THREE PRESENTATIONS OF THE HOPF ALGEBRA $\mathcal{U}_v(\widehat{\mathfrak{gl}}_n)$

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Abstract. We describe an isomorphism between the Ringel-Hall algebra of the cyclic quiver with n vertices and the positive part of Drinfeld's new presentation of the quantum group $\mathcal{U}_v(\widehat{\mathfrak{gl}}_n)$. This isomorphism extends the isomorphism from the composition algebra to Drinfeld's new presentation of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_n)$ provided by Green [10] and Beck [1]. Moreover, we show that these isomorphisms are unique with respect to compatibility with the "upper left corner" embeddings.

Key results used in the proof are the comultiplication formulae for the generators in Drinfeld's new presentation, as well as a characterisation of the centre of the positive part of the quantum group, proved using Ringel-Hall algebras.

The study of affine Kac-Moody Lie algebras and their associated quantised enveloping algebras is a central part of many areas of mathematics and mathematical physics, and the special cases of $\widehat{\mathfrak{sl}}_n$ and more generally $\widehat{\mathfrak{gl}}_n$ are of particular interest and importance. As a standard example, the quantised enveloping algebra of $\widehat{\mathfrak{gl}}_1$ has as positive part Macdonald's ring of symmetric functions.

There are many different approaches to the theory of affine quantum groups. These include the Drinfeld-Jimbo presentation [6, 13], which can be viewed as a quantisation of the Serre relations for the Chevalley generators; Drinfeld's new presentation [7], which can be viewed as a quantisation of the loop algebra construction; and the universal $\mathcal{R}$-matrix approach by Faddeev, Reshetikhin and Takhtajan [8]. An isomorphism between the Drinfeld-Jimbo presentation and Drinfeld's new presentation was described by Beck [1] for the untwisted affine Lie algebras, whereas Ding and Frenkel [5] exhibited an isomorphism between the latter two presentations for $\widehat{\mathfrak{gl}}_n$. In the case of $\widehat{\mathfrak{gl}}_n$, a survey of all three presentations and isomorphisms between them can be found in the article [9].

It is important to note, however, that until now, no explicit description of either the comultiplication or the antipode for Drinfeld's new presentation existed. Beck calculates the comultiplication of the generators $x^{\pm}_{i,r}$ in terms of the braid group action [1], and most of the comultiplication formulae are present in the preprint by Thoren [27]. The isomorphisms described above, however, are just algebra isomorphisms and the induced Hopf algebra structure for Drinfeld's new presentation has yet to be fully described.

On the other hand, one has the Ringel-Hall algebra approach [19]. Here one considers an orientation of the associated Dynkin diagram and considers the category of finite dimensional representations of this quiver over some finite field. The Ringel-Hall algebra has as basis the isomorphism classes of representations and with multiplication induced by counting extensions of representations. The composition algebra is the subalgebra generated by the simple representations, and Green [10]

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showed how to construct a generic version of the composition algebra, which is then isomorphic to the Drinfeld-Jimbo presentation of the corresponding quantum group. In fact, after adjoining the quantum torus, this becomes an isomorphism of self-dual Hopf algebras. We emphasise that the Ringel-Hall algebra approach has the benefit of having a natural basis. This has played a significant role in several important developments, for example the canonical basis [15].

In the case of $\widehat{\mathfrak{sl}}_n$, if one considers the cyclic orientation and only nilpotent representations, then the isomorphism classes are parametrised by multipartitions and the structure constants for the multiplication are given via polynomials [20]. It follows that there is a generic version of the whole Ringel-Hall algebra, not just the composition algebra, and the generic Ringel-Hall algebra is isomorphic to $\mathcal{U}_v^+(\widehat{\mathfrak{gl}}_n)$.

The existence of an algebra isomorphism was proved by Schiffmann in [22], where he showed that the Ringel-Hall algebra is isomorphic as an algebra to the tensor product of the composition algebra with some central subalgebra. Moreover, an explicit isomorphism was conjectured in [24]. In [12] the author gave an explicit set of generators for the centre of the Ringel-Hall algebra and provided a natural monomorphism of self-dual Hopf algebras from Macdonald's ring of symmetric functions to the centre. This monomorphism can be made an isomorphism by extension of scalars.

In this article, we complete the description of the comultiplication started by Thoren. We then show that there is a unique sequence of isomorphisms of self-dual Hopf algebras from the Ringel-Hall algebra to Drinfeld's new presentation for each n which extends the composition of Green's isomorphism with Beck's monomorphism and which is compatible with the "upper left corner" embeddings $\widehat{\mathfrak{gl}}_n \to \widehat{\mathfrak{gl}}_{n+1}$. In particular, we describe precisely the invers images of the Drinfeld generators $h_{i,r}$, thus proving the conjecture of Schiffmann.

One of the key steps in proving this theorem is the comultiplication formula for the Drinfeld generators $h_{i,r}$. This comultiplication formula appears to be new, and is clearly of independent interest. One should remark that the proof exhibited here makes explicit use of the Ringel-Hall algebra, and in particular the characterisation of the centre given in Appendix A.

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1. Introduction

Let C_n be the cyclic quiver with vertices $1,\dots,n$ and arrows $i \to i-1 \bmod n$. We consider the lattice $\mathbb{Z}^n$ with basis e_i. The Ringel or Euler form $\langle -,-\rangle$ on $\mathbb{Z}^n$ is given via the matrix $R=(r_{ij})$, where $r_{ij} := \delta_{i,j} - \delta_{i,j+1}$. The associated symmetric form $(-,-)$ then corresponds to the Cartan matrix $C=(c_{ij}) = R + R^t$ of the quiver C_n.

We shall work over the function field $\mathbb{Q}(v)$. We recall the quantum numbers and the quantum binomial coefficients

$$[r] := \frac{v^r - v^{-r}}{v - v^{-1}} \quad \text{and} \quad \begin{bmatrix} n \\ r \end{bmatrix} := [n][n-1]\cdots[r+1].$$

Also, given $a \in \mathbb{Q}(v)$ we define the skew-commutator $[x,y]_a := xy - ayx$.

We write $\Lambda := \mathbb{Q}(t)[\mathbf{p}_1, \mathbf{p}_2, \dots]$ for Macdonald's ring of symmetric functions of the infinite set of variables $\underline{X} = \{X_1, X_2, \dots\}$, where $\mathbf{p}_r(\underline{X}) := X_1^r + X_2^r + \cdots$ is the r-th power sum function [17]. We view elements of $\Lambda \otimes \Lambda$ as functions of the variables $\underline{X} \cup \underline{Y}$, symmetric in both the Xs and the Ys: that is, we define $(f \otimes g)(\underline{X}, \underline{Y}) := f(\underline{X})g(\underline{Y})$. This defines a natural comultiplication on Λ by setting $\Delta(f)(\underline{X}, \underline{Y}) := f(X_1, Y_1, X_2, Y_2, \ldots)$. The power sum functions are thus the only primitive elements (up to scalars). There is a counit satisfying $\epsilon(\mathbf{p}_r) = 0$ for all $r \geq 1$ and an antipode given by $S(\mathbf{p}_r) = -\mathbf{p}_r$. The Hall-Littlewood scalar product $(-,-)$ on Λ defined via $(\mathbf{p}_r, \mathbf{p}_s) := \delta_{r,s}\frac{r}{1-t^r}$ is symmetric and satisfies

$$(ab, c) = (a \otimes b, \Delta(c)), \quad (a, 1) = \epsilon(a), \quad (S(a), b) = (a, S(b)).$$

Therefore Λ is a self-dual Hopf algebra.

2. The algebras

2.1. Drinfeld-Jimbo presentation of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_n)$. The Drinfeld-Jimbo presentation of the quantum group can be viewed as a quantisation of the Serre relations, satisfied by the Chevalley generators [6, 13]. For our presentation we shall set $C = 1$ and $D = 0$, where C is the canonical central element. Thus we really have a quantisation of the derived Lie algebra.

The quantum group $\mathcal{U}_v(\widehat{\mathfrak{sl}}_n)$ is the $\mathbb{Q}(v)$-algebra with generators $X_i^{\pm}$ and K_i for $1 \leq i \leq n$ and relations

$$[K_i, K_j] = 0, \qquad K_1 \cdots K_n = 1,$$

$$K_i X_j^{\pm} = v^{\pm c_{ij}} X_j^{\pm} K_i, \qquad [X_i^+, X_j^-] = \delta_{i,j}\frac{K_i - K_i^{-1}}{v - v^{-1}}$$

together with the quantum Serre relations

$$\sum_{r+s=1-c_{ij}} \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix} (-1)^r (X_i^{\pm})^r X_j^{\pm} (X_i^{\pm})^s = 0 \quad \text{for } i \neq j$$

This is a Hopf algebra such that

$$\begin{aligned}
\Delta(K_i) &= K_i \otimes K_i, & S(K_i) &= K_i^-, & \epsilon(K_i) &= 1,\\
\Delta(X_i^+) &= X_i^+ \otimes 1 + K_i \otimes X_i^+, & S(X_i^+) &= -K_i^{-1}X_i^+, & \epsilon(X_i^+) &= 0,\\
\Delta(X_i^-) &= X_i^- \otimes K_i^{-1} + 1 \otimes X_i^-, & S(X_i^-) &= -X_i^- K_i. & \epsilon(X_i^-) &= 0.
\end{aligned}$$

The quantum group is naturally $\mathbb{Z}^n$-graded, where

$$\deg X_i^{\pm} = \pm e_i, \quad \deg K_i = 0.$$

There is also a triangular decomposition

$$\mathcal{U}_v^-(\widehat{\mathfrak{sl}}_n) \otimes \mathcal{U}_v^0(\widehat{\mathfrak{sl}}_n) \otimes \mathcal{U}_v^+(\widehat{\mathfrak{sl}}_n),$$

where $\mathcal{U}_v^{\pm}(\widehat{\mathfrak{sl}}_n)$ is generated by the $X_i^{\pm}$ and $\mathcal{U}_v^0(\widehat{\mathfrak{sl}}_n)$ is generated by the K_i. Moreover, the Borel subalgebra $\mathcal{U}_v^{\geq}(\widehat{\mathfrak{sl}}_n) = \mathcal{U}_v^+(\widehat{\mathfrak{sl}}_n) \otimes \mathcal{U}_v^0(\widehat{\mathfrak{sl}}_n)$ admits the structure of a self-dual Hopf algebra via the symmetric bilinear form $(-,-)$ defined by

$$\{E_i K_r, E_j K_s\} := \delta_{i,j}\frac{v^{c_{rs}}}{1 - v^{-2}} \qquad \text{and} \qquad \{x, yz\} = \{\Delta(x), y \otimes z\}.$$

2.2. **Drinfeld's new presentation of $\mathcal{U}_v(\widehat{\mathfrak{gl}}_n)$.** We can express the Lie algebra $\mathfrak{gl}_{n-1}$ as

$$\mathfrak{gl}_{n-1} = \mathfrak{sl}_{n-1} \oplus \mathbb{C}1.$$

Forming the loop algebras, we obtain

$$\widehat{\mathfrak{sl}}_n = \mathfrak{sl}_{n-1}[t,t^{-1}] \quad \text{and} \quad \widehat{\mathfrak{gl}}_n = \mathfrak{gl}_{n-1}[t,t^{-1}] = \widehat{\mathfrak{sl}}_n \oplus \mathbb{C}1[t,t^{-1}].$$

Note that the loop algebra is isomorphic to the derived affine Lie algebra.

Drinfeld's new presentation of the untwisted affine Lie algebras can be viewed as a quantisation of the loop algebra relations [7]. We describe this presentation for the algebra $\mathcal{U}_v(\widehat{\mathfrak{gl}}_n)$.

The quantum group $\mathcal{U}_v(\widehat{\mathfrak{gl}}_n)$ is generated as a $\mathbb{Q}(v)$-algebra by the elements $t_i^{\pm 1}$ and $g_{i,r}$ for $1 \leq i \leq n$ and $r \in \mathbb{Z} \setminus 0$ together with $x^{\pm}_{i,r}$ for $1 \leq i \leq n-1$ and $r \in \mathbb{Z}$. The relations are

$$\begin{aligned} [t_i, t_j] &= 0, & t_i x^{\pm}_{j,s} &= v^{\pm r_{ij}} x^{\pm}_{j,s} t_i \\ t_i t_i^{-1} = t_i^{-1} t_i &= 1, & [g_{i,r}, x^{\pm}_{j,s}] &= 0 \quad \text{for } i \neq j \text{ or } j+1, \\ [t_i, g_{j,r}] &= 0, & [g_{j,r}, x^{\pm}_{j,s}] &= \pm v^{-jr} \frac{[r]}{r} x^{\pm}_{j,r+s}, \\ [g_{i,r}, g_{j,s}] &= 0, & [g_{j+1,r}, x^{\pm}_{j,s}] &= \mp v^{-jr} \frac{[r]}{r} x^{\pm}_{j,r+s}, \end{aligned}$$

together with

$$\begin{aligned} [x^{+}_{i,r}, x^{-}_{j,s}] &= \delta_{i,j} \frac{\phi^{+}_{i,r+s} - \phi^{-}_{i,r+s}}{v - v^{-1}}, \\ [x^{\pm}_{i,r+1}, x^{\pm}_{j,s}]_{v^{\pm c_{ij}}} &= -[x^{\pm}_{j,s+1}, x^{\pm}_{i,r}]_{v^{\pm c_{ij}}}, \\ [x^{\pm}_{i,r}, [x^{\pm}_{j,p}, x^{\pm}_{i,s}]_v]_v &= -[x^{\pm}_{i,s}, [x^{\pm}_{j,p}, x^{\pm}_{i,r}]_v]_v \quad \text{if } |i-j| = 1, \end{aligned}$$

where the $\phi^{\pm}_{i,r}$ are given via the generating functions

$$\sum_{\pm r \geq 0} \phi^{\pm}_{i,r} T^{-r} := K_i^{\pm 1} \exp\big(\pm (v - v^{-1}) \sum_{\pm r \geq 1} h_{i,r} T^{-r} \big)$$

and we have written

$$K_i := t_i / t_{i+1} \quad \text{and} \quad h_{i,r} := v^{(i-1)r} g_{i,r} - v^{(i+1)r} g_{i+1,r} \quad \text{for } 1 \leq i < n,\ r \in \mathbb{Z} \setminus 0.$$

Note that, in particular,

$$\phi^{\pm}_{i,0} = K_i^{\pm 1} \quad \text{and} \quad \phi^{\pm}_{i,\pm 1} = \pm(v - v^{-1}) K_i^{\pm 1} h_{i,\pm 1}.$$

Also

$$K_i x^{\pm}_{j,s} = v^{\pm c_{ij}} x^{\pm}_{j,s} K_i \quad \text{and} \quad [h_{i,r}, x^{\pm}_{j,s}] = \pm \frac{[r c_{ij}]}{r} x^{\pm}_{j,r+s}.$$

Define

$$\theta_r := v^{nr}(g_{1,r} + g_{2,r} + \cdots + g_{n,r}) \quad \text{for } r \in \mathbb{Z} \setminus 0.$$

Then θ_r is central of degree $r\delta$.

