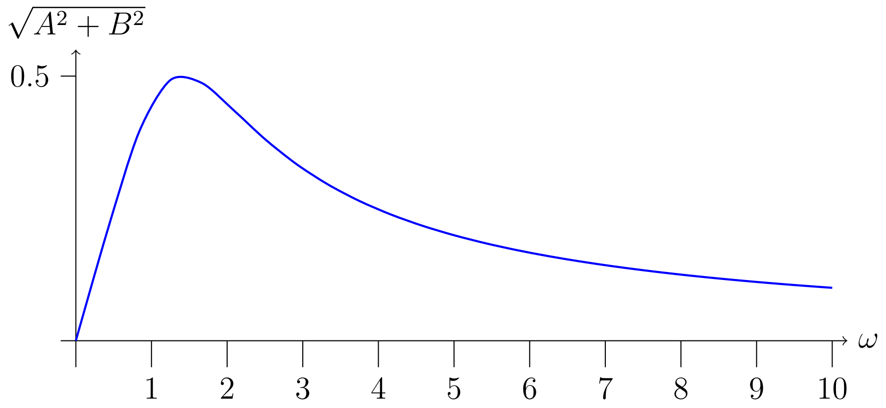
$$\frac{d^2I}{dt^2} + 2\frac{dI}{dt} + 2I = \omega\sin\omega t,$$

(roots are $-1 \pm i$) and

$$\alpha = 1 \qquad \text{and} \qquad \omega_0 = \sqrt{2}.$$

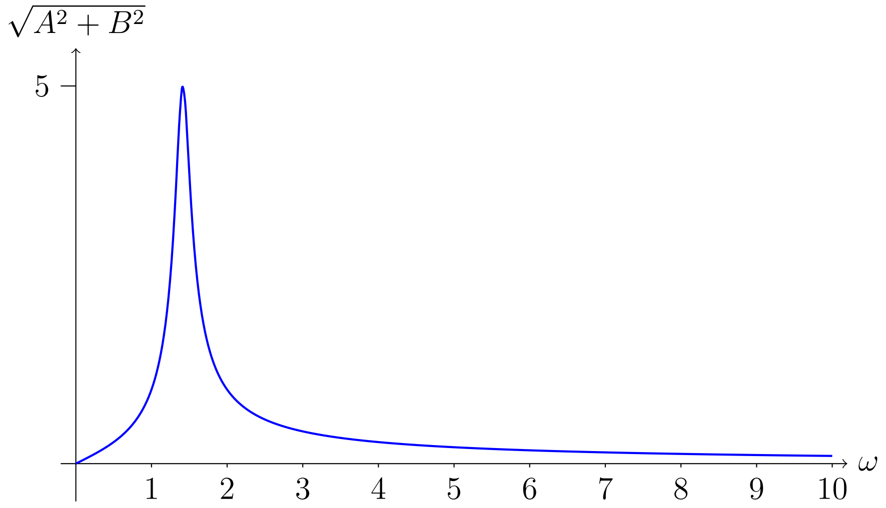
In this case, the amplitude of I_{PI} is

$$\sqrt{A^2+B^2} = \frac{\omega}{\sqrt{4\omega^2 + (\omega^2-2)^2}} = \frac{\omega}{\sqrt{\omega^4+4}}$$



As a second example, take $R = 0.2\,\Omega$ but otherwise the same as the first example. In this case, $\alpha = 0.1$ and

$$\sqrt{A^2+B^2} = \frac{5\omega}{\sqrt{25\omega^4 - 99\omega^2 + 100}}$$



With smaller α, the peak of the response is much more sharply focussed at $\omega = \omega_0$. This forms the basis of a **band pass filter**: the circuit responds only frequencies close to ω_0. *LRC* circuits are used as tuners in simple radio receivers.

5.2 Further applications

Simple supply/demand/price models; Voting model; Modelling infectious diseases; Two species radioactive decay; Gradient systems.