Derived Hall Algebras

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Ringel-Hall Algebras

- $\mathcal{E}$ a finitary, exact category.
- $\mathcal{H}(\mathcal{E})$ an associative, unital algebra.
- Structure constants of $\mathcal{H}(\mathcal{E})$ encode the exact structure of $\mathcal{E}$.

$\mathcal{H}(\mathcal{E})$ is the Ringel-Hall algebra of $\mathcal{E}$.

These were introduced by C.M. Ringel in 1990, generalising the construction of the classical Hall algebra.

The Classical Hall Algebra, I

Let (R, m, k) be a DVR with finite residue field k. Let $\mathcal{A} = \operatorname{mod} R$. This is an abelian category with finite homs and exts (as sets).

The indecomposable modules R/m^n are determined up to isomorphism by their lengths.

The isomorphism classes of modules are thus indexed by partitions. Fix a representative M_λ for each partition λ.

The Classical Hall Algebra, II

Let $\mathcal{H}(\mathcal{A})$ be the $\mathbb{Q}$-vector space with basis u_λ and multiplication

$$u_\mu u_\lambda := \sum_\xi F^\xi_{\mu\lambda} u_\xi$$

where

$$F^\xi_{\mu\lambda} := |\{U \leq M_\xi : M_\xi/U \cong M_\mu, U \cong M_\lambda\}|.$$

The Classical Hall Algebra, III

Then (Steinitz 1901, Hall 1959, Macdonald 1995)

$$\mathcal{H}(\mathcal{A}) \cong \mathbb{Q}[x_1, x_2, \ldots]$$

is an associative, unital algebra.

In fact, $\mathcal{H}(\mathcal{A})$ is naturally isomorphic to Macdonald's ring of symmetric functions

$$e_n \mapsto q^{\binom{n}{2}} u_{1^n}$$

where e_n is the n-th elementary symmetric function and u_{1^n} corresponds to the semisimple module of dimension n.

Finitary Exact Categories

Let $\mathcal{E}$ be exact.

$\mathcal{E}$ is called finitary if $\mathsf{Hom}(M, N)$ and $\mathrm{Ext}^1(M, N)$ are finite sets for all objects M and N.
For example, take $\mathcal{E}$ to be an abelian category over a finite field, and with finite dimensional Homs and Exts.

Ringel-Hall Algebras

Let $\mathcal{E}$ be a finitary, exact category.

The Ringel-Hall algebra $\mathcal{H}(\mathcal{E})$ is the $\mathbb{Q}$-algebra with

- basis u_X indexed by the isomorphism classes of objects in $\mathcal{E}$
- multiplication $u_N u_M = \sum_{[X]} F^X_{NM} u_X$.

The F^X_{NM} are called Hall numbers and are defined as follows.

Hall Numbers

Let M, N and X be objects of $\mathcal{E}$. Define

$$\mathsf{Ex}(M, X, N) := \{(f, g) : M \overset{f}{\hookrightarrow} X \overset{g}{\twoheadrightarrow} N \text{ exact}\}.$$

The group $\mathsf{Aut}(M) \times \mathsf{Aut}(N)$ acts *freely* on $\mathsf{Ex}(M, X, N)$.

We define the Hall number to be

$$F^X_{NM} := |\,\mathsf{Ex}(M, X, N)|/|\mathrm{Aut}(M) \times \mathrm{Aut}(N)|.$$

This is a finite number.

Clearly u_0 is a unit for the multiplication.

Associativity

We need to show for all M, N, P and E that

$$\sum_{[X]} F^{E}_{PX} F^{X}_{NM} = \sum_{[Y]} F^{Y}_{PN} F^{E}_{YM}.$$

This is easier to envisage as an exact commutative diagram:

$$\begin{array}{ccccc}
M & \longrightarrow & X & \longrightarrow & N \\
\| & & \downarrow & & \downarrow \\
M & \longrightarrow & E & \longrightarrow & Y \\
 & & \downarrow & & \downarrow \\
 & & P & = & P
\end{array}$$

Note that $X \hookrightarrow N \oplus E \twoheadrightarrow Y$ is exact.

Proof of Associativity

The pull-back/push-out constructions give a natural bijection

$$\bigcup_{[X]} \frac{\mathrm{Ex}(X,E,P) \times \mathrm{Ex}(M,X,N)}{\mathrm{Aut}(X)} \longleftrightarrow \bigcup_{[Y]} \frac{\mathrm{Ex}(N,Y,P) \times \mathrm{Ex}(M,E,Y)}{\mathrm{Aut}(Y)}$$

since kernels and cokernels are unique up to unique isomorphism.

