MATH0370: Introduction to Applied Mathematics 2, 2010–11
Examples covered during lectures

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Throughout, take the gravitational acceleration at the surface of the Earth as $g = 10\,\mathrm{m/s^2}$.

1. Two crows start at point A. One flies 3 km North to B, then 4 km East to C. The second flies directly to C. What is the distance travelled by each crow, and what are their final displacements?

2. Suppose a particle moves along the x-axis with $s(t) = t^3 - 2t\,\mathrm{m}$. Plot $s(t)$ and calculate the average speed from $t = 1\,\mathrm{s}$ to $t = 2\,\mathrm{s}$. Calculate the instantaneous speed at $t = 1\,\mathrm{s}$ and $t = 2\,\mathrm{s}$.

3. The position of a particle (in m) at time t (in s) is

$$\underline{r}(t) = 10 \sin\left(4t - \frac{\pi}{4}\right) \underline{i}$$

Find the velocity and the speed at $t = \pi/12\,\mathrm{s}$, and at $t = \pi/3\,\mathrm{s}$. At what time does the particle first come to rest? Where is the particle at this time?

4. The velocity of a particle is

$$\underline{v}(t) = \cos 2t \ \underline{i} + \sin 2t \ \underline{j}$$

Explain why the particle is never at rest.

5. A driver engages top gear of her car when its speed reaches 16 m/s. In this top gear, she accelerates at a constant rate of $4\,\mathrm{m/s^2}$. How long after engaging top gear does it take to reach a speed of 32 m/s? How far has the car travelled in this time?

6. A ball is propelled vertically upwards with an initial speed of 20m/s, and is subjected to the downwards acceleration due to gravity: $g = 10\text{m/s}^2$. (Actually, $g = 9.81\text{m/s}^2$, but in this course we will always take $g = 10\text{m/s}^2$ unless stated otherwise.) How high does the ball go? How long does it take to reach this maximum height? How long does it take for the ball to return to the ground?

7. A particle moves such that its position is

$$\underline{r}(t) = t\ \underline{i} + \frac{1}{t}\ \underline{j}$$

Find its velocity $\underline{v}$, its acceleration $\underline{a}$ and its speed.

8. A car starts from rest and accelerates at $3\,\mathrm{m/s^2}$ for $10\,\mathrm{s}$. It then decelerates at $1\,\mathrm{m/s^2}$ until it stops. How far has it travelled, and what is its average speed?

9. A particle moves so that its acceleration is $-10\,\underline{j}\,\mathrm{m/s^2}$. At $t = 0$, its velocity is $5\,\underline{i} + 20\,\underline{j}\,\mathrm{m/s}$ and its position is $10\,\underline{j}\,\mathrm{m}$. Find $\underline{r}(t)$. At what time is the y component of $\underline{r}$ equal to zero?

10. A particle moves so that its acceleration is $2e^{-t}\,\underline{i} + 5\cos(t)\,\underline{j} - 3\sin(t)\,\underline{k}$. At $t = 0$, its velocity is $4\,\underline{i} - 3\,\underline{j} + 2\,\underline{k}$ and its position is $\underline{i} - 3\,\underline{j} + 2\,\underline{k}$. Find $\underline{r}(t)$.

11. Suppose the position of P is $\underline{r}_P = (2t+3)\,\underline{i} + (t^2-4)\,\underline{j}$, and the position of Q is $\underline{r}_Q = (t^2-1)\,\underline{i} - (t+1)\,\underline{j}$. Find the relative velocity of P with respect to Q at time $t = 1$. Find the speeds of P and Q at time $t = 1$ and the relative speed of P with respect to Q at time $t = 1$.

12. The passenger in a car travelling due East on a straight road at 40 m/s is watching a train moving due North at 75 m/s. What is the apparent velocity (speed and direction) of the train as observed by the passenger.

13. An aeroplane travels in a Northwesterly direction at 125 m/s relative to the ground. However, there is a strong Westerly (i.e., from the West) wind blowing at 50 m/s, relative to the ground. Determine how fast, and in what direction, the aeroplane would move if there were no wind.

14. A train passes under a bridge travelling at 10 m/s. A passenger starts walking up the train (i.e., towards the front) at the instant he passes under the bridge. He accelerates at $1\,\mathrm{m/s^2}$ for 4 s, then walks at a constant speed for a while, then decelerates at the same rate so that he comes to rest (relative to the train) at the end of the 48 m train carriage.
(a) How long does this take? (b) How far from the bridge is he at this point? (c) At what time does the passenger pass under a second bridge, 104 m from the first, and what is his velocity relative to the ground at this time?

15. An escalator is broken and therefore stationary. A person can walk up the stationary escalator in 90 s. Once the escalator is repaired, the person would be carried to the top (without walking) in 60 s. How long would it take to reach the top if the person walked up the moving escalator?

16. A fly is perched on the edge of an LP (vinyl record) of diameter 30 cm, rotating at $33\frac{1}{3}$ rpm (revolutions per minute). How fast is the fly moving, in m/s?

