Chapter 5

Applications of second-order ODEs

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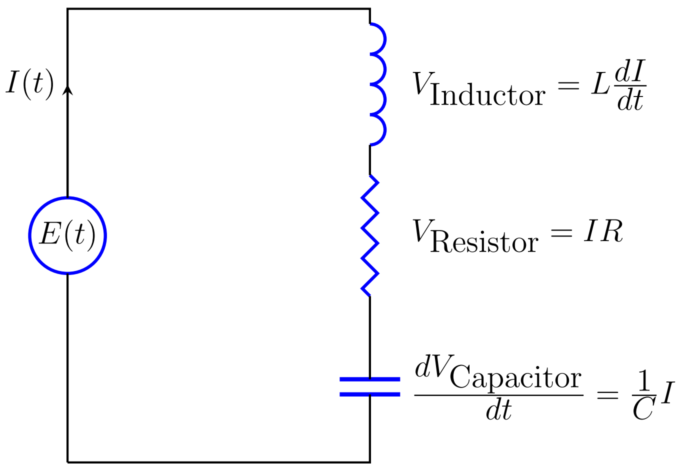
5.1 Resonant electric circuits

A resonant electric circuit (or LRC circuit) consists of an imposed voltage $E(t)$, and three circuit elements: an **inductor**, a **resistor** and a **capacitor**:



Kirchhoff's Circuit Laws state that the current $I(t)$ is the same through each element, and that the sum of the voltages across each element is equal to the imposed voltage $E(t)$.

Voltages are measured in Volts, currents (I) in Amperes, charge (Q) in Coulombs, inductance (L) in Henrys, resistance (R) in Ohms, and capacitance (C) in Faradays.

Voltage drop across an inductance

$$V_{\text{Inductor}} = L\frac{dI}{dt}.$$

Voltage drop across a resistor

$$V_{\text{Resistor}} = IR.$$

Voltage drop across a capacitor

$$V_{\text{Capacitor}} = \frac{1}{C}Q,$$

where Q is the charge in the capacitor (related to the current by $I = dQ/dt$), so

$$\frac{dV_{\text{Capacitor}}}{dt} = \frac{1}{C}I.$$

The sum of the three voltages equals the imposed $E(t)$:

$$V_{\text{Inductor}} + V_{\text{Resistor}} + V_{\text{Capacitor}} = E(t).$$

Differentiate and substitute the three relations above:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}$$

We will solve this in the case of an imposed sinusoidal voltage of amplitude E_0 and frequency ω, that is, $E(t) = -E_0 \cos \omega t$:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \omega E_0 \sin \omega t.$$

First, we find the characterstic equation by substituting $I = e^{\lambda t}$ into the homogeneous equation, and dividing by $e^{\lambda t}$:

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0, \qquad \text{or} \qquad LC\lambda^2 + RC\lambda + 1 = 0,$$

The roots are:

$$\lambda = \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC}.$$

When LRC circuits are used as resonant electric circuits, the resistance R is small, so the roots are complex. Let the roots be

$$\lambda = -\alpha \pm i\sqrt{\omega_0^2 - \alpha^2},$$

where

$$\alpha = \frac{R}{2L}$$

is called the *attenuation factor* and

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

is the (undamped) *resonant frequency*. Thus the Complementary Function is

$$I_{CF} = e^{-\alpha t}\left(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\right).$$

with $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ being the damped resonant frequency. Note that since $\alpha > 0$, we have $I_{CF} \to 0$ as $t \to \infty$.
Now we look for a Particular Integral:

$$I_{PI} = A\cos(\omega t) + B\sin(\omega t).$$

Before proceeding, divide the ODE by L and use α and ω_0 to eliminate L and R: $LC = \omega_0^{-2}$ and $RC = 2\alpha\omega_0^{-2}$:

$$\frac{d^2 I}{dt^2} + 2\alpha\frac{dI}{dt} + \omega_0^2 I = \omega\frac{E_0}{L}\sin\omega t.$$

Substitute the assumed form of the Particular Integral into the ODE:

$$\left(-A\omega^2 + 2B\alpha\omega + A\omega_0^2\right)\cos(\omega t) + \left(-B\omega^2 - 2A\alpha\omega + B\omega_0^2\right)\sin(\omega t) = \omega\frac{E_0}{L}\sin\omega t$$

Compare terms multiplying $\cos(\omega t)$ and $\sin(\omega t)$ to get a pair of equations for A and B:

$$-A\omega^2 + 2B\alpha\omega + A\omega_0^2 = 0 \qquad \text{and} \qquad -B\omega^2 - 2A\alpha\omega + B\omega_0^2 = \omega\frac{E_0}{L},$$

which can be solved:

$$A = -\frac{E_0}{L}\frac{2\alpha\omega^2}{(\omega^2-\omega_0^2)^2 + 4\alpha^2\omega^2} \qquad \text{and} \qquad B = \frac{E_0}{L}\frac{\omega(\omega^2-\omega_0^2)}{(\omega^2-\omega_0^2)^2 + 4\alpha^2\omega^2}.$$