Using that the automorphism groups act freely, we deduce the associativity law

$$\sum_{[X]} F^{E}_{PX} F^{X}_{NM} = \sum_{[Y]} F^{Y}_{PN} F^{E}_{YM}.$$

Example 1

Let A be the algebra of upper triangular matrices of size $n-1$ over a finite field k. Set $\mathcal{A} := \operatorname{mod} A$.
Then

$$\mathcal{H}(\mathcal{A}) \cong \mathcal{U}_v^+(\mathfrak{sl}_n)$$

the positive part of the quantised enveloping algebra, specialised at v, where $v^2 = |k|$.

Caveat: This is true only up to a twist of the multiplication.

Example 2

More generally, let k be a finite field, Δ a Dynkin graph and Q an orientation of Δ. Then the category $\mathcal{A}$ of finite dimensional k-representations of Q is a finitary, k-linear, abelian category.

(Ringel 1990) With the same caveat,

$$\mathcal{H}(\mathcal{A}) \cong \mathcal{U}_v^+(\mathfrak{g})$$

where $\mathfrak{g}$ is the complex Lie algebra of type Δ and $v^2 = |k|$.

This can be extended to arbitrary graphs and Kac-Moody Lie algebras (J.A. Green 1995, B. Sevenhant and M. Van den Bergh 2001).

Example 3

M. Kapranov (1998) studied the Hall algebra of the category of coherent sheaves on a smooth projective curve over a finite field. In particular, for $\mathbb{P}^1$, there is a natural subalgebra isomorphic to a non-standard 'positive part' of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_2)$.

I. Burban and O. Schiffmann (2005) studied $\mathcal{H}(\mathrm{coh}\, X)$ for an elliptic curve X. This is related to the spherical double affine Hecke algebra of type $\mathfrak{gl}_n$.

O. Schiffmann (2003) also considered $\mathcal{H}(\mathrm{coh}\, X)$ for a weighted projective line X (a 'non-commutative smooth projective curve'). In this case, the Hall algebra is related to the loop algebra of a Kac-Moody Lie algebra.

Results pre-2006, I

- The associativity of the Hall algebra of an exact category follows from pull-back/push-out constructions.
- The Octahedral Axiom is the analogue of the pull-back/push-out in a triangulated category.

In 1998, Kapranov remarked:

> *Unfortunately, a direct mimicking of the Hall algebra construction, but with exact triangles replacing exact sequences, fails to give an associative multiplication, even though the octahedral axiom looks like the right tool to establish associativity.*

Results pre-2006, II

Let $\mathcal{T}$ be a finitary, triangulated category.
Define

$$\mathrm{Ex}(M, X, N) := \{(f, g, h) : M \xrightarrow{f} X \xrightarrow{g} N \xrightarrow{h} M[1] \text{ exact}\}.$$

Then $\mathrm{Aut}(M) \times \mathrm{Aut}(N)$ acts on $\mathrm{Ex}(M, X, N)$, but not freely.

Define

$$V(M, X, N) := \mathrm{Ex}(M, X, N) / \mathrm{Aut}(M) \times \mathrm{Aut}(N).$$

The naive way of defining Hall numbers would be to take the cardinality of the sets $V(M, X, N)$.

Results pre-2006, III

In 2000, L. Peng and J. Xiao showed that the subspace spanned by the indecomposable objects of a triangulated category form a Lie algebra. The Lie bracket is given by the commutator of the naive Hall multiplication.
This only works, however, if the triangulated category is two-periodic ($[2] \cong \mathrm{id}_{\mathcal{T}}$), k-linear, and one reduces the coefficients modulo $|k| - 1$.

This result has been used by Crawley-Boevey (2005) to obtain existence results for indecomposable sheaves over weighted projective lines.

Most triangulated categories, though, are not two-periodic — and one cannot simply form the quotient! Also, we don't want to work with coefficients in $\mathbb{Z}/n\mathbb{Z}$.