17. Show that if the position of a particle is

$$\underline{r}(t) = R\cos(\omega t)\,\underline{i} + R\sin(\omega t)\,\underline{j},$$

then the particle is moving in a circle with constant speed. Find the direction and magnitude of its acceleration.

18. A man drags a boat of mass 100 kg across a beach, with force 200 N. The horizontal motion is hindered by a frictional force of 160 N. Find the total force on the boat and its acceleration. Take $g = 10\,\mathrm{m/s^2}$.

19. One person pushing a 1000 kg car can accelerate it at $0.1\,\mathrm{m/s^2}$. Two people pushing equally (with the same force as before) can accelerate it at $0.3\,\mathrm{m/s^2}$. How large is the resistance to the motion?

20. A bullet has mass $m = 10\,\text{g} = 0.01\,\text{kg}$ and a muzzle velocity of $800\,\text{m/s}$. The barrel of the gun is $0.8\,\text{m}$ long. Assuming that the bullet accelerates uniformly, find the force on the bullet.

21. What force must be supplied by the engine of a 600 kg car in order to accelerate it at $3\,\mathrm{m/s^2}$?

22. A block of mass m rests on another block of mass M, which rests on the ground.
    (1) What is the force the top block exerts on the bottom block?
    (2) What is the force the bottom block exerts on the top block?
    (3) What is the force the bottom block exerts on the ground?

23. A particle of mass $m = 2\,\text{kg}$ moves in a force field $\underline{F} = -20\,\underline{j}\,\text{N}$. Given that at $t = 0$ the particle is located at $3\,\underline{i} - \underline{j} + 4\,\underline{k}$ and has velocity $6\,\underline{i} + 15\,\underline{j} - 8\,\underline{k}$, find:

    (1) the velocity

    (2) the position at any time t.

    Is the particle ever at rest?

24. A car of mass 1000 kg is pulling a trailer of mass 200 kg, with an acceleration of $2\,\mathrm{m/s^2}$. Draw a diagram showing all the forces on the car and on the trailer.

25. A stone of mass 0.2 kg falls vertically under gravity and experiences air resistance equal to $0.0008v^2$ N, where v is the speed of the stone.

(1) Find the downwards acceleration of the stone: (a) when $v = 0$; (b) when $v = 20\,\text{m/s}$.

(2) Find the maximum speed of the stone (taking $g = 10\,\text{m/s}^2$). This maximum speed is called the terminal velocity.

26. Two planets are made of an identical material, but planet Zim has twice the radius of planet Zot. If a person weights 50 N on the surface of planet Zim, how much does she weigh on the surface of planet Zot?

27. A stone is thrown horizontally at 5 m/s from a platform 10 m above the ground.
    (1) Describe the motion of the stone.
    (2) Determine where and when it hits the ground.

28. A stone is slung from a catapult with a speed of $30\,\mathrm{m/s}$ at an angle α above the horizontal, where $\tan\alpha = 3/4$. The ground is horizontal. Find:

    (1) the greatest height reached.

    (2) the distance the stone travels before hitting the ground.

    (3) the velocity (speed and direction) when it hits the ground.

    (4) if α is varied, the value of α that would give the maximum range.

29. An inclined plane makes an angle $\beta = 30^\circ$ above the horizontal. A particle is launched from O at the bottom of the incline with speed U_0 and at an angle $\alpha = 60^\circ$ above the horizontal. Show that the range (i.e., the distance the particle lands up the plane) is $2U_0^2/3g$.

30. A particle is launched at an angle of 45° from a cliff of height H, and it falls into the sea a distance D from the base of the cliff. Show that its maximum height above sea level is $H + D^2/4(H + D)$.

31. A student in a lift carries books of mass 5 kg. The lift descends by first accelerating downwards at $3\,\mathrm{m/s^2}$ for 4 s, then travelling at a constant speed of 12 m/s, and finally decelerating at $2\,\mathrm{m/s^2}$ for 6 s. If the student judges the weight of the books by the force she has to supply to hold them still, what is the apparent weight of the books in each of the three phases of travel?

32. A cart of mass 40 kg runs smoothly on horizontal rails. A light inextensible rope is attached to the front of the cart and runs parallel to the rails until it passes over a light, frictionless pulley. The other end of the rope hangs vertically and is attached to a 10 kg load. Find:

    (1) the acceleration of the cart and the tension in the rope.

    (2) the forces on the pulley.

33. A man of mass 75 kg wishes to cross a canyon using a rope strung between the two sides, using a light chair suspended from a pulley. Halfway across he stops, with the rope making an angle θ above the horizontal on each side of the chair.

(1) Find the tension in the rope.

(2) The rope will break if the tension exceeds 2000 N. What is the threshold value of θ at which the rope breaks?

34. A block of mass 10 kg is sliding across a rough surface at an initial speed of 5 m/s. The coefficient of friction is 0.25. How far does the block travel before it stops?