This is again a Hopf algebra. The counit is easy to describe:

$$\epsilon(x^{\pm}_{i,r}) = 0, \quad \epsilon(g_{i,r}) = 0, \quad \epsilon(t_i) = 1.$$

The comultiplication is best described via the generating functions

$$X_i^+(T) := \sum_{r\geq 0} x_{i,r}^+ T^{-r}, \quad Y_i^+(T) := \sum_{r>0} x_{i,r}^- T^{-r}, \quad \Phi_i^\pm(T) := \sum_{\pm r\geq 0} \phi_{i,r}^\pm T^{-r},$$
$$X_i^-(T) := \sum_{r<0} x_{i,r}^+ T^{-r}, \quad Y_i^-(T) := \sum_{r\leq 0} x_{i,r}^- T^{-r}, \quad H_i^\pm(T) := \sum_{\pm r\geq 1} \frac{r}{[2r]} h_{i,r} T^{-r}.$$

For $F(T) := \sum_r f_r T^{-r}$ we define $\Delta(F(T)) := \sum_r \Delta(f_r) T^{-r}$. The comultiplication is now given by the five formulae

(1) $\Delta(X_i^\pm(T)) = X_i^\pm(T) \otimes 1$
$$+\, \Phi_i^\pm(T) \otimes X_i^\pm(T) \sum_{r\geq 0} \Big(-v^{-1}(v-v^{-1})^2 Y_i^\pm(v^{-2}T) \otimes X_i^\pm(T)\Big)^r.$$

(2) $\Delta(Y_i^\pm(T)) = 1 \otimes Y_i^\pm(T)$
$$+\, Y_i^\pm(T) \otimes \Phi_i^\pm(T) \sum_{r\geq 0} \Big(-v^{-1}(v-v^{-1})^2 Y_i^\pm(T) \otimes X_i^\pm(v^2T)\Big)^r.$$

(3) $\Delta(\Phi_i^\pm(T)) = \Phi_i^\pm(T) \otimes \Phi_i^\pm(T)$
$$\times \sum_{r\geq 0} [r+1] \Big(-(1-v^{-2})^2 Y_i^\pm(v^{-2}T) \otimes X_i^\pm(v^2T)\Big)^r.$$

(4) $\Delta(H_i^\pm(T)) = H_i^\pm(T) \otimes 1 + 1 \otimes H_i^\pm(T)$
$$+ \sum_{r\geq 0} \frac{1}{v^r - v^{-r}} \Big(-(v-v^{-1})^2 Y_i^\pm(T) \otimes X_i^\pm(T)\Big)^r.$$

(5) $\Delta(\theta_r) = \theta_r \otimes 1 + 1 \otimes \theta_r$.

The first three formulae follow from [27] by setting $c = 1$. Note however that Thoren uses the opposite comultiplication. The fourth formula seems to be new. For completeness we prove the first four of these formulae in Appendix B, Theorem 20. The last formula is shown in [9] Section 3.3.

There does not yet seem to be a description of the antipode in the literature.

The Hopf subalgebra generated by the $x_{i,r}^\pm$ $(1 \leq i \leq n-1, r \in \mathbb{Z})$ and the K_i $(1 \leq i \leq n)$ is Drinfeld's new presentation of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_n)$. Note that this subalgebra contains the elements $\phi_{i,r}^\pm$ and $h_{i,r}$.

The "upper left corner" algebra embedding $\iota : \mathcal{U}_v(\widehat{\mathfrak{gl}}_n) \to \mathcal{U}_v(\widehat{\mathfrak{gl}}_{n+1})$ sends

$$x_{i,r}^\pm \mapsto x_{i,r}^\pm, \quad t_i \mapsto t_i, \quad g_{i,r} \mapsto g_{i,r}$$

for all i and r. This restricts to an algebra embedding on the level of $\widehat{\mathfrak{sl}}_n$, so that we have a commutative diagram of algebra monomorphisms

(1)
$$\begin{array}{ccc} \mathcal{U}_v(\widehat{\mathfrak{sl}}_n) & \longrightarrow & \mathcal{U}_v(\widehat{\mathfrak{sl}}_{n+1}) \\ \downarrow & & \downarrow \\ \mathcal{U}_v(\widehat{\mathfrak{gl}}_n) & \longrightarrow & \mathcal{U}_v(\widehat{\mathfrak{gl}}_{n+1}) \end{array}$$

such that the vertical maps are also Hopf algebra monomorphisms.

2.3. **The Ringel-Hall algebra.** We now describe the generic Ringel-Hall algebra $\mathcal{H}_v(C_n)$ arising from the category $\operatorname{rep}^0 kC_n$ of nilpotent representations of the cyclic quiver C_n over a finite field k [20, 22, 12]. This category is hereditary and each indecomposable module is uniserial.

We recall that, over any field k, the simple representations S_i are indexed by the vertices $1 \leq i \leq n$. An indecomposable representation is completely determined (up to isomorphism) by its simple socle and its Loewy length. We write $S_i(r)$ for the indecomposable with socle S_i and Loewy length r.[1] It follows that the isomorphism classes of nilpotent representations are indexed by n-multipartitions $\lambda = (\lambda^{(1)}, \dots, \lambda^{(n)})$, with the partition $\lambda^{(i)}$ determining the Loewy lengths of the indecomposable summands with simple socle S_i.

We identify the Grothendieck group of the category with the lattice $\mathbb{Z}^n$ and write $\underline{\dim}\, S(\lambda) \in \mathbb{Z}^n$. In particular, $\underline{\dim}\, S_i = e_i$. The Euler form satisfies

$$\langle S(\lambda), S(\mu)\rangle := \dim \operatorname{Hom}(S(\lambda), S(\mu)) - \dim \operatorname{Ext}^1(S(\lambda), S(\mu))$$

and depends only on the dimension vectors of $S(\lambda)$ and $S(\mu)$. This agrees, therefore, with the form defined in the Introduction, given via the matrix R. We again write $(-,-)$ for the associated symmetric form.

Given an n-multipartition λ and a field k, choose a representative $S(\lambda, k)$ for the corresponding isomorphism class. It is shown in [20] that for all triples of n-multipartitions there exists an integer polynomial $\mathcal{E}^{\nu}_{\lambda\mu}$ such that, for any finite field k, $\mathcal{E}^{\nu}_{\lambda\mu}(|k|)$ equals the number of short exact sequences

$$0 \to S(\mu, k) \to S(\nu, k) \to S(\lambda, k) \to 0.$$

Similarly there exists an integer polynomial a_λ for each n-multipartition λ such that

$$a_\lambda(|k|) = |\operatorname{Aut} S(\lambda, k)|\,.$$

We use these polynomials to form the generic Ringel-Hall algebra $\mathcal{H}_v(C_n)$. We take a $\mathbb{Q}(v)$ vector space with basis u_λ indexed by the n-multipartitions and multiplication

$$u_\lambda u_\mu := v^{\langle S(\lambda), S(\mu)\rangle} \sum_{\nu} F^{\nu}_{\lambda\mu}(v^2) u_\nu, \quad \text{where } F^{\nu}_{\lambda\mu} := \mathcal{E}^{\nu}_{\lambda\mu}/a_\lambda a_\mu$$

N.B. To simplify the exposition, we shall use the notation of modules rather than n-multipartitions. That is, we shall write $[M]$ for the basis elements and use the polynomials $\mathcal{E}^X_{MN}$, a_M and F^X_{MN}. Also, these polynomials will always be evaluated at v^2, which we therefore omit.

The extended Ringel-Hall algebra $\mathcal{H}^{\geq}_v(C_n)$ is given by adjoining the quantum torus $\mathbb{Q}(v)[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ such that

$$t_i[M] := v^{\langle S_i, M\rangle}[M]t_i.$$

We set $K_i := t_i/t_{i+1}$, so that $K_1 \cdots K_n = 1$, and write $K^\alpha := K_1^{\alpha_1} \cdots K_n^{\alpha_n}$ for any dimension vector α. Note that $K_i[M] = v^{(S_i, M)}[M]K_i$.

The extended Ringel-Hall algebra $\mathcal{H}^{\geq}_v(C_n)$ is naturally a Hopf algebra with comultiplication given by Green's formula [10]

$$\Delta([X]) := \sum_{[M],[N]} v^{\langle M,N\rangle} \frac{\mathcal{E}^X_{MN}}{a_X}[M]K^{\underline{\dim}\, N} \otimes N.$$

[1]This notation differs form [12], where the opposite orientation was used and $S_i(r)$ denoted the module with simple top S_i and Loewy length r.

The counit satisfies $\epsilon([M]) := \delta_{[M],[0]}$ and the antipode is given via [29] $S(t_i) := t_i^{-1}$ and

$$S([X]) := K^{-\underline{\dim} X} \sum_{r\geq 1} (-1)^r \sum_{\substack{[M_1],\ldots,[M_r] \\ \text{non-zero}}} v^{\sum_{i<j}\langle M_i,M_j\rangle} \frac{\prod_i a_{M_i}}{a_X} F^X_{M_1\cdots M_r}[M_1]\cdots[M_r]$$

where $F^X_{M_1\cdots M_r}$ is the number of filtrations $0 = X_{r+1} \subset \cdots \subset X_1 = X$ with factors $X_i/X_{i+1} \cong M_i$. In particular, $F^X_X = 1$ and $F^X_{MN} = \mathcal{E}^X_{MN}/a_M a_N$ as above.

In fact, the extended Ringel-Hall algebra is a self-dual Hopf algebra via Green's symmetric bilinear form

$$\{[M]t^\alpha, [N]t^\beta\} := \delta_{[M],[N]} \frac{v^{2\dim M + \sum_i \alpha_i\beta_i}}{a_M}.$$

Note that this form is a weighted version of that defined in [10] and [12], but will agree with Lusztig's bilinear form on $\mathcal{U}_v^{\geq}(\widehat{\mathfrak{sl}}_n)$ as well as the Hall-Littlewood scalar product.

The extended composition algebra $\mathcal{C}_v^{\geq}(C_n)$ is the subalgebra generated by the simple representations together with the K_i.

Set $\mathcal{M}_{i,\alpha}$ to be the set of all isomorphism classes of modules M such that $\underline{\dim}\, M = \alpha$ and $\mathrm{soc}(M) \subseteq S_1 \oplus \cdots \oplus S_i$. Now define

$$c_{i,r} := (-1)^r v^{-2ir} \sum_{\mathcal{M}_{i,r\delta}} (-1)^{\dim \mathrm{End}(M)} a_M[M], \quad c_{i,0} := 1.$$

Let $C_i(T) := 1 + \sum_{r\geq 1} c_{i,r}T^r$ be the generating function for these elements and define $p_{i,r}$ via

$$\sum_{r\geq 1} (1 - v^{-2ir}) p_{i,r} T^{r-1} := \frac{\mathrm{d}}{\mathrm{d}T} \log C_i(T).$$

Theorem 1 ([22, 12])**.** *There is a decomposition*

$$\mathcal{H}_v^{\geq}(C_n) = \mathcal{C}_v^{\geq}(C_n) \otimes \mathcal{Z}_v(C_n) \otimes \mathbb{Q}(v)[t_n^{\pm 1}]$$

where $\mathcal{Z}_v(C_n) := \mathbb{Q}(v)[p_{n,1}, p_{n,2}, \ldots]$ is the centre of the Hall algebra and the elements $p_{n,r}$ are all primitive. Moreover the map

$$\Lambda \to \mathcal{Z}_v(C_n), \qquad t \mapsto v^{-2n} \quad \textit{and} \quad \mathbf{p}_r \mapsto p_{n,r}$$

is a Hopf algebra monomorphism respecting the bilinear forms.

This article [12] concerns the untwisted Ringel-Hall algebra, but as mentioned in the remarks following Theorem 3 [12], the result for the twisted Ringel-Hall algebra follows immediately. In fact, we prove in Appendix A, Theorem 17 a stronger characterisation of the centre by considering homogeneous elements which commute with the simples up to a power of v.