Derived Hall Algebras

In 2006, however, Toën proved the following:

Let $\mathcal{T}$ be a finitary, Krull-Schmidt, triangulated category, which is also left homologically finite; that is:

- $\mathrm{Hom}(X, Y)$ is finite for all X and Y
- $\mathrm{End}(X)$ is local for all indecomposable X
- Given X and Y we have $\mathrm{Hom}(X[i], Y) = 0$ for $i \gg 0$

Theorem (Toën, Xiao-Xu)

There is an associative, unital algebra $\mathcal{H}(\mathcal{T})$, called the derived Hall algebra, whose structure constants g_{NM}^{X} encode the triangulated structure of $\mathcal{T}$.

Toën's Formula, I

Toën takes as structure constants the numbers

$$g_{NM}^{X} := \frac{|\operatorname{Hom}(M,X)_N|\{M,X\}}{|\operatorname{Aut}(M)|\{M,M\}},$$

where

$$\operatorname{Hom}(M,X)_N := \{M \xrightarrow{f} X : \mathrm{Cone}(f) \cong N\}$$

and

$$\{M,X\} := \prod_{i\geq 1} |\operatorname{Hom}(M[i],X)|^{(-1)^i}$$
$$= \frac{1}{|\operatorname{Hom}(M[1],X)|} \cdot |\operatorname{Hom}(M[2],X)| \cdot \frac{1}{|\operatorname{Hom}(M[3],X)|} \cdots$$

He calls g_{NM}^{X} the number of virtual homomorphisms $M \to X$ with cone isomorphic to N.

Toën's Formula, II

Let $\mathcal{E}$ be an exact category. Then

$$F_{NM}^{X} := \frac{|\operatorname{Ex}(M,X,N)|}{|\operatorname{Aut}(M)||\operatorname{Aut}(N)|} = \frac{|\mathrm{Infl}(M,X)_N|}{|\operatorname{Aut}(M)|},$$

where

$$\mathrm{Infl}(M,X)_N := \frac{\operatorname{Ex}(M,X,N)}{\operatorname{Aut}(N)} = \{f : \exists g \text{ with } M \overset{f}{\hookrightarrow} X \overset{g}{\twoheadrightarrow} N \text{ exact}\}$$

and $\operatorname{Aut}(M)$ acts freely on $\mathrm{Infl}(M,X)_N$.

Toën's Formula, III

The structure constants g_{NM}^{X} can be thought of as an analogue of this for triangulated categories. In particular, we have

$$\mathrm{Hom}(M, X)_N = \mathrm{Ex}(M, X, N)/\,\mathrm{Aut}(N).$$

For, consider $\mathrm{Ex}(M, X, N) \to \mathrm{Hom}(M, X)_N$, $(f, g, h) \mapsto f$. This is surjective and we can complete an exact commutative diagram

$$\begin{array}{ccccccc}
M & \xrightarrow{\quad f \quad} & X & \xrightarrow{\quad g \quad} & N & \xrightarrow{\quad h \quad} & M[1] \\
\| & & \| & & & & \| \\
M & \xrightarrow{\quad f \quad} & X & \xrightarrow{\quad g' \quad} & N & \xrightarrow{\quad h' \quad} & M[1]
\end{array}$$

to an isomorphism of triangles.

Remarks

We remark that Toën proved his result for dg-categories $\mathcal{T}$.
There is a homotopy type $X^{(0)}(\mathcal{T})$ whose connected components correspond bijectively to quasi-isomorphism classes of perfect dg $\mathcal{T}$-modules.
For $x \in X^{(0)}(\mathcal{T})$ we have

$$\pi_1(X^{(0)}(\mathcal{T}), x) = \mathrm{Aut}(x), \quad \pi_{n+1}(X^{(0)}(\mathcal{T}), x) = \mathrm{Hom}(x[n], x) \quad n \geq 1.$$

Then

$$|\mathrm{Aut}(x)| \frac{1}{|\mathrm{Hom}(x[1], x)|} |\mathrm{Hom}(x[2], x)| \cdots$$

is the 'homotopy cardinality' of the connected component containing x.
Finally Toën defines the Hall multiplication via a convolution product.

We will discuss an elementary proof, due to J. Xiao and F. Xu (2007), which works for triangulated categories.

Radical Homomorphisms, I

A morphism $f\colon X \to Y$ is called radical provided that, given L indecomposable and morphisms $\theta\colon L \to X$ and $\phi\colon Y \to L$, $\phi f \theta \notin \mathrm{Aut}(L)$.

