Chapter 1

Kinematics

1.1 Basic ideas

$\underline{r}(t)$ is the position of a particle; $r = |\underline{r}|$ is the distance to the origin.
If $\underline{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k} = (x, y, z)$, then $r = |\underline{r}| = \sqrt{x^2 + y^2 + z^2}$.
$\underline{v}(t)$ is the velocity; $v = |\underline{v}|$ is the speed.
$\underline{a}(t)$ is the acceleration of the particle.

In one-dimensional motion, we have

$$\text{Average speed during a time interval} = \frac{\text{Distance travelled}}{\text{time taken}},$$

and if the particle is a distance $s(t)$ along the x-axis, then its instantaneous speed is:

$$v = \frac{ds}{dt} = \dot{s}.$$

In more than one dimension,

$$\underline{r}(t) = x(t)\,\underline{i} + y(t)\,\underline{j} + z(t)\,\underline{k} = (x, y, z),$$

and the instantaneous velocity is

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{\underline{r}} = \dot{x}\,\underline{i} + \dot{y}\,\underline{j} + \dot{z}\,\underline{k} = (\dot{x}, \dot{y}, \dot{z}). \tag{1}$$

A particle is *at rest* if $\underline{v} = 0$.

1.2 Acceleration

The acceleration of a particle in one or more dimension is:

$$\underline{a} = \frac{d\underline{v}}{dt} = \dot{\underline{v}} = \frac{d^2\underline{r}}{dt^2} = \ddot{\underline{r}} = (\ddot{x}, \ddot{y}, \ddot{z}). \tag{2}$$

In one dimension, let the position of a particle be $\underline{r} = s(t)\,\underline{i}$, with speed $v = \dot{s}$ and acceleration $a = \dot{v} = \ddot{s}$. Then

$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds} = \frac{d}{ds}\left(\frac{1}{2}v^2\right). \tag{3}$$

If we assume that the acceleration is uniform (a constant), and that at time $t = 0$, the particle is at position $s = 0$ and is travelling with speed u, then its speed at time t is:

$$v = u + at, \tag{4}$$

its position at time t is

$$s = ut + \frac{1}{2}at^2, \tag{5}$$

and we can eliminate t from the two equations above to get:

$$v^2 = u^2 + 2as. \tag{6}$$

With non-uniform acceleration in one or more dimensions, use (1) and (2) to calculate $\underline{v}$ and $\underline{a}$ from $\underline{r}$ (or $\underline{a}$ from $\underline{v}$) by differentiation, and integrate to find $\underline{r}$ from $\underline{v}$, or $\underline{r}$ and $\underline{v}$ from $\underline{a}$.

1.3 Relative velocity

The position of a particle P relative to a particle Q is

$$\underline{r}_{PQ} = \underline{r}_P - \underline{r}_Q. \tag{7}$$

Differentiating, we get the relative velocity

$$\underline{v}_{PQ} = \underline{v}_P - \underline{v}_Q, \tag{8}$$

and the relative acceleration

$$\underline{a}_{PQ} = \underline{a}_P - \underline{a}_Q. \tag{9}$$

1.4 Angular speed

Let θ be the angle that the line OP makes with a fixed reference line OA, so $\theta = \angle\mathrm{POA}$, where O and A are fixed and P is moving. Then the instantaneous angular speed of P is

$$\omega = \frac{d\theta}{dt} = \dot{\theta}. \tag{10}$$

If the particle is moving in a circle of radius R, then the instantaneous speed of the particle is

$$v = R\omega. \tag{11}$$

Chapter 2

Dynamics: Newton's laws of motion and gravitation

2.1 Newton's laws of motion

A **particle** is a point mass, having no internal structure. We will not consider 'rigid body dynamics', where rotation of the object would have to be included (e.g., the spin on a tennis ball). All objects that we consider (stones, balls, people, trains, cars, boats etc) will be treated as particles.

