MATH1400: Modelling with differential equations, 2011–12
Examples 5

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Course web page: `http://www.maths.leeds.ac.uk/~alastair/MATH1400/`.

There is no section 1/section 2 this week. The tutorials are on Fridays 27 April and 4 May. There will also be a quiz on **Friday 27 April**. The topics for the quiz will be announced during lectures. Hand in your answers in the grey boxes outside the School of Maths Undergraduate Office by 5pm on **Tuesday 8th May** (NB this is after lectures have ended, Monday 7th is a Bank Holiday). Be sure to put your work in **your tutor's** box. Be sure to write your name on your work. Your work will be returned in the blue trays outside the School of Maths Undergraduate Office.

1. We will solve each of the following second-order linear ODEs in two ways. First, solve the second-order ODE with the initial condition $x(0) = 0$, $\dot{x}(0) = 1$. Second, convert the ODEs into the form of two-dimensional first-order ODEs, and write down the known solution in this new coordinate system. Identify the equilibrium point. Calculate the eigenvalues of the appropriate matrix, and, if the eigenvalues are real, calculate the eigenvectors. Use all this information to classify the equilibrium point and sketch the phase portrait.

(a) $\ddot{x} - \dot{x} - 2x = 0$.

(b) $\ddot{x} + 4\dot{x} + 8x = 0$.

2. Identify the equilibrium point of the following two-dimensional first-order ODEs. Calculate the eigenvalues of the matrix, and, if the eigenvalues are real, calculate the eigenvectors and write down the general solution. Classify the equilibrium point and sketch the phase portrait.

(a) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

(b) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} -1 & 2 \\ 6 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

3. The following differential equation is a model of an electric oscillator:

$$\frac{d^2I}{dt^2} + 2\frac{dI}{dt} + 5I = 0$$

where I is the current in the oscillator and t is time. Find the general solution of this second-order linear constant coefficient homogeneous ODE. Describe what happens to this solution as $t \to \infty$.

If the oscillator is subjected to an externally applied sinusoidal forcing with frequency ω, the ODE that describes the situation is:

$$\frac{d^2I}{dt^2} + 2\frac{dI}{dt} + 5I = \omega\sin(\omega t)$$

Find the general solution of this ODE. How does the solution in the limit $t \to \infty$ depend on ω?

4. The radioactive element ^{238}Pu (Plutonium-238) decays to ^{234}U (Uranium-234) by emitting an α particle. ^{234}U is itself radioactive, and decays to form ^{230}Th (Thorium-230). The half-life of ^{238}Pu is $T_1 = 87.7$ years, and the half-life of ^{234}U is $T_2 = 246,000$ years. Let $x(t)$ be the amount of ^{238}Pu in a sample, and let $y(t)$ be the amount of ^{234}U. The equations governing the radioactive decay are:

$$\frac{dx}{dt} = -k_1 x, \qquad \frac{dy}{dt} = k_1 x - k_2 y,$$

where $k_1 = \log(2)/T_1$ and $k_2 = \log(2)/T_2$.

(a) Identify the equilibrium point of the ODEs. Calculate the eigenvalues and eigenvectors of the matrix, write down the general solution, classify the equilibrium point and sketch the phase portrait.

(b) The initial sample has 1 unit of ^{238}Pu (so $x(0) = 1$) and no ^{234}U (so $y(0) = 0$). Solve the first equation, $\dot{x} = -k_1 x$, with the initial condition $x(0) = 1$. Substitute this solution into the second equation for $\dot{y}$ and solve this one as well. Sketch the functions $x(t)$ and $y(t)$. What is the maximum value of y? What happens as $t \to \infty$?

5. House buyers base their demand not only on the current selling price, but also on the rate of change of price (how quickly the price has been rising or falling), and possibly also on whether this rate of change is slowing down or speeding up (the second derivative of price). For example, if a buyer sees that prices are falling quickly, they may wait before buying, and so demand is reduced. Conversely sellers, or potential sellers, will also base their decision put their house on the market on the current price and its rate of change, and possibly on whether the rate of change is slowing down or speeding up. For example, if prices are rising, potential sellers may decide to wait before putting their house on the market, and so supply is reduced.

Suppose that the demand for houses is given by:

$$9 - 2p + 6\frac{dp}{dt} - 2\frac{d^2p}{dt^2},$$

where $p(t)$ is the price. Note that demand goes down when the price is higher ($-2p$), that demand increases if prices are rising ($+6\frac{dp}{dt}$ – buy now before it is too late!), and that demand decreases if the rate of price increase is itself increasing ($-2\frac{d^2p}{dt^2}$).
Suppose also that the supply of houses on the market is given by:

$$-3 + 4p - \frac{dp}{dt} - \frac{d^2p}{dt^2},$$

Note the signs are mainly opposite to those in the expression for demand.
If we assume that the market is in equilibrium, that is, assume that the supply equals demand, we obtain an equation for the market price:

$$9 - 2p + 6\frac{dp}{dt} - 2\frac{d^2p}{dt^2} = -3 + 4p - \frac{dp}{dt} - \frac{d^2p}{dt^2},$$

or

$$\frac{d^2p}{dt^2} - 7\frac{dp}{dt} + 6p = 12.$$

Convert this into the form of a two-dimensional first-order ODE. Identify the equilibrium point of the ODE. Calculate the eigenvalues and eigenvectors of the matrix, classify the equilibrium point and sketch the phase portrait. What does this model tell you about house prices? Is it a sensible model?
(This example is from *Mathematics for economics and finance: methods and modelling*, by Martin Anthony and Norman Biggs, CUP 1996.)