For any field k there is a natural fully faithful functor $\iota : \mathrm{rep}^0\, kC_n \to \mathrm{rep}^0\, kC_{n+1}$ whose image is the full subcategory with objects those nilpotent representations X such that $\mathrm{Hom}(X, S_n) = 0 = \mathrm{Hom}(S_{n+1}, X)$ (see for example [2]). Moreover, by considering dimension vectors and using the Ringel form, we see that $\mathrm{Ext}^1_{C_n}(M,N) \cong \mathrm{Ext}^1_{C_{n+1}}(\iota(M), \iota(N))$ for all $M, N \in \mathrm{rep}^0\, kC_n$.

Using this, we obtain an algebra embedding $\mathcal{H}_v(C_n) \to \mathcal{H}_v(C_{n+1})$ which sends $t_i \mapsto t_i$, $[S_i] \mapsto [S_i]$ for $1 \leq i < n$ and $[S_n] \mapsto [S_n(2)] = v[S_{n+1}][S_n] - [S_n][S_{n+1}]$. In particular, we again obtain a commutative diagram of algebra embeddings

$$(2)\qquad \begin{array}{ccc} \mathcal{C}_v(C_n) & \longrightarrow & \mathcal{C}_v(C_{n+1}) \\ \downarrow & & \downarrow \\ \mathcal{H}_v(C_n) & \longrightarrow & \mathcal{H}_v(C_{n+1}) \end{array}$$

such that the vertical maps are also Hopf algebra monomomorphisms.

We shall also need the following result, proved in [12].

Proposition 2. *The difference $p_{n,r} - p_{1,r}$ lies in the composition algebra $\mathcal{C}_v(C_n)$.*

3. Beck's Isomorphism

In [1], Beck constructed an explicit isomorphism between the Drinfeld-Jimbo presentation and Drinfeld's new presentation of $\mathcal{U}_v(\widehat{\mathfrak{g}})$ for any untwisted affine Kac-Moody Lie algebra $\widehat{\mathfrak{g}}$. We recall this isomorphism for $\widehat{\mathfrak{sl}}_n$.

Consider the following subquiver of the cyclic quiver C_n of type A_{n-1}:

$$1 \leftarrow 2 \leftarrow \cdots \leftarrow n-1.$$

Let W be the corresponding Weyl group, with generators the simple reflections $s_1, \ldots, s_{n-1}$. Let P and Q the weight and root lattices, respectively, and write $\omega_1, \ldots, \omega_{n-1}$ for the simple weights. We form the semi-direct product $\widetilde{W} := P \rtimes W$, with multiplication $(x,w)(x',w') := (x + w(x'), ww')$, and let $\widehat{W} := Q \rtimes W$ be the natural subgroup.[2] We identify W and P with their natural subgroups inside $\widetilde{W}$.

We have a decomposition $\widetilde{W} = \widehat{W} \rtimes \mathbb{Z}/n\mathbb{Z}$, where the cyclic group has generator

$$\tau := (\omega_1, s_1 s_2 \cdots s_{n-1}).$$

The group $\widehat{W}$ is the affine Weyl group of type of type $\widetilde{A}_{n-1}$ and has generators $s_1, \ldots, s_n$, where $s_n := (\omega_1 + \omega_{n-1}, s_1 s_2 \cdots s_{n-1} \cdots s_2 s_1)$. Note that $\tau s_i = s_{i+1}\tau$ (indices modulo n). The length function on $\widetilde{W}$ satisfies $\ell(s_i) = 1$ for all i and $\ell(\tau) = 0$.

The simple weights have the following reduced expressions in terms of the generators s_i and τ:

$$\omega_i = \tau^i (s_{n-i} \cdots s_{n-1}) \cdots (s_2 \cdots s_{i+1})(s_1 \cdots s_i).$$

We shall also need the Braid group associated to $\widetilde{W}$. This has generators T_ω for $\omega \in \widetilde{W}$ and relations $T_\omega T_{\omega'} = T_{\omega\omega'}$ whenever $l(\omega) + l(\omega') = l(\omega\omega')$. The braid group acts on the Drinfeld-Jimbo presentation of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_n)$ via

$$T_i X_i^+ = -X_i^- K_i, \qquad T_i X_j^+ = \sum_{r+s=-c_{ij}} (-1)^r v^{-r} (X_i^+)^{(s)} X_j^+ (X_i^+)^{(r)} \quad \text{for } i \neq j,$$

$$T_i X_i^- = -K_i^{-1} X_i^+, \quad T_i X_j^- = \sum_{r+s=-c_{ij}} (-1)^r v^{r} (X_i^-)^{(r)} X_j^- (X_i^-)^{(s)} \quad \text{for } i \neq j,$$

$$T_i K^\alpha = K^{s_i(\alpha)}, \qquad T_\tau K_i = K_{i+1}, \qquad T_\tau X_i^\pm = X_{i+1}^\pm,$$

where we have used the divided powers $X^{(r)} := X^r/[r]!$.

[2] This is the presentation needed for Lemma 3.1 of [1], but the opposite of that given in the introduction Section 1.2 of *loc. cit.*.

Define

$$\varepsilon_n^+ := (-1)^n [x_{n-1,0}^-, \cdots, [x_{3,0}^-, [x_{2,0}^-, x_{1,1}^-]_{v^{-1}}]_{v^{-1}} \cdots]_{v^{-1}} K_n$$
$$\varepsilon_n^- := (-1)^n K_n^{-1} [\cdots [[x_{1,-1}^+, x_{2,0}^+]_v, x_{3,0}^+]_v, \cdots, x_{n-1,0}^+]_v$$

Note that for $n = 2$, $\varepsilon_2^+ = x_{1,1}^- K_2 = x_{1,1}^- K_1^{-1}$ and $\varepsilon_2^- = K_2^{-1} x_{1,-1}^+ = K_1 x_{1,-1}^+$.

Theorem 3 (Beck)**.** *There is a monomorphism of Hopf algebras* $\mathcal{U}_v(\widehat{\mathfrak{sl}}_n) \to \mathcal{U}_v(\widehat{\mathfrak{gl}}_n)$ *such that*

$$K_i \mapsto K_i, \quad X_i^\pm \mapsto x_{i,0}^\pm \quad \text{for } 1 \le i < n \quad \text{and} \quad X_n^\pm \mapsto \varepsilon_n^\pm.$$

This satisfies

$$(-1)^{ir} v^{nr} T_{\omega_i}^{\mp r}(X_i^\pm) \mapsto x_{i,r}^\pm.$$

Note that we have weighted Beck's original isomorphism by v^n. This we can do since for any non-zero scalar a there is a Hopf algebra automorphism of Drinfeld's new presentation given by

$$x_{i,r}^\pm \mapsto a^r x_{i,r}^\pm, \quad g_{i,r} \mapsto a^r g_{i,r}, \quad t_i \mapsto t_i.$$

Lemma 4. *The upper left corner embedding* $\iota : \mathcal{U}_v(\widehat{\mathfrak{gl}}_n) \to \mathcal{U}_v(\widehat{\mathfrak{gl}}_{n+1})$ *satisfies*

$$\iota(\varepsilon_n^+) = v\varepsilon_{n+1}^+ x_{n,0}^+ - x_{n,0}^+ \varepsilon_{n+1}^+ \quad \text{and} \quad \iota(\varepsilon_n^-) = v^{-1} x_{n,0}^- \varepsilon_{n+1}^- - \varepsilon_{n+1}^- x_{n,0}^-.$$

Proof. Let us write $y = [x_{n-1}^-, \cdots, [x_{2,0}^-, x_{1,1}^-]_{v^{-1}} \cdots]_{v^{-1}}$, considered as an element of either $\mathcal{U}_v(\widehat{\mathfrak{gl}}_n)$ or $\mathcal{U}_v(\widehat{\mathfrak{gl}}_{n+1})$. Then $\varepsilon_n^+ = (-1)^n y K_n$ and so $\iota(\varepsilon_n^+) = (-1)^n y K_n K_{n+1}$, since $K_n = (K_1 \cdots K_{n-1})^{-1}$ and hence $\iota(K_n) = K_n K_{n+1}$.

On the other hand, $\varepsilon_{n+1}^+ = (-1)^{n+1} [x_{n,0}^-, y]_{v^{-1}} K_{n+1}$. Since y and $x_{n,0}^+$ commute and $K_n y = vyK_n$, we see that

$$\begin{aligned} v\varepsilon_{n+1}^+ x_{n,0}^+ - x_{n,0}^+ \varepsilon_{n+1}^+ &= (-1)^{n+1} [[x_{n,0}^-, x_{n,0}^+], y]_{v^{-1}} K_{n+1} \\ &= (-1)^n (v - v^{-1})^{-1} [K_n - K_n^{-1}, y]_{v^{-1}} K_{n+1} \\ &= (-1)^n y K_n K_{n+1} = \iota(\varepsilon_n^+). \end{aligned}$$

The second formula is proved similarly. □

We define the Borel subalgebra $\mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_n)$ to be the subalgebra generated by the $x_{i,r}^+$ for $r \ge 0$, the $x_{i,r}^-$ and the $g_{i,r}$ for $r \ge 1$, ε_n^+ and the $t_i^\pm$. The above lemma shows that the upper left corner map restricts to a map on the Borel subalgebras.

4. Green's Isomorphism and the Main Theorem

The following theorem was proved by Green in [10] for all hereditary algebras, but was already known for the Dynkin quivers and the cyclic quiver by work of Ringel [19, 20]. Dynkin quivers reference.

Theorem 5. *There is a natural Hopf algebra isomorphism*

$$\mathcal{C}_v^{\geq}(C_n) \cong \mathcal{U}_v^{\geq}(\widehat{\mathfrak{sl}}_n),$$

sending $K_i \mapsto K_i$ *and* $[S_i] \mapsto X_i^+$*. Moreover,*

$$\{S_i K^\alpha, S_j K^\beta\} = \delta_{i,j} \frac{v^{(\alpha,\beta)}}{1 - v^{-2}} = \{E_i K^\alpha, E_j K^\beta\}.$$

Thus the algebras are isomorphic as self-dual Hopf algebras.

For the quantum group $\mathcal{U}_v(\widehat{\mathfrak{sl}}_n)$, we can view $\mathrm{ad}X_i^- = [X_i^-, -]$ as a derivation

$$\mathrm{ad}X_i^- : \mathcal{U}_v^+(\widehat{\mathfrak{sl}}_n) \to \mathcal{U}_v^{\geq}(\widehat{\mathfrak{sl}}_n).$$

By definition, $\mathrm{ad}X_i^-(X_i)^+ = \delta_{ij}\frac{K_i^{-1}-K_i}{v-v^{-1}}$.

On the other hand, following Kashiwara [14], Chen and Xiao [3] defined the skew-derivations ${}_i\delta$ and δ_i on the Ringel-Hall algebra, where

$${}_i\delta[X] := \sum_{[N]} v^{\langle S_i,N\rangle}\frac{\mathcal{E}^X_{S_iN}}{a_X}[N] \quad \text{and} \quad \delta_i[X] := \sum_{[M]} v^{\langle M,S_i\rangle}\frac{\mathcal{E}^X_{MS_i}}{a_X}[M].$$

Note that

$$\Delta([X]) = \delta_i[X]\cdot K_i \otimes [S_i] + [S_i]K^{\underline{\dim}\,X-e_i} \otimes {}_i\delta[X] + \text{other terms}.$$

It follows that ${}_i\delta$ and δ_i are skew-derivations in the sense that

$$\begin{aligned}
{}_i\delta([X][Y]) &= {}_i\delta[X]\cdot[Y] + v^{(S_i,X)}[X]\cdot{}_i\delta[Y]\\
\delta_i([X][Y]) &= v^{(S_i,Y)}\delta_i[X]\cdot[Y] + [X]\cdot\delta_i[Y].
\end{aligned}$$

Lemma 6. *The map $d_i : \mathcal{H}_v(C_n) \to \mathcal{H}_v^{\geq}(C_n)$ defined via*

$$d_i([M]) := \frac{K_i^{-1}\cdot{}_i\delta[M] - \delta_i[M]\cdot K_i}{v-v^{-1}}$$

is a derivation, and its restriction to $\mathcal{C}_v(C_n)$ coincides via Green's isomorphism with $\mathrm{ad}X_i^-$.

We can compose Green's isomorphism with Beck's monomorphism to obtain a Hopf algebra monomorphism

$$\eta_n : \mathcal{C}_v^{\geq}(C_n) \to \mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_n)$$

sending $K_i \mapsto K_i$ and $[S_i] \mapsto x_{i,0}^+$ for $1 \leq i < n$ and $[S_n] \mapsto \varepsilon_n^+$.

The main aim of this paper is to extend η_n to a Hopf algebra isomorphism

$$\eta_n : \mathcal{H}_v^{\geq}(C_n) \xrightarrow{\sim} \mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_n)$$

which furthermore respects the natural inclusions ι. We also wish to describe the inverse images of the Drinfeld generators in terms of elements of the Ringel-Hall algebra.

We remark that any extension of η_n must identify $p_{n,r}$ and θ_r up to scalars, since these elements are central and primitive of degree $r\delta$. Since we require that η_n respects the natural inclusions ι, these scalars are uniquely determined.

We define

$$\pi_{i,r} := \frac{[ir]}{r}p_{i,r} \quad \text{for } 1 \leq i \leq n \text{ and } r \geq 1, \quad \pi_{0,r} := 0 \quad \text{for } r \geq 1.$$

These elements can be defined recursively in terms of the $c_{i,r}$ via

$$rc_{i,r} = (v-v^{-1})\sum_{s=1}^{r} sv^{-is}\pi_{i,s}c_{i,r-s}.$$

Recall that $\mathcal{M}_{i,\alpha}$ denotes the set of all isomorphism classes of modules M such that $\underline{\dim}\,M = \alpha$ and $\mathrm{soc}(M) \subseteq S_1 \oplus \cdots \oplus S_i$.

