Geometry of Quiver Varieties

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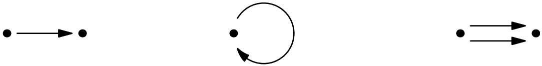
Quiver Varieties

A quiver consists of vertices and arrows connecting them:



Given a field K we obtain a variety together with a group action by choosing dimensions at each vertex:

Quiver Varieties

$$a \longrightarrow b$$

gives

$$\mathrm{Rep}(a,b) := \mathbb{M}_{b\times a}(K), \quad \mathrm{GL}(a,b) := \mathrm{GL}_a(K) \times \mathrm{GL}_b(K)$$

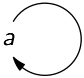
with the action

$$(g,h)\cdot x := hxg^{-1}.$$

The orbits are given by the rank of a matrix, so finitely many.

There are nice representatives for the orbits.

Quiver Varieties



gives

$$\mathrm{Rep}(a) := \mathbb{M}_{a\times a}(K), \quad \mathrm{GL}(a) := \mathrm{GL}_a(K)$$

with the action

$$g \cdot x := gxg^{-1}$$

Representatives for the orbits are matrices in Jordan normal form, so infinitely many orbits.

Quiver Varieties

$$a \rightrightarrows b$$

gives

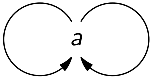
$$\mathrm{Rep}(a,b) := \mathbb{M}_{b\times a}(K)\times\mathbb{M}_{b\times a}(K), \quad \mathrm{GL}(a,b) := \mathrm{GL}_a(K)\times\mathrm{GL}_b(K)$$

with the action

$$(g,h)\cdot(x,y) := (hxg^{-1}, hyg^{-1}).$$

This gives ‘matrix pencils’, as studied by Kronecker. We again have nice representatives for the orbits, of which there are infinitely many.

Quiver Varieties



gives

$$\mathrm{Rep}(a) := \mathbb{M}_{a\times a}(K) \times \mathbb{M}_{a\times a}(K), \quad \mathrm{GL}(a) := \mathrm{GL}_a(K)$$

with the action

$$g \cdot (x, y) := (gxg^{-1}, gyg^{-1}).$$

In this case, there is no natural choice for orbit representatives, and it seems to be a hopeless task to try and describe the orbits in general.

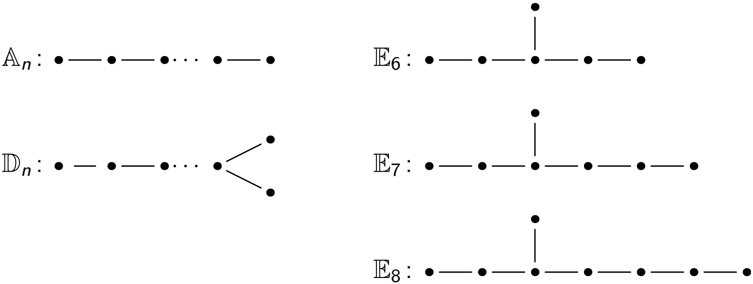
Quiver Varieties

The points of a quiver variety Rep(α) parameterise quiver representations of dimension vector α. Moreover, the orbits correspond to isomorphism classes of representations.

It has proved fruitful to play off geometric and representation-theoretic techniques against each other to obtain interesting results.

Finite Type

Gabriel (1972) showed that for a quiver Q, there are only finitely many orbits in each quiver variety $\text{Rep}(\alpha)$ if and only if Q is of Dynkin type:



Degenerations

Each orbit $\mathcal{O}_x$ is locally-closed and irreducible (in the Zariski topology). Moreover, its closure $\overline{\mathcal{O}_x}$ is a union of orbits.

We define a partial order on orbits (isomorphism classes of representations) via

$$x \leq_{\deg} y \quad \text{if} \quad \mathcal{O}_y \subseteq \overline{\mathcal{O}_x}.$$

How can we understand this in terms of representations?

Degenerations

Riedtmann (1986) and Zwara (2000) proved that $x \leq_{\mathrm{deg}} y$ if and only if there exists a representation z and a short exact sequence

$$0 \to z \to z \oplus x \to y \to 0.$$

However, there are no known bounds on the size of z.