The Particular Integral $I_{PI} = A\cos(\omega t) + B\sin(\omega t)$ can also be written in the form

$$I_{PI} = \sqrt{A^2+B^2}\sin(\omega t + \arctan(A/B))$$

(elementary trigonometry), so we call $\sqrt{A^2+B^2}$ the **amplitude** of I_{PI}:

$$\sqrt{A^2+B^2} = \frac{E_0}{L}\frac{\omega}{\sqrt{(\omega^2-\omega_0^2)^2 + 4\alpha^2\omega^2}}.$$

The general solution is $I_{CF} + I_{PI}$, but since $I_{CF} \to 0$ as $t \to \infty$, only I_{PI} remains after I_{CF} has decayed away. As a function of ω, this is maximum at $\omega = \omega_0$.
We can plot the amplitude of I_{PI} as a function of ω in the case (for example) $L = 1\,\mathrm{H}$, $R = 2\,\Omega$ and $C = 0.5\,\mathrm{F}$, with $E_0 = 1\,\mathrm{V}$, so the original ODE is

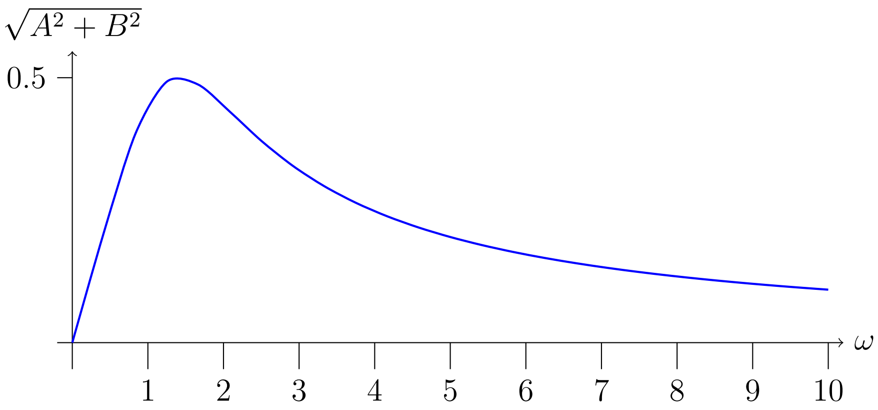
$$\frac{d^2 I}{dt^2} + 2\frac{dI}{dt} + 2I = \omega\sin\omega t,$$

(roots are $-1 \pm i$) and

$$\alpha = 1 \qquad \text{and} \qquad \omega_0 = \sqrt{2}.$$

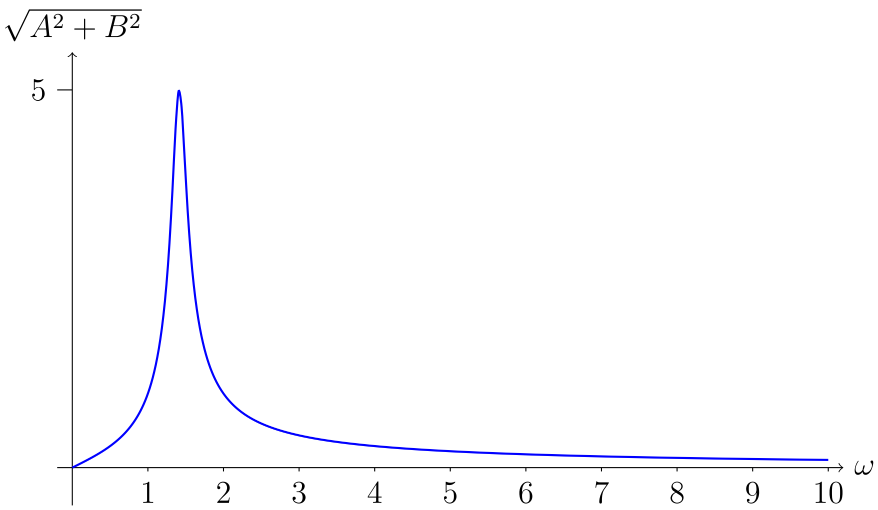
In this case, the amplitude of I_{PI} is

$$\sqrt{A^2 + B^2} = \frac{\omega}{\sqrt{4\omega^2 + (\omega^2 - 2)^2}} = \frac{\omega}{\sqrt{\omega^4 + 4}}$$



As a second example, take $R = 0.2\,\Omega$ but otherwise the same as the first example. In this case, $\alpha = 0.1$ and

$$\sqrt{A^2 + B^2} = \frac{5\omega}{\sqrt{25\omega^4 - 99\omega^2 + 100}}$$



With smaller α, the peak of the response is much more sharply focussed at $\omega = \omega_0$. This forms the basis of a **band pass filter**: the circuit responds only frequencies close to ω_0. LRC circuits are used as tuners in simple radio receivers.

5.2 Further applications

Simple supply/demand/price models; Voting model; Modelling infectious diseases; Two species radioactive decay; Gradient systems.