Main Theorem. *For each $n \geq 1$ there is an isomorphism of Hopf algebras $\eta_n : \mathcal{H}_v^{\geq}(C_n) \xrightarrow{\sim} \mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_n)$ extending the monomorphism $\mathcal{C}_v^{\geq}(C_n) \to \mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_n)$. This sends*

$$[S_i] \mapsto x_{i,0}^+ \quad \text{for } 1 \leq i < n, \quad [S_n] \mapsto \varepsilon_n^+, \quad \pi_{n,r} \mapsto -\theta_r, \quad t_i \mapsto t_i.$$

The inverse isomorphism satisfies

$$\begin{aligned} x_{i,0}^+ &\mapsto X_i^+ \\ x_{i,-1}^+ K_i^{-1} &\mapsto v^{1-i} \sum_{\mathcal{M}_{i+1,\delta-e_i}} (1-v^2)^{\dim \operatorname{End}(M)-1}[M] \\ v^{ir} g_{i,r} &\mapsto -\pi_{i,r} + v^r \pi_{i-1,r} \\ h_{i,r} &\mapsto \pi_{i+1,r} - (v^r + v^{-r})\pi_{i,r} + \pi_{i-1,r} \\ \theta_r &\mapsto -\pi_{n,r}. \end{aligned}$$

Moreover, the η_n respect the inclusions ι given in (1) *and* (2).

5. Proof of the Main Theorem

We begin with some general results about the monomorphism $\eta_n : \mathcal{C}_v^{\geq}(C_n) \to \mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_n)$. We then prove the Main Theorem for the case $n = 2$, and finally in the general case. For simplicity we shall identify $\mathcal{C}_v^{\geq}(C_n)$ with its image in $\mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_n)$.

We note that all the structure constants in the Ringel-Hall algebra lie in the ring $\mathbb{Z}[v, v^{-1}]$. In this section, we will use the notation $x \equiv \sum \lambda_{[M]}[M]$ if all other modules in x occur with coefficient divisible by $v - v^{-1}$.

Proposition 7. *For $n \geq 2$ we have*

$$x_{i,1}^- K_i^{-1} = v^{1-i} \sum_{\mathcal{M}_{i+1,\delta-e_i}} (1-v^2)^{\dim \operatorname{End}(M)-1}[M] \equiv v^{1-i}[S_{i+1}(n-1)]$$

$$h_{i,1} = \pi_{i+1,1} - (v+v^{-1})\pi_{i,1} + \pi_{i-1,1} \equiv v^{1-i}\big(v^{-1}[S_{i+1}(n)] - v[S_i(n)]\big).$$

We divide the proof into several lemmas. Recall that $x_{i,1}^- = (-1)^i v^n T_{\omega_i}(X_i^-)$ where

$$\omega_i = \tau^i(s_{n-i} \cdots s_{n-1}) \cdots (s_2 \cdots s_{i+1})(s_1 \cdots s_i).$$

Lemma 8. *For $n \geq 2$ and $1 \leq i < n$ we have*

$$T_{\omega_i}(X_i^-) = -K_i[X_n^+, X_{n-1}^+, \ldots, X_{i+1}^+, X_1^+, X_2^+, \ldots, X_{i-1}^+]_{v^{-1}},$$

where $[a,b,c]_{v_{-1}} = [[a,b]_{v^{-1}}, c]_{v^{-1}}$.

Proof. The result is trivial if $n = 2$ so assume that $n \geq 3$. Then for $|i-j| = 1$ we have $T_i(X_j^+) = [X_i^+, X_j^+]_{v^{-1}}$ and $T_i T_j(X_i^+) = X_j^+$.

By induction, we obtain for $0 \leq r < n-i$

$$\begin{aligned} &(T_{r+1} \cdots T_{r+i}) \cdots (T_1 \cdots T_i)(X_i^-) \\ &\qquad = -(K_1 \cdots K_{r+i})^{-1}[\big(T_{r+1} \cdots T_2(X_1)^+\big), X_{r+2}^+, \ldots, X_{r+i}^+]_{v^{-1}} \\ &\qquad = -(K_1 \cdots K_{r+i})^{-1}[X_{r+1}^+, \ldots, X_1^+, X_{r+2}^+, \ldots, X_{r+i}^+]_{v^{-1}}. \end{aligned}$$

Thus

$$\begin{aligned} T_{\omega_i}(X_i^-) &= -T_\tau^i\Big((K_1 \cdots K_{n-1})^{-1}[X_{n-i}^+, \ldots, X_1^+, X_{n-i+1}^+, \ldots, X_{n-1}^+]_{v^{-1}}\Big) \\ &= -K_i[X_n^+, \ldots, X_{i+2}^+, X_{i+1}^+, X_1^+, X_2^+, \ldots, X_{i-1}^+]_{v^{-1}}, \end{aligned}$$

where we have used that $K_1 \cdots K_n = 1$. □

Lemma 9. *For $n \geq 2$ and $1 \leq i < n$ we have*

$$T_{\omega_i}(X_i^-)K_i^{-1} = (-1)^i v^{1-n-i} \sum_{\mathcal{M}_{i+1,\delta-e_i}} (1-v^2)^{\dim \operatorname{End}(M)-1}[M].$$

Proof. By induction,

$$[X_n^+, \dots, X_{n-r}^+]_{v^{-1}} = v^{-r}[S_{n-r}(r+1)],$$

so that

$$T_{\omega_i}(X_i^-) = -v^{-n+i+1}K_i\big[[S_{i+1}(n-i)],[S_1],\dots,[S_{i-1}]\big]_{v^{-1}}.$$

We may therefore reduce to the quiver C_{i+1}, using the natural inclusions of Ringel-Hall algebras. By applying T_τ, we can then reduce to considering the product $\big[[S_1],[S_2],\dots,[S_i]\big]_{v^{-1}}$ for the linear quiver $1 \leftarrow 2 \leftarrow \cdots \leftarrow i$. We show by induction that

$$\big[[S_1],[S_2],\dots,[S_i]\big]_{v^{-1}} = (-v^{-2})^{i-1}\sum(1-v^2)^{\dim \operatorname{End}(M)-1}[M],$$

where the sum is taken over all isomorphism classes of modules M such that $\underline{\dim}\, M = \alpha_i := e_1 + \cdots + e_i$.

This is clear for $i = 1$. Suppose that it is true for $i-1$. Then

$$\big[[S_1],[S_2],\dots,[S_i]\big]_{v^{-1}} = (-v^{-2})^{i-2} \sum_{\underline{\dim}\, M=\alpha_{i-1}} (1-v^2)^{\dim \operatorname{End}(M)-1}\big[[M],[S_i]\big]_{v^{-1}}.$$

Since S_i is injective and not a composition factor of M we know that $[M][S_i] = [M \oplus S_i]$. On the other hand, let $M = M' \oplus S_j(i-j)$. Then

$$[S_i][M] = v^{-1}[M' \oplus S_j(i-j+1)] + v^{-1}[M \oplus S_i].$$

Thus $\big[[S_1],[S_2],\dots,[S_i]\big]_{v^{-1}}$ equals

$$(-v^{-2})^{i-2} \sum_{\substack{\underline{\dim}\, M=\alpha_{i-1}\\ M=M'\oplus S_j(i-j)}} (1-v^2)^{\dim \operatorname{End}(M)-1} \times \big((1-v^{-2})[M \oplus S_i] - v^{-2}[M' \oplus S_j(i-j+1)]\big)$$

$$= (-v^{-2})^{i-1} \sum_{\underline{\dim}\, M=\alpha_i} (1-v^2)^{\dim \operatorname{End}(M)-1}[M].$$

Therefore

$$T_{\omega_i}(X_i^-) = (-1)^i v^{3-n-i} K_i \sum_{\mathcal{M}_{i+1,\delta-e_i}} (1-v^2)^{\dim \operatorname{End}(M)-1}[M]$$

and $K_i T_{\omega_i}(X_i^-) = v^{-2} T_{\omega_i}(X_i^-) K_i$. □

Lemma 10. *For $n \geq 2$ and $1 \leq i < n$ we have*

$$[X_i^+, T_{\omega_i}(X_i^-)] = (-1)^i v^{-n} K_i(\pi_{i+1,1} - (v+v^{-1})\pi_{i,1} + \pi_{i-1,1}).$$

Proof. For M occurring in $T_{\omega_i}(X_i^-)$ then either $M \cong M' \oplus S_j(i-j) \oplus S_{i+1}(r)$, where $1 \leq j < i$ and $n-i \leq r < n+j-i$, or else $M \cong S_{i+1}(n-1)$. In the first case we have

$$[S_i]K_i[M] = v^{-3}K_i\big([M' \oplus S_j(i-j+1) \oplus S_{i+1}(r)] + [M \oplus S_i]\big)$$
$$K_i[M][S_i] = v^{-1}K_i\big([M' \oplus S_j(i-j) \oplus S_i(r+1)] + [M \oplus S_i]\big),$$

whereas in the second case we have

$$[S_i]K_i[S_{i+1}(n-1)] = v^{-3}K_i\big([S_{i+1}(n)] + [S_{i+1}(n-1)\oplus S_i]\big)$$
$$K_i[S_{i+1}(n-1)][S_i] = v^{-1}K_i\big([S_i(n)] + [S_{i+1}(n-1)\oplus S_i]\big).$$

It follows that

$$[X_i^+, T_{\omega_i}(X_i^-)] = (-1)^i v^{-n-i} K_i \sum_{\substack{[M]\in\mathcal{M}_{i+1,\delta}\\ S_{i+1}\subseteq \mathrm{soc}(M)}} (1-v^2)^{\dim \mathrm{End}(M)-1}[M]$$
$$- (-1)^i v^{2-n-i} K_i \sum_{\substack{[M]\in\mathcal{M}_{i,\delta}\\ S_{i}\subseteq \mathrm{soc}(M)}} (1-v^2)^{\dim \mathrm{End}(M)-1}[M].$$

On the other hand, for any $[M]\in\mathcal{M}_{i,\delta}$ we have that

$$\dim \mathrm{End}(M) = \dim \mathrm{soc}(M), \quad a_M = (v^2-1)^{\dim \mathrm{End}(M)}.$$

Therefore

$$\pi_{i,1} = \frac{v^{i+1}}{v^2-1} c_{i,1} = v^{1-i}\sum_{\mathcal{M}_{i,\delta}} (1-v^2)^{\dim \mathrm{End}(M)-1}[M].$$

We deduce that

$$\pi_{i+1,1} - (v+v^{-1})\pi_{i,1} + \pi_{i-1,1}$$
$$= v^{-i}\sum_{\substack{[M]\in\mathcal{M}_{i+1,\delta}\\ S_{i+1}\subseteq \mathrm{soc}(M)}} (1-v^2)^{\dim \mathrm{End}(M)-1}[M] - v^{2-i}\sum_{\substack{[M]\in\mathcal{M}_{i,\delta}\\ S_{i}\subseteq \mathrm{soc}(M)}} (1-v^2)^{\dim \mathrm{End}(M)-1}[M].$$

□

This completes the proof of Proposition 7.

Proposition 11. *The η_n are compatible with the natural inclusions.*

Proof. We shall denote by ι both of the inclusion maps $\mathcal{H}_v^{\geq}(C_n)\to\mathcal{H}_v^{\geq}(C_{n+1})$ and $\mathcal{U}_v(\widehat{\mathfrak{gl}}_n)\to\mathcal{U}_v(\widehat{\mathfrak{gl}}_{n+1})$.

We know that $\eta_n([S_i]) = x_{i,0}^+$ for $1\le i\le n-1$, so that $\iota\eta_n = \eta_{n+1}\iota$ on the generators $[S_i]$ for $1\le i<n$. Furthermore,

$$\eta_{n+1}\iota([S_n]) = \eta_{n+1}\big(v[S_{n+1}][S_n] - [S_n][S_{n+1}]\big) = v\varepsilon_{n+1}^+ x_{n,0}^+ - x_{n,0}^+\varepsilon_{n+1}^+.$$

Conversely, using Lemma 4, we know that

$$\iota\eta_n([S_n]) = \iota(\varepsilon_n^+) = v\varepsilon_{n+1}^+ x_{n,0}^+ - x_{n,0}^+\varepsilon_{n+1}^+.$$

□

Lemma 12. $\pi_{i,r} \equiv v^{r(1-i)}\frac{[r]}{r}\big([S_1(nr)]+\cdots+[S_i(nr)]\big)$ *and all other terms correspond to decomposable modules.*

Proof. We have the recursive formula

$$\frac{rc_{i,r}}{v-v^{-1}} = \sum_{s=1}^{r} s v^{-is}\pi_{i,s}c_{i,r-s}$$

and

$$\frac{c_{i,s}}{v-v^{-1}} \equiv v^{2s(1-i)-1}\big([S_1(ns)]+\cdots+[S_i(ns)]\big).$$

We also know that

$$[S_j(ns)][S_j(n(r-s))] \equiv [S_j(nr)]$$

and this is the only contribution to $[S_j(nr)]$ in $\pi_{i,s}c_{i,r-s}$. Therefore the coefficient in $rv^{-ir}\pi_{i,r}$ of a decomposable module is divisible by $v-v^{-1}$, and by induction the coefficient of $[S_j(nr)]$ for $1 \le j \le i$ is

$$v^{2r(1-i)-1}\Big(r-(v-v^{-1})\sum_{s=1}^{r-1} v^{-s}[s]\Big) = v^{r-2ir}[r],$$

finishing the proof. □

Lemma 13. *We have the following identities, where all the other terms correspond to decomposable modules.*

(1) $x^+_{i,r} \equiv v^{r(1-i)}[S_i(nr+1)]$.
(2) $\phi^+_{i,r}K_i^{-1}/(v-v^{-1}) \equiv v^{r(1-i)}\big(v^{-1}[S_{i+1}(nr)] - v[S_i(nr)]\big)$.
(3) $h_{i,r} \equiv v^{r(1-i)}\frac{[r]}{r}\big(v^{-r}[S_{i+1}(nr)] - v^r[S_i(nr)]\big)$.