Orbit Closures

In the other direction, Zwara and coauthors have proved several results about orbit closures using representation theory:

- (2001) If Q is of type $\mathbb{A}$ or $\mathbb{D}$, then $\overline{\mathcal{O}_x}$ is a normal variety.
- (2002) If Q is of Dynkin type, then $\overline{\mathcal{O}_x}$ is a unibranch variety.
- (2007) $\overline{\mathcal{O}_x} \subset \mathrm{Rep}(\alpha)$ is smooth if and only if

$$KQ/\mathrm{Ann}(x) \cong KQ' \quad \text{and} \quad \overline{\mathcal{O}_x} = \mathrm{Rep}_{Q'}(\alpha).$$

Quiver Grassmannians

The Grassmannian

$$\mathrm{Gr}\binom{n}{d} := \{\text{subspaces } U \subset K^n \text{ of dimension } d\}$$

is a smooth projective variety.

If we set

$$\mathrm{Ex}(d,e) := \{(f,g) : 0 \to K^d \xrightarrow{f} K^{d+e} \xrightarrow{g} K^e \to 0 \text{ exact}\},$$

then we have a map

$$\mathrm{Ex}(d,e) \to \mathrm{Gr}\binom{d+e}{d}, \quad (f,g) \mapsto \mathrm{Im}(f) = \mathrm{Ker}(g)$$

and this is a principal $\mathrm{GL}_d(K) \times \mathrm{GL}_e(K)$-bundle.

Quiver Grassmannians

We can emulate this construction for quiver varieties.

Let $x \in \mathrm{Rep}(\gamma)$, so we have matrices

$$x_{i \to j} \colon K^{\gamma_i} \to K^{\gamma_j}$$

for each arrow $i \to j$ in the quiver.

We set

$$\mathrm{Gr}_Q \binom{x}{\alpha} := \{(U_i)_i : U_i \in \mathrm{Gr}\binom{\gamma_i}{\alpha_i}, \quad x_{i \to j}(U_i) \subset U_j\}$$

and call this a Quiver Grassmannian.

Quiver Grassmannians

If $\alpha + \beta = \gamma$, set

$$\mathrm{Ex}(\alpha, \beta; x) := \{(f_i, g_i)_i : (f_i, g_i) \in \mathrm{Ex}(\alpha_i, \beta_i), \quad g_j x_{i\to j} f_i = 0\}.$$

In terms of diagrams we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^{\alpha_i} & \xrightarrow{f_i} & K^{\alpha_i+\beta_i} & \xrightarrow{g_i} & K^{\beta_i} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle x_{i\to j}} & & & & \\ 0 & \longrightarrow & K^{\alpha_j} & \xrightarrow{f_j} & K^{\alpha_j+\beta_j} & \xrightarrow{g_j} & K^{\beta_j} & \longrightarrow & 0 \end{array}$$

Then we have a map

$$\mathrm{Ex}(\alpha, \beta; x) \to \mathrm{Gr}_Q \binom{x}{\alpha}, \quad (f_i, g_i)_i \mapsto (\mathrm{Im}(f_i) = \mathrm{Ker}(g_i))_i,$$

and this is a principal $\mathrm{GL}(\alpha) \times \mathrm{GL}(\beta)$-bundle.

Example

Consider the quiver

$$\bullet \longrightarrow \bullet$$

and the representation

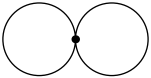
$$x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Rep}(2,2).$$

Then the Quiver Grassmannian $\mathrm{Gr}_Q\binom{x}{(1,1)}$ consists of pairs of one dimensional subspaces

$$U = \mathrm{span}\{\tbinom{a}{b}\} \quad \text{and} \quad V = \mathrm{span}\{\tbinom{c}{d}\}$$

such that $x(U) \subset V$; that is, $\binom{a}{0} \in V$.

Therefore $\mathrm{Gr}_Q\binom{x}{(1,1)}$ is two copies of $\mathbb{P}^1$ meeting at a point



Positivity Conjecture

Quiver Grassmannians occur in cluster algebras, since Caldero and Chapoton (2006) showed that every cluster variable

$$c \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

in the cluster algebra of type Q has coefficients which are (sums of) Euler characteristics of Quiver Grassmannians.

The Positivity Conjecture claims that all these coefficients are non-negative.

Counting Polynomials

Let X be a quasi-projective variety defined over $\mathbb{Z}$.