I: Newton's First Law of Motion:
A particle moves at a constant velocity (perhaps zero) unless it is acted up on by a force.
The **momentum** of a particle is its mass times its velocity:

$$\underline{p} = m\underline{v} \tag{12}$$

II: Newton's Second Law of Motion:
The rate of change of momentum of a particle is equal to the total force acting on the particle:

$$\underline{F} = \frac{d\underline{p}}{dt} = \frac{d}{dt}\left(m\underline{v}\right) = \frac{dm}{dt}\underline{v} + m\frac{d\underline{v}}{dt} = \dot{m}\underline{v} + m\dot{\underline{v}}.$$

The total force $\underline{F}$ is the vector sum of all forces acting on the particle. If the mass of the particle is constant ($\dot{m} = 0$) , then

$$\underline{F} = m\frac{d\underline{v}}{dt} = m\dot{\underline{v}} = m\underline{a} \tag{13}$$

III: Newton's Third Law of Motion:
If a particle A exerts a force $\underline{F}$ on particle B, then particle B exerts an equal and opposite force $-\underline{F}$ on particle A.

One consequence of Newton's Third Law of Motion is the **Principle of Conservation of Momentum**: the total momentum of two particles interacting though equal and opposite forces is constant.

2.2 Newton's Law of Gravitation

Two particles of masses M and m a distance r apart exert a mutually attractive force on each other:

$$\text{Gravitational Force} = \frac{GMm}{r^2} \tag{14}$$

where G is the Universal Gravitational Constant $G = 6.67 \times 10^{-11}\,\mathrm{m}^3/\mathrm{kg}/\mathrm{s}^2$ in S.I. units.

For an object close to the surface of a planet of mass M and radius R, the weight of the object is a force directed towards the centre of the planet, of magnitude

$$\text{Weight} = \frac{GM}{R^2}m = mg \qquad \text{where} \qquad g = \frac{GM}{R^2} \tag{15}$$

Chapter 3

Projectiles, lifts, ropes and pulleys, friction, collisions, circular motion

3.1 Projectiles

A projectile starting from the origin at time $t = 0$ and with an initial speed U_0, launched with an angle α above the horizontal, moves under the influence of gravity:

$$\underline{a} = -g\,\underline{j},$$

where the x-direction is horizontal and the y-direction is vertically upwards, and g is the gravitational acceleration. We integrate to get the velocity:

$$\underline{v} = U_0 \cos\alpha\,\underline{i} + (U_0 \sin\alpha - gt)\;\underline{j},$$

(using the given initial velocity), and integrate again to get the position:

$$\underline{r} = x\,\underline{i} + y\,\underline{j} = (U_0\cos\alpha)\,t\,\underline{i} + \left((U_0\sin\alpha)\,t - \frac{1}{2}gt^2\right)\,\underline{j},$$

(using the given initial position). Putting this together, let $x(t)$ be the horizontal position, let $y(t)$ be the vertical position, and let $v = \dot{y}$ be the vertical component of the velocity (note that v is not the magnitude of $\underline{v}$ in this case), we have:

$$x = (U_0\cos\alpha)\,t \tag{16}$$

$$v = U_0\sin\alpha - gt \tag{17}$$

$$y = (U_0\sin\alpha)\,t - \frac{1}{2}gt^2 \tag{18}$$

$$v^2 = (U_0\sin\alpha)^2 - 2gy \tag{19}$$

Time t can be eliminated by writing t in terms of x using (16) and then substituting into (18). With a different initial velocity of postion, the expressions may be different, but the idea is the same. The range can be found by solving $y = 0$ (or whatever condition specifies when the particle comes back to the ground.

3.2 Lifts

If a lift (elevator) accelerates upwards with an acceleration a, then the apparent weight of an object of mass m in the lift is $m(g+a)$.

3.3 Ropes and pulleys

We will treat only massless inextensible ropes and strings, and the *tension* in a rope is the force that rope applies to the particles that are connected to either end of the rope. We consider only massless frictionless pulleys, so the tension in the rope on either side of the pulley is the same. Ropes can only pull particles, not push. If a rope connects two particles, then the speeds and accelerations of the two particles are the same (thought the directions will be different).