In particular, the coefficients of $[S_j(nr)]$ *in* $h_{i,r}$ *and in* $\pi_{i+1,r}-(v^r+v^{-r})\pi_{i,r}+\pi_{i-1,r}$ *agree.*

Proof. We first note that $x^+_{i,r}$, viewed as an element of $\mathcal{C}^{\geq}_v(C_n)$ has coefficients in the ring $\mathbb{Z}[v,v^{-1}]$ since it can be expressed in terms of the braid group acting on the simple representations.

Let $P_j(r)$ denote the truth of the j-th formula up to level r. Clearly $P_1(0)$ is true.

$P_1(r-1)$ implies $P_2(r)$:

$$\begin{aligned}\phi^+_{i,r}K_i^{-1}/(v-v^{-1}) &= [x^+_{i,r-1}, x^-_{i,1}]K_i^{-1}\\ &\equiv v^{r(1-i)}\big[[S_i(n(r-1)+1)],[S_{i+1}(n-1)]\big]_{v^2}\\ &\equiv v^{r(1-i)}\big(v^{-1}[S_{i+1}(nr)] - v[S_i(nr)]\big).\end{aligned}$$

$P_2(r)$ implies $P_3(r)$:

$$\begin{aligned}rh_{i,r} &= \frac{r\phi^+_{i,r}K_i^{-1}}{v-v^{-1}} - \sum_{s=1}^{r-1} sh_{i,s}\phi^+_{i,r-s}K_i^{-1}\\ &\equiv rv^{r(1-i)}\big(v^{-1}[S_{i+1}(nr)] - v[S_i(nr)]\big)\\ &\qquad - v^{r(1-i)}(v-v^{-1})\sum_{s=1}^{r-1}[s]\big(v^{-s}[S_{i+1}(ns)] - v^s[S_i(ns)]\big)\\ &\qquad\qquad \times\big(v^{-1}[S_{i+1}(n(r-s))] - v[S_i(n(r-s))]\big)\\ &\equiv rv^{r(1-i)}\big(v^{-1}[S_{i+1}(nr)] - v[S_i(nr)]\big)\\ &\qquad - v^{r(1-i)}(v-v^{-1})\sum_{s=1}^{r-1}[s]\big(v^{-s-1}[S_{i+1}(nr)] + v^{s+1}[S_i(nr)]\big)\\ &\equiv v^{r(1-i)}[r]\big(v^{-r}[S_{i+1}(nr)] - v^r[S_i(nr)]\big).\end{aligned}$$

N.B. We have implicitly used that $P_2(j)$ implies $P_3(j)$ for $1 \le j < r$.

$P_3(r)$ implies $P_1(r)$:

$$\frac{[2r]}{[r]}x^+_{i,r} = [h_{i,r}, x^+_{i,0}] \equiv v^{r(1-i)}\big[v^{-r}[S_{i+1}(nr)] - v^r[S_i(nr)], [S_1]\big]$$
$$\equiv v^{r(1-i)}(v^r + v^{-r})[S_i(nr+1)].$$

The lemma now follows by induction. □

5.1. **The case $n = 2$.** In this section we deal with the quiver $C_2 : 1 \rightleftarrows 2$.
We shall simplify the notation by writing

$$x_r := x^+_{1,r}, \quad y_r := x^-_{1,r}, \quad h_r := h_{1,r}, \quad \phi^\pm_r = \phi^\pm_{1,r}, \quad K_1 = K.$$

The results of the previous section give

$$x_0 = X^+_1 = [S_1]$$
$$y_1 = X^+_2 K = [S_2]K$$
$$h_1 = \pi_{2,1} - (v + v^{-1})\pi_{1,1} = -v[S_1(2)] + v^{-1}[S_2(2)] - (v - v^{-1})[S_1 \oplus S_2],$$

using that

$$\pi_{1,1} = vc_{1,1}/(v - v^{-1}) = [S_1(2)]$$
$$\pi_{2,1} = v^2 c_{2,1}/(v - v^{-1}) = v^{-1}\big([S_1(2)] + [S_2(2)]\big) - (v - v^{-1})[S_1 \oplus S_2].$$

In fact, we know from Hall's Theorem that

$$\frac{r}{[r]}\pi_{1,r} = p_{1,r} = \sum_{\lambda \vdash r}(1 - v^2)\cdots(1 - v^{2\ell(\lambda)-2})[S_1(2\lambda)],$$

where 2λ denotes the partition $(2\lambda_1 \geq 2\lambda_2 \geq \cdots)$ [12].

Proposition 14. *$h_r = \pi_{2,r} - (v^r + v^{-r})\pi_{1,r}$ for all $r \geq 1$.*

Proof. We first note that $p_{2,r} - p_{1,r} \in \mathcal{C}_v(C_n)$ implies that $\pi_{2,r} - (v^r + v^{-r})\pi_{1,r} \in \mathcal{C}_v(C_n)$.

Let $\lambda = (\lambda_1 \leq \cdots \leq \lambda_a)$ be an increasing sequence of non-negative integers and let μ and ν be partitions, so decreasing sequences of positive integers. As usual, write $|\lambda| = \sum_j \lambda_j$ and $\ell(\lambda) = a$. Define $x_\lambda = x_{\lambda_1}\cdots x_{\lambda_a}$ and similarly for h_μ and y_ν. Then the products $x_\lambda h_\mu y_\nu$ for triples (λ, μ, ν) form a $\mathcal{U}^0_v(\widehat{\mathfrak{sl}}_2)$-basis for $\mathcal{U}^{\geq}_v(\widehat{\mathfrak{sl}}_2)$. In particular, the triples (λ, μ, ν) such that $|\lambda| + |\mu| + |\nu| = r$ and $\ell(\lambda) = \ell(\nu)$ form a basis for the degree $r\delta$ part.

Using this basis, let us write

$$\pi_{2,r} - (v^r + v^{-r})\pi_{1,r} = \sum m_{(\lambda,\mu,\nu)} x_\lambda h_\mu y_\nu.$$

Consider the action of the comultiplication. We know that $\pi_{2,r}$ is primitive, and that $\pi_{1,r}$ is primitive when viewed as an element of $\mathcal{H}^{\geq}_v(C_1)$. Suppose that $[M]K^{\underline{\dim} N} \otimes [N]$ occus in $\Delta(\pi_{1,r})$, where $[M]$ and $[N]$ are both non-zero. Then there exists some partition $\lambda \vdash r$ and a short extact sequence

$$0 \to N \to S_1(2\lambda) \to M \to 0.$$

It follows that $\mathrm{soc}(N)$ is isotypic of type S_1 and hence that $\underline{\dim}\, N = a\delta + be_1$ with $a, b \geq 0$. Furthermore, $b \neq 0$, otherwise both M and N lie in the image of $\mathcal{H}^{\geq}_v(C_1)$, where we know that $\pi_{1,r}$ is primitive.

On the other hand, consider the coefficient of the basis element $x_\lambda h_\mu \otimes y_\nu$ in $\Delta(\sum m_{(\lambda,\mu,\nu)} x_\lambda h_\mu y_\nu)$. By considering the formulae for the comultiplication, we see that the coefficient is precisely $m_{(\lambda,\mu,\nu)}$ and this must be zero whenever $\ell(\nu) \neq 0$ by the considerations above. Therefore we may write $\pi_{2,r} - (v^r + v^{-r})\pi_{1,r} = \sum m_\mu h_\mu$. Now consider the coefficient of the basis element $h_{\mu'} \otimes h_{\mu''}$ in $\Delta(\sum m_\mu h_\mu)$. The only contribution comes from $\Delta(h_\mu)$ where $\mu = \mu' \cup \mu''$ and again this is zero whenever μ' and μ'' are both non-zero. We deduce that $m_\mu = 0$ whenever $\ell(\mu) \geq 2$.

Hence $\pi_{2,r} - (v^r + v^{-r})\pi_{1,r} = m_r h_r$. By equating coefficients of $[S_i(2r)]$ we see from Lemma 13 that $m_r = 1$. □

Recall that $\theta_r = v^{2r}(g_{1,r} + g_{2,r})$ is central and primitive.

Theorem 15. *We can extend η_2 to an isomorphism of Hopf algebras*

$$\eta_2 : \mathcal{H}_v^{\geq}(C_2) \xrightarrow{\sim} \mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_2).$$

This satisfies

$$\begin{aligned}
x_r &= \sum_{a,\,\lambda\vdash r-a} (1-v^2)\cdots(1-v^{2\ell(\lambda)})[S_1(2a+1)\oplus S_1(2\lambda)], \quad r\geq 0\\
y_{r+1}K^{-1} &= \sum_{a,\,\lambda\vdash r-a} (1-v^2)\cdots(1-v^{2\ell(\lambda)})[S_2(2a+1)\oplus S_1(2\lambda)], \quad r\geq 0\\
h_r &= \pi_{2,r} - (v^r + v^{-r})\pi_{1,r}, \quad r\geq 1\\
v^{ir} g_{i,r} &= -\pi_{i,r} + v^r \pi_{i-1,r}, \quad i = 1,2 \quad r\geq 1\\
\theta_r &= -\pi_{2,r}, \quad r\geq 1.
\end{aligned}$$

Moreover this is compatible via the natural inclusions with the isomorphism $\eta_1 : \mathcal{H}_v^{\geq}(C_1) \xrightarrow{\sim} \mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_1)$ such that $v^r g_{1,r} = -\pi_{1,r}$.

Proof. We have already shown that $h_r \mapsto \pi_{2,r} - (v^r + v^{-r})\pi_{1,r}$. Also, we know that $\pi_{2,r}$ is central, so

$$x_r = \frac{r}{[2r]}[h_r, x_0] = -\frac{r}{[r]}[\pi_{1,r}, x_0] = -\big[p_{1,r}, [S_1]\big],$$

and that $p_{1,r} = \sum_{\lambda\vdash r}(1-v^2)\cdots(1-v^{2\ell(\lambda)-2})[S_1(2\lambda)]$.

Any extension of S_1 by $S_1(2\lambda)$ must be of the form $S_1(2a+1)\oplus S_1(2\mu)$, and since $\lambda = \mu \cup (a)$, such a module occurs only for this value of λ. The corresponding Hall number (structure constant) is clearly 1, so

$$\begin{aligned}
[S_1]p_{1,r} = &\sum_{\lambda\vdash r}(1-v^2)\cdots(1-v^{2\ell(\lambda)-2})[S_1\oplus S_1(2\lambda)]\\
&+ \sum_{a\geq 1,\,\lambda\vdash r-a}(1-v^2)\cdots(1-v^{2\ell(\lambda)})[S_1(2a+1)\oplus S_1(2\lambda)].
\end{aligned}$$

Since $[S_1(2\lambda)][S_1] = v^{2\ell(\lambda)}[S_1\oplus S_1(2\lambda)]$ the result for x_r follows.

We next note that

$$y_{r+1}K^{-1} = -\frac{r}{[2r]}[h_r, y_1K^{-1}] = \big[p_{1,r}, [S_2]\big].$$

A similar analysis of the products involved yields

$$\begin{aligned}
p_{1,r}[S_2] = &\sum_{\lambda\vdash r}(1-v^2)\cdots(1-v^{2\ell(\lambda)-2})[S_2\oplus S_1(2\lambda)]\\
&+ \sum_{a\geq 1,\,\lambda\vdash r-a}(1-v^2)\cdots(1-v^{2\ell(\lambda)})[S_2(2a+1)\oplus S_1(2\lambda)],
\end{aligned}$$

whereas $[S_2]p_{1,r} = \sum_{\lambda\vdash r}(1-v^2)\cdots(1-v^{2\ell(\lambda)-2})v^{2\ell(\lambda)}[S_2\oplus S_1(2\lambda)]$, proving the formula for $y_{r+1}K^{-1}$.

We can extend η_2 to $\mathcal{H}_v^{\geq}(C_n)$ by setting $\theta_r = -a_r\pi_{2,r}$ for some non-zero scalars a_r, and any extension must be of this form since θ_r and $\pi_{2,r}$ are both central and primitive of degree $r\delta$. It follows that

$$v^r g_{1,r} = -\pi_{1,r} + \frac{1-a_r}{v^r+v^{-r}}\pi_{2,r}, \quad v^{2r}g_{2,r} = v^r\pi_{1,r} - \frac{v^r+a_rv^{-r}}{v^r+v^{-r}}\pi_{2,r}.$$

Similarly we have an isomorphism of Hopf algebras $\eta_1 : \mathcal{H}_v^{\geq}(C_1) \xrightarrow{\sim} \mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_1)$ sending $v^rg_{1,r} \to -b_r\pi_{1,r}$ for any $b_r \neq 0$.

Finally we have the diagram of algebra maps

$$\begin{array}{ccc} \mathcal{H}_v^{\geq}(C_1) & \longrightarrow & \mathcal{H}_v^{\geq}(C_2) \\ \downarrow & & \downarrow \\ \mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_1) & \longrightarrow & \mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_2), \end{array}$$

where the maps from left to right are the natural inclusions. This diagram will commutate if and only if $a_r = b_r = 1$ for all $r \geq 1$. □

5.2. **The general case.** We now construct a sequence of Hopf algebra isomorphisms $\mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_n) \to \mathcal{H}_v^{\geq}(C_n)$ extending the η_n such that the natural diagrams of algebra maps

$$\begin{array}{ccc} \mathcal{H}_v^{\geq}(C_{n-1}) & \longrightarrow & \mathcal{H}_v^{\geq}(C_n) \\ \downarrow & & \downarrow \\ \mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_{n-1}) & \longrightarrow & \mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_n) \end{array}$$

commute.