So there are homogeneous polynomials $f_i, g_j \in \mathbb{Z}[x_0, \ldots, x_n]$ with

$$X(K) = \{a \in \mathbb{P}^n(K) : f_i(a) = 0,\ g_j(a) \neq 0\}.$$

Suppose there exists a polynomial $\xi \in \mathbb{C}(t)$ with

$$|X(\mathbb{F}_q)| = \xi(q) \quad \text{for all finite fields.}$$

Then $\xi \in \mathbb{Z}[t]$ and $\chi(X) = \xi(1)$.

We call ξ a counting polynomial for X.

Examples

$X = \mathbb{A}^n \qquad \xi = t^n$

$X = \mathbb{P}^{n-1} \qquad \xi = [n] := t^{n-1} + \cdots + t + 1$

$X = \mathrm{Gr}\binom{n}{d} \qquad \xi = \begin{bmatrix} n \\ d \end{bmatrix} := \frac{[n]!}{[d]![n-d]!}$

$X = \mathrm{GL}_n \qquad \xi = t^{\binom{n}{2}}(t-1)^n[n]!$

Quiver Grassmannians

Caldero and Reineke (2008) showed that if $\overline{\mathcal{O}_x} = \mathrm{Rep}(\gamma)$, then $\mathrm{Gr}_Q\binom{x}{\alpha}$ is a smooth projective variety with a counting polynomial.

It now follows from the Weil Conjectures that $\chi\big(\mathrm{Gr}_Q\binom{x}{\alpha}\big) > 0$, thus proving the Positivity Conjecture.

Hall Varieties

Recall that $\mathrm{Ex}(\alpha, \beta; x)$ is given by diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^{\alpha_i} & \xrightarrow{f_i} & K^{\alpha_i+\beta_i} & \xrightarrow{g_i} & K^{\beta_i} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle y_{i\to j}} & & \downarrow{\scriptstyle x_{i\to j}} & & \downarrow{\scriptstyle z_{i\to j}} & & \\ 0 & \longrightarrow & K^{\alpha_j} & \xrightarrow{f_j} & K^{\alpha_j+\beta_j} & \xrightarrow{g_j} & K^{\beta_j} & \longrightarrow & 0 \end{array}$$

It follows that there exist unique $y \in \mathrm{Rep}(\alpha)$ and $z \in \mathrm{Rep}(\beta)$ making the diagram commute.

Hall Varieties

It follows that we have maps

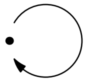
$$\operatorname{Rep}(\alpha)\times\operatorname{Rep}(\beta) \xleftarrow{\ \pi\ } \operatorname{Ex}(\alpha,\beta;x) \xrightarrow{\ p\ } \operatorname{Gr}_Q\binom{x}{\alpha}$$

Thus, given $y \in \operatorname{Rep}(\alpha)$ and $z \in \operatorname{Rep}(\beta)$ we can form the Hall variety

$$\begin{aligned}\operatorname{Gr}_Q\binom{x}{y,z} &:= p\big(\pi^{-1}(\mathcal{O}_y \times \mathcal{O}_z)\big)\\ &= \pi^{-1}\big(\mathcal{O}_y \times \mathcal{O}_z\big)/GL(\alpha)\times \mathrm{GL}(\beta)\end{aligned}$$

Classical Case

Consider the quiver



and take $x \in \mathbb{M}_{n\times n}(K)$ nilpotent.

Then x is determined by a partition λ, giving the sizes of the Jordan blocks.

If y and z are also nilpotent, corresponding to partitions μ and ν, then the Hall variety

$$\mathrm{Gr}_Q\binom{x}{y,z} = \mathrm{Gr}_Q\binom{\lambda}{\mu,\nu}$$

has a counting polynomial $g^{\lambda}_{\mu\nu} \in \mathbb{Z}[t]$.