3.4 Friction

If an object of mass m is moving across a rough horizontal surface, then there is a vertical normal force N that balances the object's weight mg, and there is an opposing frictional resistance R:

$$R = \mu N, \tag{20}$$

where μ is called the coefficient of friction. If the object is not moving, but some horizontal forces are being applied, there may still be an opposing frictional force $R < \mu N$.

3.5 Impulse and collisions

Newton's Second Law (II) is:

$$\underline{F} = m\frac{d\underline{v}}{dt} = m\dot{\underline{v}} = m\underline{a};$$

If we integrate this from time 0 to t, we get:

$$\int_0^t \underline{F}(t)\,dt = \int_0^t m\frac{d\underline{v}}{dt}\,dt = m\underline{v}(t) - m\underline{v}(0)$$

In a collision, where the force is non-zero only for a very short time, from 0 to Δt, the *Impulse* is defined to be:

$$\underline{I} = \int_0^{\Delta t} \underline{F}(t)\,dt = m\underline{v}_1 - m\underline{v}_0, \tag{21}$$

where $\underline{v}_0$ is the velocity before the collision, and $\underline{v}_1$ is the velocity after the collision. Thus, the Impulse acting on a particle in a collision is equal to the change in the particle's momentum during the collision. The units of Impulse are Ns (Newton seconds), and it is a vector.

3.5.1 Inelastic collisions

Suppose two particles collide and stick together. Then the total momentum before the collision is equal to the total momentum after the collision. This is called *Conservation of momentum.*

In the case of one-dimensional motion, consider two particles of masses m_A and m_B moving to the right with speeds u_A and u_B respectively. If one of the particles is moving to the left, take its speed to be negative. The total momentum before any collision is therefore $m_A u_A + m_B u_B$. The particles stick together in an inelastic collision, then the new particle has mass $m_A + m_B$, and travels to the right with speed v, and so has total momentum $(m_A + m_B)v$. Equating these, we find v:

$$v = \frac{m_A u_A + m_B u_B}{m_A + m_B} \tag{22}$$

3.5.2 Elastic collisions

If two particles collide and then bounce apart, then the total momentum is conserved: the total momentum before the collision is equal to the total momentum after the collision.

In the case of one-dimensional motion, consider two particles of masses m_A and m_B moving to the right with speeds u_A and u_B respectively. If one of the particles is moving to the left, take its speed to be negative. After the collision, the particles have speeds v_A and v_B respectively, where again positive speeds mean that the particle is travelling to the right. The total momentum is the same before and after:

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B \tag{23}$$

In addition, we have *Newton's Elastic Law* (NEL), which says that when two particles of the same material collide directly, then their relative velocity after impact is in direct proportion to their relative velocity before impact, but in the opposite direction:

$$\text{NEL:} \qquad v_B - v_A = -e(u_B - u_A), \qquad \text{or} \qquad \frac{v_B - v_A}{u_B - u_A} = -e \tag{24}$$

3.6 Circular motion

A particle P whose position vector is

$$\underline{r}(t) = R\cos(\omega t)\,\underline{i} + R\sin(\omega t)\,\underline{j} = x(t)\,\underline{i} + y(t)\,\underline{j}$$

is moving in a circle of radius R, and the line OP makes an angle θ with the x-axis, where $\tan\theta = y(t)/x(t) = \tan(\omega t)$, so $\theta = \omega t$. The particle has *angular velocity* ω – see equation (10). The period of the oscillation is $T = 2\pi/\omega$

The velocity and acceleration of the particle are:

$$\underline{v} = \dot{\underline{r}} = -R\omega\sin(\omega t)\,\underline{i} + R\omega\cos(\omega t)\,\underline{j} \quad \text{and} \quad \underline{a} = \dot{\underline{v}} = -R\omega^2\cos(\omega t)\,\underline{i} - R\omega^2\sin(\omega t)\,\underline{j} = -\omega^2\underline{r}.$$