Properties:

- $\mathrm{Rad}(X, Y)$ is a subspace of $\mathrm{Hom}(X, Y)$.
- If X and Y are indecomposable, then $\mathrm{Rad}(X, Y)$ is the set of non-isomorphisms $X \to Y$.
- If $X = \bigoplus_i X_i$ and $Y = \bigoplus_j Y_j$ are finite decompositions into indecomposable objects, then $\mathrm{Rad}(X, Y) := \prod_{i,j} \mathrm{Rad}(X_i, Y_j)$.

Radical Homomorphisms, II

Lemma

Given $f\colon X \to Y$, there exist objects A, X' and Y', a radical morphism $f'\colon X' \to Y'$ and isomorphisms x and y such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow x & & \downarrow y \\ A \oplus X' & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & f' \end{pmatrix}} & A \oplus Y' \end{array}$$

One proof is by induction on the number of indecomposable summands of $X \oplus Y$.

Decomposing Exact Triangles

Lemma

The morphism $\begin{pmatrix} 1 & 0 \\ 0 & f' \end{pmatrix} : A \oplus X' \to A \oplus Y'$ *can be completed to an exact triangle of the form*

$$A \oplus X' \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & f' \end{pmatrix}} A \oplus Y' \xrightarrow{(0 \ g')} Z \xrightarrow{\begin{pmatrix} 0 \\ h' \end{pmatrix}} A[1] \oplus X[1].$$

and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z \xrightarrow{h'} X'$ *is also exact.*

Functorial Interpretation

Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be exact. Let G be the coherent subfunctor of $\mathrm{Hom}(-, Z)$ such that $G(U) := g\,\mathrm{Hom}(U, Y)$.

Lemma

The factorisation $f = \begin{pmatrix} 1 & 0 \\ 0 & f' \end{pmatrix}$ with f' radical is equivalent to taking a projective cover of G.

- $G(U) = g'\,\mathrm{Hom}(U, Y)$
- if $g'y = g'$ for $y \in \mathrm{End}(Y')$, then $y \in \mathrm{Aut}(Y')$.

Reformulation of the Hall Numbers, I

Recall

$$V(M,X,N) = \mathrm{Hom}(M,X)_N / \mathrm{Aut}(M).$$

Let $\alpha \in V(M,X,N)$. This has a representative

$$A[-1] \oplus M' \xrightarrow{(0\ f')} X \xrightarrow{\binom{0}{g'}} A \oplus N \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & h' \end{pmatrix}} A \oplus M'[1]$$

with h' radical. We write $A(\alpha) := A$.

Lemma (Xiao-Xu)

$$g_{NM}^{X} = \{M,N\} \sum_{\alpha \in V(M,X,N)} \frac{|\,\mathrm{End}(A(\alpha))|}{|\,\mathrm{Aut}(A(\alpha))|}.$$

Reformulation of the Hall Numbers, II

We also have the dual statements

$$\operatorname{Hom}(X,N)_{M[1]} = \operatorname{Ex}(M,X,N)/\operatorname{Aut}(M)$$
$$V(M,X,N) = \operatorname{Hom}(X,N)_{M[1]}/\operatorname{Aut}(N).$$

The Xiao-Xu Lemma shows that we have the dual expression

$$g_{NM}^{X} = \frac{|\operatorname{Hom}(X,N)_{M[1]}|\{X,N\}}{|\operatorname{Aut}(N)|\{N,N\}}.$$

Thus there is no bias in the definition, as one would expect.

Proof of the Xiao-Xu Lemma, I

We use the Orbit-Stabiliser Theorem

$$\frac{|\operatorname{Hom}(M,X)_N|}{|\operatorname{Aut}(M)|} = \sum_{\alpha \in V(M,X,N)} \frac{1}{|\mathrm{Stab}(f)|},$$

where (f,g,h) is any representative of α and $\mathrm{Stab}(f) \leq \operatorname{Aut}(M)$.

In particular, we can take the representative

$$A[-1] \oplus M' \xrightarrow{(0\ f')} X \xrightarrow{\binom{0}{g'}} A \oplus N' \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & h' \end{pmatrix}} A \oplus M'[1]$$

Proof of the Xiao-Xu Lemma, II

Take $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{End}(A \oplus M'[1])$ such that

$$\begin{pmatrix} 0 & f'[1] \end{pmatrix} = \begin{pmatrix} 0 & f'[1] \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} f'[1]c & f'[1]d \end{pmatrix}.$$