Theorem 16. *For each $n \geq 1$ we have an isomorphism*

$$\eta_n : \mathcal{H}_v^{\geq}(C_n) \xrightarrow{\sim} \mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_n)$$

extending the natural monomorphism from $C_v^{\geq}(C_n)$ such that

$$v^{ir}g_{i,r} = -\pi_{i,r} + v^r\pi_{i-1,r} \quad \textit{for all } 1 \leq i \leq n \textit{ and all } r \geq 1.$$

In particular,

$$h_{i,r} = \pi_{i+1,r} - (v^r+v^{-r})\pi_{i,r} + \pi_{i-1,r} \quad \textit{and} \quad \theta_r = -\pi_{n,r}.$$

Furthermore, these η_n commute with the natural inclusions of algebras.

Proof. Let $n \geq 3$. Since $\eta_n : \mathcal{C}_v^{\geq}(C_n) \to \mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_n)$ is compatible with the inclusions, we may assume by induction that

$$h_{i,r} = \pi_{i+1,r} - (v^r+v^{-r})\pi_{i,r} + \pi_{i-1,r} \quad \text{for } 1 \leq i \leq n-2 \text{ and } r \geq 1.$$

For non-zero scalars a_r define $\theta_r = -a_{n,r}\pi_{n,r}$. Since θ_r and $\pi_{n,r}$ are both central and primitive, this induces a Hopf algebra isomorphism $\mathcal{H}_v^{\geq}(C_n) \to \mathcal{U}_v^{\geq}(\widehat{\mathfrak{gl}}_n)$ and any Hopf algebra isomorphism must be of this form.

Using the formulae

$$h_{i,r} = -v^{(i+1)r}g_{i+1,r} + v^{(i-1)r}g_{i,r},$$

we see that

$$\sum_{i=1}^{n-1}\frac{[(n-i)r]}{[r]}h_{i,r}=-\theta_r+\frac{[nr]}{[r]}v^r g_{1,r}.$$

It follows that

$$h_{n-1,r}=\big(a_{n,r}\pi_{n,r}-(v^r+v^{-r})\pi_{n-1,r}+\pi_{n-2,r}\big)+\frac{[nr]}{[r]}\big(v^r g_{1,r}+\pi_{1,r}\big).$$

We claim that $g_{1,r}+\pi_{1,r}$ is central.

Clearly $[\pi_{1,r},x^+_{j,0}]=\big[\pi_{1,r},[S_j]\big]=0$ for $2\leq j\leq n-1$. Also, since $(v^r+v^{-r})\pi_{1,r}=\pi_{2,r}-h_{1,r}$ and $\big[\pi_{2,r},[S_1]\big]=0$, we have that $\big[\pi_{1,r},[S_1]\big]=-\frac{[r]}{r}x^+_{1,r}$. Finally, by analogy with the $n=2$ case,

$$\big[\pi_{1,r},[S_n]\big]=\sum_{a,\,\lambda\vdash r-1}(1-v^2)\cdots(1-v^{2\ell(\lambda)})[S_1(n\lambda)\oplus S_n(na+1)].$$

On the other hand, $[v^r g_{1,r},x^+_{j,0}]=0$ for $2\leq j\leq n-1$ and $[v^r g_{1,r},x^+_{1,0}]=\frac{[r]}{r}x^+_{1,r}$. Note that we can write

$$\varepsilon^+_n=(-1)^n[x^-_{n-1,0},\cdots,[x^-_{3,0},[x^-_{2,0},x^-_{1,1}K_1^{-1}]K_2^{-1}]\cdots]K^-_{n-1}.$$

Then

$$\begin{aligned}[v^r g_{1,r},\varepsilon^+_n]&=(-1)^n[x^-_{n-1,0},\cdots,[x^-_{3,0},[x^-_{2,0},[v^r g_{1,r},x^-_{1,1}K_1^{-1}]]K_2^{-1}]\cdots]K^{-1}_{n-1}\\&=\frac{[r]}{r}(-1)^{n-1}[x^-_{n-1,0},\cdots,[x^-_{3,0},[x^-_{2,0},x^-_{1,r+1}K_1^{-1}]K_2^{-1}]\cdots]K^{-1}_{n-1}.\end{aligned}$$

Consider this as an element of $\mathcal{H}_v^{\geq}(C_n)$. We know that

$$x^-_{1,r+1}K_1^{-1}=\sum_{a,\,\lambda\vdash r-a}(1-v^2)\cdots(1-v^{2\ell(\lambda)})[S_1(n\lambda)\oplus S_2(na+n-1)].$$

Applying the derivation $d_2=\mathrm{ad}X_2^-$ introduced in Lemma 6 yields

$$\begin{aligned}[x_{2,0},x^-_{1,r+1}K_1^{-1}]K_2^{-1}&=d_2(x^-_{1,r+1}K_1^{-1})K_2^{-1}=-\delta_2(x^-_{1,r+1}K_1^{-1})\\&=-\sum_{a,\,\lambda\vdash r-a}(1-v^2)\cdots(1-v^{2\ell(\lambda)})[S_1(n\lambda)\oplus S_3(na+n-2)].\end{aligned}$$

By induction we obtain

$$\big[v^r g_{1,r},[S_n]\big]=-\sum_{a,\,\lambda\vdash r-a}(1-v^2)\cdots(1-v^{2\ell(\lambda)})[S_1(n\lambda)\oplus S_n(na+1)].$$

Hence we have shown that $\pi_{1,r}+v^r g_{1,r}$ commutes with each $[S_i]$. Theorem 17 now applies, giving that $\pi_{1,r}+v^r g_{1,r}$ is central.

We can now write $\frac{[nr]}{[r]}\big(\pi_{1,r}+v^r g_{1,r}\big)=\sum_{\lambda\vdash r}b_{n,\lambda}\pi_{n,\lambda}$, and hence

$$h_{n-1,r}=a_{n,r}\pi_{n,r}-(v^r+v^{-r})\pi_{n-1,r}+\pi_{n-2,r}+\sum_{\lambda\vdash r}b_{n,\lambda}\pi_{n,\lambda}.$$

Here we have written $\pi_{n,\lambda}$ for the product $\pi_{n,\lambda_1}\pi_{n,\lambda_2}\cdots$.

Consider the comultiplication Δ. Since each $\pi_{i,s}$ is primitive as an element of $\mathcal{H}_v^{\geq}(C_i)$ we know that $\Delta(\pi_{i,s})$ contains no term of the form $[M]\otimes[N]$ where $\underline{\dim}\,M$ and $\underline{\dim}\,N$ are both positive multiples of δ. The same conclusion holds for

$\Delta(h_{n-1,r})$ by the comultiplication formulae. Since $\pi_{n,\lambda'} \otimes \pi_{n,\lambda''}$ occurs in $\Delta(\pi_{n,\lambda})$ for $\lambda = \lambda' \cup \lambda''$ we see that $b_{n,\lambda} = 0$ whenever $\ell(\lambda) \geq 2$. Therefore

$$h_{n-1,r} = (a_{n,r} + b_{n,r})\pi_{n,r} - (v^r + v^{-r})\pi_{n-1,r} + \pi_{n-2,r}.$$

Equating coefficients of $[S_i(nr)]$ using Lemma 13 reveals $a_{n,r} + b_{n,r} = 1$.

Hence the extension of η_n to $\mathcal{H}_v^{\geq}(C_n)$ given via $\theta_r \mapsto -a_{n,r}\pi_{n,r}$ satisfies $v^r g_{1,r} = -\pi_{1,r} + \frac{[r]}{[nr]}(1 - a_{n,r})\pi_{n,r}$. These isomorphisms will commute with the natural inclusions if and only if $a_{n,r} = 1$ for all n and r. □

Appendix A. The centre of $\mathcal{U}_v^+(\widehat{\mathfrak{gl}}_n)$

The following is a generalisation of Proposition 8 of [12].

Theorem 17. *Let $x \in \mathcal{H}_v(C_n)$ be homogeneous such that $[S_i]x = v^{r_i}x[S_i]$ with $\sum r_i = 0$. Then $x \in \mathcal{Z}_v(C_n)$ and each $r_i = 0$. In particular, the centre of $\mathcal{H}_v(C_n)$ is $\mathcal{Z}_v(C_n)$ and the centre of $\mathcal{C}_v(C_n)$ is trivial.*

Recall that we can associate a partition $\mu(M)$ to any module M by considering the Loewy lengths of the indecomposable summands of M. We endow the set of partitions with the reverse lexicographic ordering.

The Auslander-Reiten translate τ is an autoequivalence of the category $\operatorname{rep}^0 kC_n$ such that ${}^{\tau}S_i = S_{i-1}$. Moreover this satisfies the Serre duality $D\operatorname{Ext}^1(N, M) \cong \operatorname{Hom}(M, {}^{\tau}N)$.

Lemma 18. *For a module M the following are equivalent.*

(1) *There exists a linear functional $f_M : \mathbb{Z}^n \to \mathbb{Z}$ such that for all N*

$$f_M(\underline{\dim}\, N) = \dim \operatorname{Hom}(M, N) - \dim \operatorname{Hom}(M, {}^{\tau}N) + \dim \operatorname{Hom}({}^{\tau^{-1}}N, M) - \dim \operatorname{Hom}(N, M);$$

(2) *There exists a partition λ such that*

$$M \cong Z_\lambda := \bigoplus_{i=1}^{n} \bigoplus_{r \geq 1} S_i(\lambda_r).$$

In this case, the linear functional f is 0.

Proof. Suppose that (2) holds, that is, $M \cong Z_\lambda$ for some partition λ. Then $M \cong {}^{\tau}M$ and (1) is satisfied for $f = 0$.

Conversely suppose that (1) holds. Let $m_i(r)$ be the multiplicity of $S_i(r)$ in M and write $d_i(r) := m_i(r) - m_{i+1}(r)$.

Considering the sequence

$$0 \to S_i \to S_i(r+1) \to S_{i+1}(r) \to 0,$$

we obtain as in [12] that

$$\dim \operatorname{Hom}(M, S_i(r+1)) = \dim \operatorname{Hom}(M, S_i \oplus S_{i+1}(r)) - \big(m_{i+1}(1) + m_{i+1}(2) + \cdots + m_{i+1}(r)\big).$$

Similarly

$$\dim \operatorname{Hom}(S_i(r+1), M) = \dim \operatorname{Hom}(S_i \oplus S_{i+1}(r), M) - \big(m_i(1) + m_i(2) + \cdots + m_i(r)\big).$$

Therefore

$$f_M(\underline{\dim}\, S_i(r+1)) = f_M(\underline{\dim}\, S_i \oplus S_{i+1}(r)) + 2\big(d_i(1) + d_i(2) + \cdots + d_i(r)\big).$$

Thus $d_i(1) + \cdots + d_i(r) = 0$ for all $1 \leq i \leq n$ and all $r \geq 1$. That is, $d_i(r) = 0$ for all i and r, and hence $M \cong Z_\lambda$, where $\lambda = (1^{m(1)} 2^{m(2)} \cdots)$ in exponential form and $m(r) = m_i(r)$ for any i. □

Proof of Theorem 17. Let x be homogeneous such that $[S_i]x = v^{r_i} x[S_i]$ with $\sum r_i = 0$. Define $f : \mathbb{Z}^n \to \mathbb{Z}$ via $e_i \mapsto r_i$ and note that $f(\delta) = 0$. It follows that for any homogeneous $y \in \mathcal{C}_v(C_n)$,

$$yx = v^{f(\underline{\dim}\, y)} xy.$$

Moreover, we know that $\mathcal{Z}_v(C_n)$ is central and graded, existing only in degrees the multiples of δ. Therefore, for any module N, we have

$$[N]x = v^{f(\underline{\dim}\, N)} x[N].$$

Write

$$x = \sum_{\mu(M)=\pi} a_{[M]}[M] + \sum_{\mu(M)<\pi} a_{[M]}[M].$$

Fix a module N and set $\sigma = \pi \cup \mu(N)$. Using Lemma 5 of [12] we obtain

$$[N]x = v^{\langle \underline{\dim}\, N, \underline{\dim}\, x\rangle} \sum_{\mu(M)=\pi} a_{[M]} F^{M\oplus N}_{NM} [M \oplus N] + \sum_{\mu(X)<\sigma} a'_{[X]}[X].$$

Similarly

$$x[N] = v^{\langle \underline{\dim}\, x, \underline{\dim}\, N\rangle} \sum_{\mu(M)=\pi} a_{[M]} F^{M\oplus N}_{MN} [M \oplus N] + \sum_{\mu(X)<\sigma} a''_{[X]}[X].$$

Using Riedtmann's formula [18] (see also [21] for the finite field case), we know that

$$F^{M\oplus N}_{MN} = \frac{|\mathrm{Aut}(M \oplus N)|}{|\mathrm{Aut}(M)|\,|\mathrm{Aut}(N)|} v^{-2\dim \mathrm{Hom}(M,N)}.$$

Equating coefficients of $[M \oplus N]$ we see that for all M occuring in x with $\mu(M) = \pi$,

$$\begin{aligned}\langle \underline{\dim}\, N, \underline{\dim}\, x\rangle &- 2\dim \mathrm{Hom}(N, M) \\ &= f(\underline{\dim}\, N) + \langle \underline{\dim}\, x, \underline{\dim}\, N\rangle - 2\dim \mathrm{Hom}(M, N).\end{aligned}$$

Since x is homogeneous, $\underline{\dim}\, M = \underline{\dim}\, x$. Therefore, for all modules N,

$$\begin{aligned}f(\underline{\dim}\, N) = \dim \mathrm{Hom}(M, N) &- \dim \mathrm{Ext}^1(N, M) \\ &+ \dim \mathrm{Ext}^1(M, N) - \dim \mathrm{Hom}(N, M).\end{aligned}$$

It follows from Serre duality and the previous lemma that $M \cong Z_\lambda$ and $f \equiv 0$, so x is central. Also, since $\pi = \mu(M) = \lambda^{\cup n}$, the n-fold cup product of λ with itself, there exists a unique maximal term occurring in x. Subtracting a suitable multiple of the central element $c_\lambda = c_{\lambda_1} c_{\lambda_2} \cdots$ yields another central element whose terms all satisfy $\mu(M) < \pi$. The result now follows by induction on π. □

Appendix B. The comultiplication

Let $\hat{\mathfrak{g}}$ be any untwisted affine Kac-Moody Lie algebra and consider Drinfeld's new presentation of the quantised enveloping algebra $\mathcal{U}_v(\hat{\mathfrak{g}})$.