35. A man pushes a sledge of mass 20 kg across the snow with a force of 50 N angled downwards at an angle α (with $\tan\alpha = 3/4$). He gets tired, and switches to pulling the sledge with the same force, now angled upwards at angle α. The coefficient of friction is $\mu = 0.1$. Find the acceleration in both cases.

36. A man pushes a block up a rough plane angled at 30° to the horizontal with a force of 60 N parallel to the plane. The mass of the block is 10 kg and it moves at a constant speed. Find the coefficient of friction.

37. A block of mass 10 kg is on a rough horizontal table (with coefficient of friction $\mu = 0.08$). It is attached to a rope that goes parallel to the table and the over a pulley at the edge. A 2 kg weight is attached to the other end of the rope.

    (1) Find the tension in the rope and the acceleration of the blocks.

    (2) If the upper block starts from rest, 8 m from the pulley, how long does it take for it to hit the pulley?

38. A squash ball of mass 35 g is travelling horizontally with speed 20 m/s when it strikes a vertical wall and rebounds horizontally with speed 16 m/s. Find the impulse.

39. A boy on rollerskates (mass 60 kg) is travelling at 8 m/s when he collides with a pedestrian (mass 80 kg) travelling at 1 m/s in the same direction.

    (1) Assuming the boy carries the pedestrian off with him, what is their speed after the collision?

    (2) Find the impulses acting on the pedestrian and on the boy.

40. Two skateboarders collide in a head-on collision and stick together. One has mass 50 kg and is travelling at speed 8 m/s immediately before the collision. The other has mass 75 kg and is travelling at 10 m/s in the opposite direction immediately before the collision.

    (1) Find their joint speed and direction of motion immediately after the collision.

    (2) If they slide together for 5 m before stopping, find the coefficient of friction.

41. Calculate v_A and v_B if $m_A = 1\,\mathrm{kg}$, $m_B = 5\,\mathrm{kg}$, $u_A = 10\,\mathrm{m/s}$, $u_B = 3\,\mathrm{m/s}$ and $e = 3/4$.

42. A vertically falling ball hits a horizontal surface and bounces back. The ball is dropped from $5\,\mathrm{m}$ above the surface. Take $e = 1/4$ and $g = 10\,\mathrm{m/s^2}$.

    (1) How fast is it moving just before it hits the surface?

    (2) How fast is it moving just after it hits the surface?

    (3) How high does it bounce?

43. A 1 kg particle slides over smooth ice at 5 m/s (assume no friction). It strikes a stationary 2 kg particle and rebounds at 1 m/s. Calculate the final velocity for the second particle and the coefficient of restitution e.

44. A toy train (mass 1 kg) goes around a circular track of diameter 3 m with constant angular velocity $\omega = 0.5\,\text{rad/s}$.

    (1) How long does it take to complete one circuit?

    (2) What force is needed to keep the train moving in a circle?

45. A pilot (mass 60 kg) is flying in a vertical loop of radius 1 km. The speed of the plane is 360 km/h.

    (1) What are the forces on the pilot at the top of the loop?

    (2) What are the forces on the pilot at the bottom of the loop?

46. A car has a total mass of 1000 kg, and is driving on a horizontal road with coefficient of friction $\mu = 0.15$. The road goes around a circular bend with radius 100 m. What is the maximum speed the car can go without skidding?

47. A satellite is in orbit around the Earth.

(1) At what radius is the satellite geostationary (i.e., always stays above the same location on Earth)?

(2) How fast does a geostationary satellite move?

48. A child pushes a frictionless cart ($m = 10\,\text{kg}$) starting from rest, with a constant horizontal force of $20\,\text{N}$.

    (1) How far has the cart travelled in $5\,\text{s}$?

    (2) How much work has been done during that time?

    (3) What is the cart's kinetic energy at the end?

49. A stone is dropped from the top of a 20 m cliff. Use conservation of energy to calculate its speed just before it hits the ground.

50. A stone is thrown upwards with speed v_1. Use conservation of energy to calculate its maximum height and to show that it has the same speed when it returns to the ground.

51. A cart of mass 40 kg runs smoothly on horizontal rails. A light inextensible rope is attached to the front of the cart and runs parallel to the rails until it passes over a light, frictionless pulley. The other end of the rope hangs vertically and is attached to a 10 kg load. The cart starts from rest. The block starts from rest 4 m above the ground. Calculate the speed of the cart at the point when the block just hits the ground.

52. A block of mass m, starting from rest, slides a distance s down a rough plane inclined at an angle α to the horizontal. The coefficient of friction is μ.

(1) Calculate the work done:

(a) by the normal reaction force.

(b) by friction.

(c) by gravity.

(2) Find the speed of the block at the bottom of the plane.

53. A crane lifts a load of 20 000 kg (20 t) to a height of 25 m in 5 s. Calculate the power required.

54. A force of 2 kN pushes a cart up a rough slope (inclination α, with $\tan\alpha = \frac{3}{4}$) at a constant speed of 5 m/s. Find the power developed by the force in the case where:

    (1) the force is parallel to the slope.

    (2) the force is horizontal.