MATH1400MOCK

This question paper consists of 2 printed pages, each of which is identified by the reference MATH1400MOCK

Only approved basic scientific calculators may be used.

Examination for the Module MATH1400

(December 2008)

Modelling with Differential Equations

Time allowed: 2 hours

Do not attempt more than **four** questions.
All questions carry equal marks.

1. **(a)** Solve the Initial Value Problem

$$xy(1+x^2)\,\frac{dy}{dx} - y^2 = 1 \qquad \text{with } y(1) = 1.$$

(b) Find the general solution of the ODE:

$$\frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{x^2}$$

(c) Solve the Initial Value Problem

$$\frac{dy}{dx} + \frac{3}{x}\,y = \frac{e^{2x}}{x^3} \qquad \text{with } y(1) = 1.$$

2. **(a)** **(i)** A radioactive substance decays at a rate proportional to the amount of the substance present. If $k > 0$ is the constant of proportionality, write down the differential equation satisfied by $x(t)$, the amount of substance present at time t.

(ii) At time $t = 0$, the amount of substance present is x_0. Show that at any later time t,

$$x(t) = x_0 e^{-kt}$$

(iii) Define the **half-life** T_h of the substance, and obtain an expression for T_h in terms of k. Find the time (expressed in terms of T_h) at which 80% of the original quantity remains.

(b) Find the solution of the following Initial Value Problem:

$$\frac{d\Theta}{dt} = -k(\Theta - A) \qquad \text{with } \Theta(t_0) = \Theta_0$$

where A and k are constants. Explain how this equation can be used to model how a hot object cools down.

continued ...

3. **(a)** Find the general solution of the following ODE:

$$y'' - y' - 2y = e^x - e^{-x}$$

(b) Solve the following Boundary Value Problem

$$y'' + 4y = \sin x \qquad \text{with } y(0) = 0 \text{ and } y(\tfrac{\pi}{4}) = 0.$$

(c) Find the general solution of the following ODE:

$$y'' + 6y' + 13y = 13x^2 - x + 22$$

4. **(a)** Solve the following Initial Value Problem

$$y'' + 4y' + 4y = 4x + e^{2x} \qquad \text{with } y(0) = 0 \text{ and } y'(0) = 0.$$

(b) Given that $y = e^x$ is one solution of the differential equation

$$(1 + x)y'' - (2x + 3)y' + (x + 2)y = 0$$

find the general solution of

$$(1 + x)y'' - (2x + 3)y' + (x + 2)y = e^x(1 + x)^2$$

5. **(a)** Write down the differentials dF of the following functions $F(x, y)$:

(i) $x^3 - 3xy^2$ (ii) $y^2e^x + \sin(x + y)$ (iii) $\log(x^2 + y^2) - 2xy$

(b) Write down the conditions under which a function $F(x, y)$ has a stationary point, and the conditions under which the stationary point is a maximum, minimum or saddle point.

(c) Find and classify the stationary points of

$$F(x, y) = y(x^2 + y^2 - 1)$$

END