We know that for any i, the subalgebra generated by $K_i, h_{i,r}$ for $r \in \mathbb{Z} \setminus 0$ and $x^{\pm}_{i,r}$ for $r \in \mathbb{Z}$ is isomorphic to $\mathcal{U}_{v_i}(\widehat{\mathfrak{sl}}_2)$ (see [1]). Therefore it is enough to prove the comultiplication formulae for $\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$. We simplify the notation by ommitting the index i and writing

$$x_r := x^+_{i,r}, \quad y_r := x^-_{i,r}.$$

We introduce two $\mathbb{Q}$-algebra involutions σ and ρ of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_2)$, defined via

$$\begin{aligned} &\sigma : v \mapsto v^{-1}, \quad x_r \mapsto y_{r+1}, \quad \phi^{\pm}_r \mapsto \phi^{\pm}_r, \quad h_r \mapsto -h_r \\ &\rho : v \mapsto v, \quad x_r \mapsto y_{-r}, \quad \phi^{\pm}_r \mapsto \phi^{\mp}_{-r}, \quad h_r \mapsto h_{-r}. \end{aligned}$$

These satisfy

$$\tau\Delta\sigma = (\sigma\otimes\sigma)\Delta \quad \text{and} \quad \tau\Delta\rho = (\rho\otimes\rho)\Delta,$$

where τ is the flip[3] $\tau(x\otimes y) = y \otimes x$.

We set

$$\begin{aligned} X^+(T) &:= \sum_{r\geq 0} x_r T^{-r}, \quad Y^+(T) := \sum_{r>0} y_r T^{-r}, \quad \Phi^{\pm} := \sum_{\pm r\geq 0} \phi^{\pm}_r T^{-r}, \\ X^-(T) &:= \sum_{r<0} x_r T^{-r}, \quad Y^-(T) := \sum_{r\leq 0} y_r T^{-r}, \quad H^{\pm}(T) := \sum_{\pm r\geq 1} \frac{r}{[2r]} h_r T^{-r}. \end{aligned}$$

We extend the comultiplication to the generating functions via

$$\Delta(\sum_r b_r T^{-r}) := \sum_r \Delta(b_r) T^{-r}.$$

We also set $\sigma(T) := T$ and $\rho(T) := T^{-1}$, so that for example

$$\sigma(X^+(T)) = Y^+(T)T \quad \text{and} \quad \rho(X^+(T)) = Y^-(T).$$

Lemma 19. *We have the following identities.*

(i) $X^+(T)\Phi^+(T) + \Phi^+(T)X^+(T) = (1+v^{-2})\Phi^+(T)X^+(v^2T)$.
(ii) $X^+(v^{-2}T)\Phi^+(T) = v^{-2}\Phi^+(T)X^+(v^2T)$.
(iii) $(v^{2r}-1)X^+(T)^{r+1} = v^{2r}X^+(T)^r x_0 - x_0 X^+(T)^r$.
(iv) $v[r+1]X^+(T)X^+(v^2T)^r = v^{r+1}X^+(T)^{r+1} + [r]X^+(v^2T)^{r+1}$.
(v) $(v-v^{-1})[X^+(T)^{r+1}, y_1]T^{-1} = v^{-r}[r+1]\big(\Phi^+(T)X^+(v^2T)^r - KX^+(T)^r\big)$.

Proof. For the first two formulae we consider the recurrence relation

$$r\phi^+_r = (v-v^{-1})\sum_{s=1}^{r} s h_s \phi^+_{r-s}.$$

By induction we have that for all m and all $r \geq 0$,

$$x_m\phi^+_r - v^{-2}\phi^+_r x_m = -(v^2-v^{-2})\sum_{s=1}^{r} v^{-2s}\phi^+_{r-s}x_{m+s}.$$

[3] We shall not need the Auslander-Reiten translate in this section, so the overlapping notation should not cause any confusion

Multiplying by T^{-r} and summing over $r \geq 0$ we see that

$$x_m\Phi^+(T) - v^{-2}\Phi^+(T)x_m = -(v^2 - v^{-2})\Phi^+(T)\sum_{s\geq 1} v^{-2s}x_{m+s}T^{-s}. \tag{3}$$

If we now multiply by T^{-m} and sum over $m \geq 0$ we obtain

$$X^+(T)\Phi^+(T) - v^{-2}\Phi^+(T)X^+(T) = (1+v^{-2})\Phi^+(T)\big(X^+(v^2T) - X^+(T)\big),$$

which proves (i). Alternatively, we can multiply by $v^{2m}T^{-m}$ and sum over $m \geq 0$ to get

$$X^+(v^{-2}T)\Phi^+(T) - v^{-2}\Phi^+(T)X^+(v^{-2}T) = v^{-2}\Phi^+(T)\big(X^+(v^2T) - X^+(v^{-2}T)\big),$$

proving (ii).

For (iii) and (iv) we use the relation

$$x_{r+1}x_s - v^2x_sx_{r+1} = -x_{s+1}x_r + v^2x_rx_{s+1}.$$

Multiplying by T^{-s-1} and summing over $s \geq 0$ yields

$$(x_{r+1}T^{-1} - v^2x_r)X^+(T) + X^+(T)(x_r - v^2x_{r+1}T^{-1}) = x_0x_r - v^2x_rx_0.$$

If we multiply by T^{-r} and sum over $r \geq 0$, we have

$$(1-v^2)X^+(T)^2 = x_0X^+(T) - v^2X^+(T)x_0,$$

which proves (iii) for $r=1$. Alternatively, we can multiply by $v^{-2r}T^{-r}$ and sum over $r \geq 0$, giving

$$\begin{aligned}(1-v^4)&X^+(T)X^+(v^2T)\\ &= v^2\big(x_0X^+(T) - v^2X^+(T)x_0\big) + \big(x_0X^+(v^2T) - v^{-2}X^+(v^2T)x_0\big).\end{aligned}$$

Applying (iii) for $r=1$ to the right hand side yields

$$(1-v^4)X^+(T)X^+(v^2T) = v^2(1-v^2)X^+(T)^2 + (1-v^2)X^+(v^2T)^2,$$

proving (iv) when $r=1$.

Assume (iii) for r and post-multiply by $X^+(T)$. Then

$$\begin{aligned}(v^{2r}-1)X^+(T)^{r+2} + x_0X^+(T)^{r+1} &= v^{2r}X^+(T)^rx_0X^+(T)\\ &= v^{2(r+1)}X^+(T)^{r+1}x_0 - v^{2r}(v^2-1)X^+(T)^{r+2},\end{aligned}$$

using the formula for $r=1$. This proves (iii).

Similarly, assume that (iv) is true for r and pre-multiply by $vX^+(T)$. Then

$$\begin{aligned}v^{r+2}X^+(T)^{r+2} + v[r]X^+(T)X^+(v^2T)^{r+1} &= v^2[r+1]X^+(T)^2X^+(v^2T)^r\\ &= v[2][r+1]X^+(T)X^+(v^2T)^{r+1} - [r+1]X^+(v^2T)^{r+2},\end{aligned}$$

using the formula for $r=1$. This proves (iv).

The fifth formula is true for $r=0$ from the defining relations, since $\phi_0^+ = K$. Using the identity $[a^{r+2},b] = [a,b]a^{r+1} + a[a^{r+1},b]$, we see by induction that

$$\begin{aligned}(v-v^{-1})[X^+(T)^{r+2},y_1] &= \big(\Phi^+(T) - K\big)X^+(T)^{r+1}\\ &\quad + v^{-r}[r+1]X^+(T)\big(\Phi^+(T)X^+(v^2T)^r - KX^+(T)^r\big).\end{aligned}$$

Using (i), this equals

$$\Phi^+(T)\big(X^+(T)^{r+1} - v^{-r}[r+1]X^+(T)X^+(v^2T)^r\big) \\ + v^{-(r+1)}[r+1][2]\Phi^+(T)X^+(v^2T)^{r+1} - v^{-(r+1)}[r+2]KX^+(T)^{r+1}.$$

Applying (iv) then gives the required result, proving (v). □

Theorem 20. *The comultiplication on the Drinfeld generators is given as follows.*

(1) $\Delta(X^\pm(T)) = X^\pm(T)\otimes 1 + \Phi^\pm(T)\otimes X^\pm(T)\sum_{r\geq 0}\Big(-v^{-1}(v-v^{-1})^2Y^\pm(v^{-2}T)\otimes X^\pm(T)\Big)^r.$

(2) $\Delta(Y^\pm(T)) = 1\otimes Y^\pm(T) + Y^\pm(T)\otimes \Phi^\pm(T)\sum_{r\geq 0}\Big(-v^{-1}(v-v^{-1})^2Y^\pm(T)\otimes X^\pm(v^2T)\Big)^r.$

(3) $\Delta(\Phi^\pm(T)) = \Phi^\pm(T)\otimes\Phi^\pm(T) \times\sum_{r\geq 0}[r+1]\Big(-(1-v^{-2})^2Y^\pm(v^{-2}T)\otimes X^\pm(v^2T)\Big)^r.$

(4) $\Delta(H^\pm(T)) = H^\pm(T)\otimes 1 + 1\otimes H^\pm(T) + \sum_{r\geq 0}\frac{1}{v^r-v^{-r}}\Big(-(v-v^{-1})^2Y^\pm(T)\otimes X^\pm(T)\Big)^r.$

The rest of this section is devoted to the proof of this theorem.

For the first three formulae we follow the proof in [27]. Note however that he uses the opposite comultiplication $\tau\Delta$.

We define $\bar{h}_r := \frac{r}{[2r]}h_r$, so that $H^\pm(T) = \sum_{\pm r\geq 1}\bar{h}_rT^{-r}$. We record the following calculations

$$\Delta(x_0) = x_0\otimes 1 + K\otimes x_0, \quad \Delta(y_1) = y_1\otimes K + 1\otimes y_1$$
$$\Delta(\bar{h}_1) = \bar{h}_1\otimes 1 + 1\otimes\bar{h}_1 - (v-v^{-1})y_1\otimes x_0,$$

which follow from

$$x_0 = X_1^+, \quad y_1 = X_2^+K, \quad h_1 = [x_0, y_1]K^{-1}.$$

Proof for $\Delta(X^+(T))$. Consider the formula

$$[\bar{h}_1, X^+(T)]T^{-1} = X^+(T) - x_0.$$

This leads to the functional equation

$$\xi^+(T) = \big[\bar{h}_1\otimes 1 + 1\otimes\bar{h}_1 - (v-v^{-1})y_1\otimes x_0, \xi^+(T)\big]T^{-1} + x_0\otimes 1 + K\otimes x_0.$$

We show that

$$\xi^+(T) = X^+(T)\otimes 1 + \sum_{r\geq 0}(-1)^rv^{-r}(v-v^{-1})^{2r}\Phi^+(T)Y^+(v^{-2}T)^r\otimes X^+(T)^{r+1}$$

is a solution of this functional equation. Since the constant terms of $\Delta(X^+(T))$ and $\xi^+(T)$ agree, the result follows.