Example

Take

$$x = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \iff (2,1) \quad \text{and} \quad y = (0) \iff (1)$$

There is a $\mathbb{P}^1$-family of subrepresentations of x isomorphic to y, given by

$$u_{[a,b]} := \mathrm{span}\left\{ \begin{pmatrix} a \\ 0 \\ b \end{pmatrix} \right\}.$$

- $x/u_{[1,0]} \cong \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iff (1,1)$ and $\left| \mathrm{Gr}_Q \binom{(2,1)}{(1),(1,1)}(\mathbb{F}_q) \right| = 1$
- $x/u_{[a,1]} \cong \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \iff (2)$ and $\left| \mathrm{Gr}_Q \binom{(2,1)}{(1),(2)}(\mathbb{F}_q) \right| = q.$

Classical Case

Hall (1959) used the polynomials $g^{\lambda}_{\mu\nu}$ as structure constants for an algebra. He took a $\mathbb{Q}(t)$-vector space with basis u_λ indexed by partitions, and multiplication

$$u_\mu u_\nu := \sum_\lambda g^{\lambda}_{\mu\nu}(t) u_\lambda.$$

This is isomorphic to Macdonald's ring of symmetric functions

$$\Lambda := \mathbb{Q}(t)[e_1, e_2, e_3, \ldots].$$

Moreover, the degree of $g^{\lambda}_{\mu\nu}$ is $n(\lambda) - n(\mu) - n(\nu)$, where

$$n(\lambda_1 \geq \lambda_2 \geq \cdots) = \sum_i (i-1)\lambda_i,$$

and the leading coefficient equals the Littlewood-Richardson coefficient.

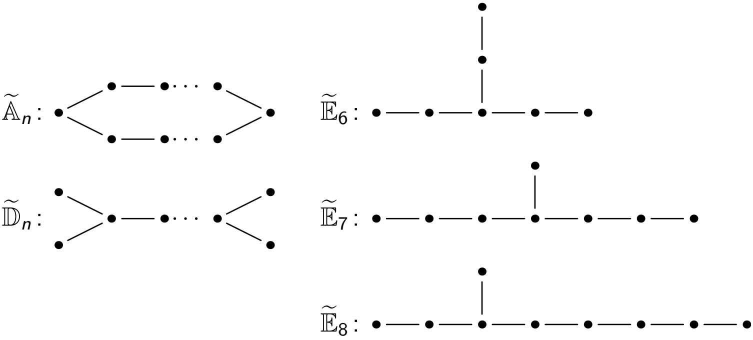
Finite Type

Ringel (1990) proved that if Q is of Dynkin type, then every Hall variety has a counting polynomial.

He used these as structure constants in the Ringel-Hall algebra, and showed that this is isomorphic to the quantum group of the same type as Q.

Furthermore, Guhe (2000) and Nörenberg (2002) proved that these polynomials all lie in $\mathbb{N}[t-2]$, and in type $\mathbb{A}$ even lie in $\mathbb{N}[t-1]$.

Extended Dynkin Diagrams

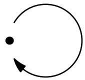


Tame Type

Donovan and Freislich (1973) and Nazarova (1973) classified all orbits of quiver varieties for quivers of extended Dynkin type.

The indecomposable representations come in at most one-parameter families.

For example, for the quiver

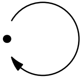


the indecomposables are those having a single Jordan block, and these form a one-parameter family according to their eigenvalues.

Segre Decomposition

By grouping together these one-parameter families , we get a courser stratification of the quiver variety.

For example, for the quiver



we remember only the partitions giving the sizes of the Jordan blocks for each eigenvalue. The strata are therefore indexed by multipartitions, which we call Segre symbols.

Arnold (1971), Gibson (1976), and Bongartz and Dudek (2001) showed that these strata are smooth, rational and GL-invariant. Moreover, they each admit a smooth, rational geometric quotient.

Example

The stratum $S\big((2,1),(1)\big)$ consists of conjugacy classes of matrices

$$x = \left(\begin{array}{ccc|c} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ \hline 0 & 0 & 0 & \mu \end{array}\right) \quad \text{for some } \lambda \neq \mu.$$

The geometric quotient is given by sending a matrix to its invariant factors. For x, the invariant factors are

$$P_1(t) = (t-\lambda)^2(t-\mu) = t^3 - (2\lambda+\mu)t^2 + (\lambda^2 + 2\lambda\mu)t - \lambda^2\mu$$

and

$$P_2(t) = (t-\lambda).$$

The quotient map is then

$$S\big((2,1),(1)\big) \to \mathbb{A}^4, \quad x \mapsto \big(2\lambda+\mu, -(\lambda^2 - 2\lambda\mu), \lambda^2\mu, \lambda\big).$$

Geometry Is Not Enough

Consider the stratum $S\big((1),(1)\big)$. This consists of all 2×2 matrices having distinct eigenvalues.