The velocity $\underline{v}$ is tangent to the circle (since $\underline{r}\cdot\underline{v}=0$) and the acceleration $\underline{a}$ is directed towards the centre of the circle (since $\underline{a}$ is a negative constant times $\underline{r}$). The magnitudes of these vectors are constant:

$$r = |\underline{r}| = R, \qquad v = |\underline{v}| = R\omega \qquad \text{and} \qquad a = |\underline{a}| = R\omega^2 = \frac{v^2}{R}. \tag{25}$$

The force required to keep the particle moving in a circle is the *Centripetal Force*. From Newton's second law, we have $\underline{F} = m\underline{a} = -m\omega^2\underline{r}$, and

$$F = |\underline{F}| = mR\omega^2 = \frac{mv^2}{R}. \tag{26}$$

This force could be provided from a variety of sources: gravity, friction, a string...

Chapter 4

Work, energy and power

4.1 Work and Kinetic Energy

A particle moving in one dimension obeys Newton's Second Law (II). Using equation (3), we get:

$$F = ma = m\frac{dv}{dt} = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds} = m\frac{d}{ds}\left(\frac{1}{2}v^2\right).$$

If we integrate $F\,ds$ from time 0 to t, where the particle moves from 0 to s in this time, we get:

$$\int_0^s F\,ds = \int_0^s m\frac{d}{ds}\left(\frac{1}{2}v^2\right)\,ds = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2, \tag{27}$$

where v_0 is the velocity at time 0 and v_1 is the velocity at time t. We define the *Kinetic Energy* (K.E.) of a particle of mass m, travelling at speed v, to be

$$\text{K.E.} = \frac{1}{2}mv^2 \tag{28}$$

and the *work done* by a force acting on a particle to be:

$$\text{Work done} = \int F\,ds \tag{29}$$

so equation (27) can also be written as Work done on a particle = increase in the particle's Kinetic Energy. The units of work and Kinetic Energy are Newton-metres (Nm), or Joules (J). In the case where the force is a constant, we get

$$\text{Constant force:}\quad \text{Work done} = F \times s = \text{Force} \times \text{distance}, \tag{30}$$

where force and distance are measured in the same sense.

4.2 Gravitational Potential Energy

Consider a particle moving under the influence of gravity, so the force acting on it is $F = -mg$ (negative because downwards). If the particle falls from h_1 to h_2 (with

$h_1 > h_2$), the work done by gravity is

$$\text{Work done} = \int_{h_1}^{h_2} -mg\,ds = -mg(h_2 - h_1) = mg(h_1 - h_2).$$

This is equal to the increase in Kinetic Energy: $mg(h_1 - h_2) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$, where v_1 is the initial speed (at height h_1) and v_2 is the final speed (at height h_2). Rearrange this to give the *Principal of conservation of Energy*:

$$mgh_1 + \frac{1}{2}mv_1^2 = mgh_2 + \frac{1}{2}mv_2^2. \tag{31}$$

The sum of either side of this equation is called the *total mechanical energy*, and the first part of the sum is defined to be the Gravitational *Potential Energy*: the Potential Energy (P.E.) of a particle of mass m a distance h above the ground is:

$$\text{P.E.} = mgh. \tag{32}$$

Mechanical energy is lost to friction and in collisions, but otherwise the total mechanical energy (Potential Energy + Kinetic Energy) is conserved.

4.3 Power

Power is the rate at which work is done. If a force F acting on a particle is constant, and it travels with constant speed v, then the distance it travels in time t is $s = vt$, the work done is Fs, and the power is

$$\text{Constant force and speed:} \quad \text{Power} = \frac{\text{Work done}}{\text{Time taken}} = \frac{F \times s}{t} = Fv = \text{Force} \times \text{speed}. \tag{33}$$

The units of power are Watts (W): $1\,\text{W} = 1\,\text{J/s} = 1\,\text{Nm/s} = 1\,\text{kg}\,\text{m}^2/\text{s}^3$.