Clearly

$$[\bar{h}_1\otimes 1 + 1\otimes\bar{h}_1, X^+(T)\otimes 1]T^{-1} = X^+(T)\otimes 1 - x_0\otimes 1.$$

Also,

$$-(v-v^{-1})[y_1\otimes x_0, X^+(T)\otimes 1]T^{-1} = \Phi^+(T)\otimes x_0 - K\otimes x_0.$$

By induction, using Lemma 19, we see that

$$\begin{aligned}[\bar{h}_1, X^+(T)^{r+1}]T^{-1} &= v^{-r}[r+1]\big(X^+(T)^{r+1} - x_0X^+(T)^r\big)\\ &= v^r[r+1]\big(X^+(T)^{r+1} - X^+(T)^r x_0\big).\end{aligned}$$

Applying the $\mathbb{Q}$-algebra involution σ gives

$$[\bar{h}_1, Y^+(T)^{r+1}] = v^{-r}[r+1]\big(Y^+(T)^r y_1 - Y^+(T)^{r+1}T\big).$$

Since $\bar{h}_1$ and $\Phi^+(T)$ commute, we deduce that

$$\begin{aligned}&[\bar{h}_1, \Phi^+(T)Y^+(v^{-2}T)^r]T^{-1}\\ &\qquad = v^{-r+1}[r]\Phi^+(T)\Big(Y^+(v^{-2}T)^{r-1}y_1T^{-1} - v^{-2}Y^+(v^{-2}T)^r\Big).\end{aligned}$$

Therefore

$$\begin{aligned}&\big[\bar{h}_1\otimes 1 + 1\otimes\bar{h}_1, \Phi^+(T)Y^+(v^{-2}T)^r\otimes X^+(T)^{r+1}\big]T^{-1}\\ &= \Phi^+(T)Y^+(v^{-2}T)^r\otimes X^+(T)^{r+1} - v^{-r}[r+1]\Phi^+(T)Y^+(v^{-2}T)^r\otimes x_0X^+(T)^r\\ &\qquad + v^{-r+1}[r]\Phi^+(T)Y^+(v^{-2}T)^{r-1}y_1T^{-1}\otimes X^+(T)^{r+1}.\end{aligned}$$

Finally

$$\begin{aligned}&-(v-v^{-1})\big[y_1\otimes x_0, \Phi^+(T)Y^+(v^{-2}T)^r\otimes X^+(T)^{r+1}\big]T^{-1}\\ &\qquad = -(v-v^{-1})y_1\Phi^+(T)Y^+(v^{-2}T)^rT^{-1}\otimes x_0X^+(T)^{r+1}\\ &\qquad\qquad + (v-v^{-1})\Phi^+(T)Y^+(v^{-2}T)^ry_1T^{-1}\otimes X^+(T)^{r+1}x_0.\end{aligned}$$

Applying σ to equation (3) for $m=0$ gives

$$y_1\Phi^+(T) - v^2\Phi^+(T)y_1 = (v^2-v^{-2})\Phi^+(T)\big(v^{-2}Y^+(v^{-2}T)T - y_1\big).$$

Therefore

$$y_1\Phi^+(T) = v^{-2}\Phi^+(T)y_1 + (1-v^{-4})\Phi^+(T)Y^+(v^{-2}T)T. \tag{4}$$

Also, applying σ to Lemma 19 (iii) yields

$$v^{-2}(v^{2r}-1)Y^+(v^{-2}T)^{r+1}T = v^{2r}y_1Y^+(v^{-2}T)^r - Y^+(v^{-2}T)^ry_1. \tag{5}$$

Hence

$$\begin{aligned}&-(v-v^{-1})\big[y_1\otimes x_0, \Phi^+(T)Y^+(v^{-2}T)^r\otimes X^+(T)^{r+1}\big]T^{-1}\\ &= v^{-2}(v-v^{-1})\Phi^+(T)\big(v^{-2r}Y^+(v^{-2}T)^ry_1 - y_1Y^+(v^{-2}T)^r\big)T^{-1}\otimes x_0X^+(T)^{r+1}\\ &\quad - v^{-2}(v-v^{-1})(v^2-v^{-2})\Phi^+(T)Y^+(v^{-2}T)^{r+1}\otimes x_0X^+(T)^{r+1}\\ &\qquad - v^{-(r+1)}(v-v^{-1})^2[r+1]\Phi^+(T)Y^+(v^{-2}T)^ry_1T^{-1}\otimes X^+(T)^{r+2}\\ &= v^{-(r+2)}(v-v^{-1})^2[r+2]\Phi^+(T)Y^+(v^{-2}T)^{r+1}\otimes x_0X^+(T)^{r+1}\\ &\quad + v^{-(r+1)}(v-v^{-1})^2[r+1]\Phi^+(T)Y^+(v^{-2}T)^ry_1T^{-1}\otimes X^+(T)^{r+2}.\end{aligned}$$

We deduce that

$$\big[\Delta(\bar{h}_1), \sum_{r\geq 0}(-1)^r v^{-r}(v-v^{-1})^{2r}\Phi^+(T)Y^+(v^{-2}T)^r \otimes X^+(T)^{r+1}\big]T^{-1}$$
$$= -\Phi^+(T)\otimes x_0 + \sum_{r\geq 0}(-1)^r v^{-r}(v-v^{-1})^{2r}\Phi^+(T)Y^+(v^{-2}T)^r \otimes X^+(T)^{r+1},$$

and hence that

$$[\Delta(\bar{h}_1), \xi^+(T)]T^{-1} = \xi^+(T) - \Delta(x_0).$$

This proves that $\Delta(X^+(T)) = \xi^+(T)$. □

Proof for $\Delta(Y^+(T))$. Applying σ immediately gives the formula for $\Delta(Y^+(T))$. □

Proof for $\Delta(\Phi^+(T))$. We note that

$$(v-v^{-1})[X^+(T), y_1]T^{-1} = \Phi^+(T) - K,$$

and so

$$\Delta(\Phi^+(T)) = K\otimes K + (v-v^{-1})[\Delta(X^+(T)), \Delta(y_1)]T^{-1}.$$

Clearly

$$K\otimes K + (v-v^{-1})[X^+(T)\otimes 1, \Delta(y_1)] = \Phi^+(T)\otimes K,$$

whereas

$$(v-v^{-1})[\Phi^+(T)Y^+(v^{-2}T)^r \otimes X^+(T)^{r+1}, \Delta(y_1)]T^{-1}$$
$$= (v-v^{-1})v^{-2(r+1)}\Phi^+(T)Y^+(v^{-2}T)^r y_1 T^{-1} \otimes KX^+(T)^{r+1}$$
$$- (v-v^{-1})y_1\Phi^+(T)Y^+(v^{-2}T)^r T^{-1}\otimes KX^+(T)^{r+1}$$
$$+ (v-v^{-1})\Phi^+(T)Y^+(v^{-2}T)^{\otimes}[X^+(T)^{r+1}, y_1]T^{-1}.$$

Using formulae (4) and (5) this reduces to

$$v^{-(r+2)}(v-v^{-1})[r+2]\Phi^{(}T)Y^+(v^{-2}T)^{r+1}\otimes KX^+(T)^{r+1}$$
$$- v^{-r}[r+1]\Phi^+(T)Y^+(v^{-2}T)^r \otimes KX^+(T)^r$$
$$+ v^{-r}[r+1]\Phi^+(T)Y^+(v^{-2}T)^r \otimes \Phi^+(T)X^+(v^2T)^r.$$

Therefore

$$(v-v^{-1})\big[\sum_{r\geq 0}(-1)^r v^{-r}(v-v^{-1})^{2r}\Phi^+(T)Y^+(v^{-2}T)^r \otimes X^+(T)^{r+1}, \Delta(y_1)\big]T^{-1}$$
$$= -\Phi^+(T)\otimes K$$
$$+ \sum_{r\geq 0}(-1)^r(1-v^{-2})^{2r}[r+1]\Phi^+(T)Y^+(v^{-2}T)^r \otimes \Phi^+(T)X^+(v^2T)^r.$$

This proves the formula for $\Delta(\Phi^+(T))$. □

Proof for $\Delta(H^+(T))$. We cannot use the same procedure as above for $\Delta(H^+(T))$ since there is no easy recursion for the elements h_r. Instead, from the implicit formulae

$$[H^+(T), x_0] = \sum_{r\geq 1}[\bar{h}_r, x_0]T^{-r} = X^+(T) - x_0, \quad [H^+(T), y_1] = y_1 - Y^+(T)T$$

we consider the equations

$$[\theta^+(T), \Delta(x_0)] = \Delta(X^+(T)) - \Delta(x_0), \quad [\theta^+(T), \Delta(y_1)] = \Delta(y_1) - \Delta(Y^+(T))T.$$

We first show that

$$\theta^+(T) = H^+(T)\otimes 1 + 1\otimes H^+(T) + \sum_{r\geq 1}\frac{(v-v^{-1})^{2r}}{v^r-v^{-r}}Y^+(T)^r\otimes X^+(T)^r$$

is a solution to both of these equations. Then, applying Theorem 17, we show that the difference $\Delta(H^+(T)) - \theta^+(T)$ is zero.

Clearly

$$[H^+(T)\otimes 1 + 1\otimes H^+(T), \Delta(x_0)] = X^+(T)\otimes 1 + K\otimes X^+(T) - \Delta(x_0).$$

Also

$$\begin{aligned}&[Y^+(T)^r\otimes X^+(T)^r, x_0\otimes 1 + K\otimes x_0]\\ &\qquad = [Y^+(T)^r, x_0]\otimes X^+(T)^r + KY^+(T)^r\otimes\Big(v^{2r}X^+(T)x_0 - x_0X^+(T)^r\Big).\end{aligned}$$

Applying σ to Lemma 19 (v) gives

$$(v-v^{-1})[Y^+(T)^{r+1}, x_0] = [r+1]\Big(v^rKY^+(T)^r - v^{-r}\Phi^+(T)Y^+(v^{-2}T)^r\Big).$$

Therefore

$$\begin{aligned}&[Y^+(T)^r\otimes X^+(T)^r, \Delta(x_0)] = (v^{2r}-1)KY^+(T)^r\otimes X^+(T)^{r+1}\\ &\qquad + v^{r-1}\frac{v^r-v^{-r}}{(v-v^{-1})^2}\Big(KY^+(T)^{r-1} - \Phi^+(T)Y^+(v^{-2}T)^{r-1}\Big)\otimes X^+(T)^r,\end{aligned}$$

and so

$$\begin{aligned}&\Big[\sum_{r\geq 1}(-1)^r\frac{(v-v^{-1})^{2r}}{(v^r-v^{-r})}Y^+(T)^r\otimes X^+(T)^r, \Delta(x_0)\Big]\\ &\qquad = -K\otimes X^+(T) + \sum_{r\geq 0}(-1)^rv^{-r}(v-v^{-1})^{2r}\Phi^+(T)Y^+(v^{-2}T)^r\otimes X^+(T)^{r+1}.\end{aligned}$$

Thus we see that

$$[\theta^+(T), \Delta(x_0)] = \Delta(X^+(T)) - \Delta(x_0).$$

Since $\tau(\sigma\otimes\sigma)(\theta^+(T)) = -\theta^+(T)$, we immediately get that

$$[\theta^+(T), \Delta(y_1)] = \Delta(y_1) - \Delta(Y^+(T))T.$$

Therefore $[\Delta(H^+(T)), -]$ and $[\theta^+(T), -]$ agree on $\Delta(x_0)$ and $\Delta(y_1)$.

We now consider the difference $\Delta(H^+(T)) - \theta^+(T)$. We can express this as a sum of terms of the form $a\otimes bT^{-r}$, with a and b homogeneous elements in $\mathcal{U}_v^{\geq}(\widehat{\mathfrak{sl}}_2)$. We use the natural $\mathbb{Z}^2$-grading given by dimension vectors, and use the lexicographic ordering. Since Δ respects the grading, $\underline{\dim}\,a + \underline{\dim}\,b$ is a multiple of δ for all such terms $a\otimes bT^{-r}$. Finally, $\underline{\dim}\,a$ and $\underline{\dim}\,b$ are both non-zero, since $\theta^+(T)$ includes the terms $H^+(T)\otimes 1 + 1\otimes H^+(T)$.

Let $r\geq 1$ be minimal such that the coefficient of T^{-r} is non-zero. Let (α,β) be the smallest pair with respect to the lexicographic ordering on $\mathbb{Z}^2\times\mathbb{Z}^2$ such that $(\alpha,\beta) = (\underline{\dim}\,a, \underline{\dim}\,b)$ for some homogeneous $a\otimes b$ occurring in the coefficient of T^{-r}. Choose linearly independent $a_1,\cdots,a_s$ in the subspace $\mathcal{U}_v^{\geq}(\widehat{\mathfrak{sl}}_2)_\alpha$ such that

$$\Delta(H^+(T)) - \theta^+(T) = \sum_{i=1}^s a_i\otimes b_iT^{-r} + \text{higher order terms}.$$

More precisely, any other term $a \otimes bT^{-m}$ satisfies either $m > r$ or else $m = r$ and $(\underline{\dim}\, a, \underline{\dim}\, b) > (\alpha, \beta)$.

Consider in $[\Delta(H^+(T)) - \theta^+(T), x_0 \otimes 1 + K \otimes x_0] = 0$ the coefficient of T^{-r} of degree $(\alpha, \beta + e_1)$. Using that the $a_i K$ are still linearly independent, we deduce that

$$b_i x_0 = v^{(e_1,\alpha)} x_0 b_i \quad \text{for all} \quad 1 \le i \le s.$$

Similarly, by considering in $[\Delta(H^+(T)) - \theta^+(T), y_1 \otimes K + 1 \otimes y_1] = 0$ the coefficient of T^{-r} of degree $(\alpha, \beta + e_2)$, we obtain that

$$b_i y_1 = y_1 b_i \quad \text{for all} \quad 1 \le i \le s.$$

Let us write $b_i = \bar{b}_i K^{m_i}$ for some integers m_i so that $\bar{b}_i \in \mathcal{U}_v^+(\widehat{\mathfrak{sl}}_2)$. Then

$$\bar{b}_i X_1^+ = v^{(e_1,\alpha)-2m_i} X_1^+ \bar{b}_i, \qquad \bar{b}_i X_2^+ = v^{2m_i + (e_1,\beta)} X_2^+ \bar{b}_i.$$

Since $\alpha + \beta$ is a multiple of δ, we deduce that

$$\bar{b}_i X_1^+ = v^{r_i} X_1^+ \bar{b}_i, \qquad \bar{b}_i X_2^+ = v^{-r_i} X_2^+ \bar{b}_i,$$

where $r_i = (e_1, \alpha) - 2m_i$. Therefore we can apply Theorem 17 to deduce that $\bar{b}_i$ is a scalar, so of degree 0. This is a contradiction, and hence $\Delta(H^+(T)) = \theta^+(T)$. $\square$

The remaining four formulae, with everywhere a minus sign, follow from the first four by applying the $\mathbb{Q}(v)$-algebra involution ρ.

This completes the proof of Theorem 20

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