Hence $\begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} h'c' & 1 + h'd' \end{pmatrix}$.
Since h' is radical, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible if and only if $a \in \mathrm{Aut}(A)$.
Thus

$$\begin{aligned} |\mathrm{Stab}\begin{pmatrix} 0 & f'[1] \end{pmatrix}| &= \left| \begin{pmatrix} \mathrm{Aut}(A) & \mathrm{Hom}(M'[1], A) \\ h'\,\mathrm{Hom}(A, N') & h'\,\mathrm{Hom}(M'[1], N') \end{pmatrix} \right| \\ &= \frac{|\mathrm{Aut}(A)|}{|\mathrm{End}(A)|} \times \left| \begin{pmatrix} 1 & 0 \\ 0 & h' \end{pmatrix} \begin{pmatrix} \mathrm{End}(A) & \mathrm{Hom}(M'[1], A) \\ \mathrm{Hom}(A, N') & \mathrm{Hom}(M'[1], N') \end{pmatrix} \right| \\ &= \frac{|\mathrm{Aut}(A)|}{|\mathrm{End}(A)|} \times |h\,\mathrm{Hom}(M[1], N)|. \end{aligned}$$

Proof of the Xiao-Xu Lemma, III

Using the exactness of the sequence

$$\cdots \to \mathrm{Hom}(M[1], X) \to \mathrm{Hom}(M[1], N) \xrightarrow{h} \mathrm{End}(M[1])$$

we see that $|h\,\mathrm{Hom}(M[1], N)|$ equals

$$\frac{|\,\mathrm{Hom}(M[1], M \oplus N)|}{|\,\mathrm{Hom}(M[1], X)|} \cdot \frac{|\,\mathrm{Hom}(M[1], X[-1])|}{|\,\mathrm{Hom}(M[1], (M \oplus N)[-1])|} \cdots$$

In other words,

$$|h\,\mathrm{Hom}(M[1], N)| = \frac{\{M, X\}}{\{M, M \oplus N\}}.$$

Proof of the Xiao-Xu Lemma, IV

We have thus shown that

$$|\mathrm{Stab}(f)| = \frac{|\operatorname{Aut}(A)|}{|\operatorname{End}(A)|} \cdot \frac{\{M, X\}}{\{M, M \oplus N\}}.$$

Hence

$$\begin{aligned} g^X_{NM} &= \frac{|\operatorname{Hom}(M, X)_N|\{M, X\}}{|\operatorname{Aut}(M)|\{M, M\}} \\ &= \{M, N\} \sum_{\alpha \in V(M,X,N)} \frac{|\operatorname{End}(A(\alpha))|}{|\operatorname{Aut}(A(\alpha))|}. \end{aligned}$$

Proof of Associativity, I

We now prove the associativity of the multiplication:

$$\sum_{[X]} g_{PX}^{E} g_{NM}^{X} = \sum_{[Y]} g_{PN}^{Y} g_{YM}^{E}.$$

As expected, this will require the Octahedral Axiom.

Proof of Associativity, II

Note first that

$$g_{PX}^{E} g_{NM}^{X} = \frac{|\operatorname{Hom}(X,E)_P \times \operatorname{Hom}(X,N)_{M[1]}|\{X, E \oplus N\}}{|\operatorname{Aut}(X) \times \operatorname{Aut}(N)|\{X,X\}\{N,N\}}.$$

Consider the diagram

$$\begin{array}{ccc}
\operatorname{Hom}(X,E) \times \operatorname{Hom}(X,N) & \xrightarrow{\sim} & \operatorname{Hom}(X, E \oplus N) \\
\| & & \| \\
\bigcup_{[P],[M]} \operatorname{Hom}(X,E)_P \times \operatorname{Hom}(X,N)_{M[1]} & \xrightarrow{\sim} & \bigcup_{[Y]} \operatorname{Hom}(X, E \oplus N)_Y
\end{array}$$

Let $\operatorname{Hom}(X, E \oplus N)_Y^{M,P}$ denote the set of morphisms $\left(\begin{smallmatrix} u \\ -g \end{smallmatrix}\right)$ with

$$\operatorname{Cone}(u) \cong P, \quad \operatorname{Cone}(g) \cong M[1], \quad \operatorname{Cone}\left(\begin{smallmatrix} u \\ -g \end{smallmatrix}\right) \cong Y.$$

Proof of Associativity, III

We have the subset of $V(X, E \oplus N, Y)$

$$V(X, E \oplus N, Y)_l^{M,P} := \mathrm{Hom}(X, E \oplus N)_Y^{M,P} / \mathrm{Aut}(X).$$

Dually, we can define

$$V(X, E \oplus N, Y)_r^{M,P} := \mathrm{Hom}(E \oplus N, Y)_{X[1]}^{M,P} / \mathrm{Aut}(Y).$$

Lemma

$$V(X, E \oplus N, Y)_l^{M,P} = V(X, E \oplus N, Y)_r^{M,P}.$$

We write $V(X, E \oplus N, Y)^{M,P}$ for this subset.