Over $\mathbb{C}$ we have

$$x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cong \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that $x, y \in S\big((1),(1)\big) \cap \mathbb{M}_2(\mathbb{R})$.

However, as real matrices they have very different properties.

- x has no proper subrepresentations, but y has two such.
- the algebra of matrices commuting with x is isomorphic to $\mathbb{C}$, but for y we get $\mathbb{R}\times\mathbb{R}$.

We therefore need to distinguish between such representations.

Generalised Segre Decomposition

We do this by remembering over which field the representation splits.

For finite fields, it is enough to remember the degree of the field extension.

We therefore consider generalised Segre symbols, which are lists of pairs (d, λ).

Companion Matrices

Given an (irreducible) polynomial

$$f = t^n - (a_{n-1}t^{n-1} + \cdots + a_1 t + a_0),$$

its companion matrix is

$$C(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & a_{n-2} \\ 0 & \cdots & 0 & 1 & a_{n-1} \end{pmatrix}.$$

This represents multiplication by t in $K[t]/(f)$ with respect to the basis $\{1, t, \ldots, t^{n-1}\}$.

Rational Canonical Form

The rational canonical form is like Jordan normal form, but it is now built out of block-matrices having $C(f)$ on the diagonal and the standard matrix E_{1n} on the upper diagonal.

For example, the set $S_K\big((2,(2))\big)$ contains the orbits of matrices of the form

$$\begin{pmatrix} C(f) & E_{12} \\ 0 & C(f) \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & b & 0 & 1 \\ 1 & a & 0 & 0 \\ \hline 0 & 0 & 0 & b \\ 0 & 0 & 1 & a \end{array}\right)$$

where $f = t^2 - (at + b) \in K[t]$ is irreducible.

Hall Polynomials

Theorem (2010).

Let Q be of extended Dynkin type.

For any three generalised Segre symbols ρ, σ, τ there exists a polynomial $g^{\rho}_{\sigma\tau} \in \mathbb{Q}[t]$ such that, for any finite field $\mathbb{F}_q$ and any $x \in S_{\mathbb{F}_q}(\rho)$,

$$\left| \operatorname{Gr}_Q \binom{x}{\sigma, \tau} (\mathbb{F}_q) \right| = g^{\rho}_{\sigma\tau}(q).$$

Hall Polynomials

In particular, for the Quiver Grassmannian $\mathrm{Gr}_Q\binom{x}{\alpha}$, the number of points over $\mathbb{F}_q$ depends only on the generalised Segre symbol ρ such that $x \in S_{\mathbb{F}_q}(\rho)$.

Put another way, let ρ be an ordinary Segre symbol and $x \in S(\rho)$. Then x splits over some finite field $\mathbb{F}_q$, and for $m \geq 1$

$$|\mathrm{Gr}_Q\binom{x}{\alpha}(\mathbb{F}_{q^m})|$$

is given by a counting polynomial.

Example

Consider the stratum $S\big((1),(1)\big)$.

Let $d, e \in \mathbb{F}_q$ be distinct, and let $c \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, so

$$f(t) = (t-c)(t-c^q) = t^2 - at - b \in \mathbb{F}_q[t]$$

is irreducible. Then the following are both in $S\big((1),(1)\big)(\mathbb{F}_q)$:

$$x = \begin{pmatrix} 0 & b \\ 1 & a \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} d & 0 \\ 0 & e \end{pmatrix}.$$

Now, $\mathrm{Gr}_Q \binom{y}{(1),(1)}(\mathbb{F}_{q^m})$ has precisely two points for all m.

On the other hand, $\mathrm{Gr}_Q \binom{x}{(1),(1)}(\mathbb{F}_{q^m})$ is empty if 2 does not divide m, and two points otherwise.

However, if we view these Grassmannians as varieties over $\mathbb{F}_{q^2}$ instead, then we always get two points.

Extensions

Stefan Wolf, in his PhD thesis (2009) extended these results to quiver flag varieties. He gave a criterion for them to be smooth and geometrically irreducible. Using the existence of Hall polynomials, he then proved that certain of these varieties have positive Euler characteristics.

Julia Sauter has shown that for quivers of Dynkin type, Quiver Grassmannians are generically smooth, and found a counter example to this when the quiver is of extended Dynkin type. Using this result, she can also describe the irreducible components.

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