The Octahedral Axiom

Given an exact triangle $X \xrightarrow{\binom{u}{-g}} E \oplus N \xrightarrow{(g' \;\; u')} Y \xrightarrow{\delta} X[1]$, there exists a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
M & \xrightarrow{f} & X & \xrightarrow{g} & N & \xrightarrow{h} & M[1] \\
\| & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} & & \| \\
M & \xrightarrow{f'} & E & \xrightarrow{g'} & Y & \xrightarrow{h'} & M[1] \\
 & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v'} & & \\
 & & P & = & P & & \\
 & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w'} & & \\
 & & X[1] & \xrightarrow{g[1]} & N[1] & &
\end{array}$$

and such that $\delta = f[1]h' = wv'$.

Proof of Lemma

- Given $\left(\begin{smallmatrix} u \\ -g \end{smallmatrix}\right) \in \mathrm{Hom}(X, E \oplus N)_Y^{P,M}$.
- Complete to an exact triangle.
- Obtain $\begin{pmatrix} g' & u' \end{pmatrix} : E \oplus N \to Y$.
- Octahedral Axiom: $\begin{pmatrix} g' & u' \end{pmatrix} \in \mathrm{Hom}(E \oplus N, Y)_{X[1]}^{M,P}$.

This induces a map

$$\mathrm{Hom}(X, E \oplus N)_Y^{M,P} \longrightarrow V(X, E \oplus N, Y)_r^{M,P},$$

which in turn induces a map

$$V(X, E \oplus N, Y)_l^{M,P} \longrightarrow V(X, E \oplus N, Y)_r^{M,P}.$$

This is just the identity map between subsets of $V(X, E \oplus N, Y)$.

Proof of Associativity, IV

We now have the analogue of the Xiao-Xu Lemma:

$$
\begin{aligned}
g_{YX}^{E\oplus N}(M,P) &:= \frac{|\operatorname{Hom}(X, E\oplus N)_Y^{M,P}|\{X, E\oplus N\}}{|\operatorname{Aut}(X)\{X,X\}} \\
&= \{X,Y\} \sum_{\alpha\in V(X,E\oplus N,Y)^{M,P}} \frac{|\operatorname{End}(A(\alpha))|}{|\operatorname{Aut}(A(\alpha))|} \\
&= \frac{|\operatorname{Hom}(E\oplus N, Y)_{X[1]}^{M,P}|\{E\oplus N, Y\}}{|\operatorname{Aut}(Y)|\{Y,Y\}}.
\end{aligned}
$$

Proof of Associativity, V

Now,

$$g_{PX}^{E} g_{NM}^{X} = \frac{|\operatorname{Hom}(X,E)_P \times \operatorname{Hom}(X,N)_{M[1]}|\{X, E \oplus N\}}{|\operatorname{Aut}(X) \times \operatorname{Aut}(N)|\{X,X\}\{N,N\}}$$

$$= \sum_{[Y]} \frac{|\operatorname{Hom}(X, E \oplus N)_Y^{M,P}|\{X, E \oplus N\}}{|\operatorname{Aut}(X)|\{X,X\}|\operatorname{Aut}(N)|\{N,N\}}$$

$$= \sum_{[Y]} \frac{g_{YX}^{E\oplus N}(M,P)}{|\operatorname{Aut}(N)|\{N,N\}}.$$

Dually

$$g_{PN}^{Y} g_{YM}^{E} = \sum_{[X]} \frac{g_{YX}^{E\oplus N}(M,P)}{|\operatorname{Aut}(N)|\{N,N\}}.$$

Thus

$$\sum_{[X]} g_{PX}^{E} g_{NM}^{X} = \sum_{[X],[Y]} \frac{g_{YX}^{E\oplus N}(M,P)}{|\operatorname{Aut}(N)|\{N,N\}} = \sum_{[Y]} g_{PN}^{Y} g_{YM}^{E}.$$

The Hereditary Case, I

- $\mathcal{A}$ a k-linear, finitary, hereditary, abelian category.
- $\mathcal{D}$ the bounded derived category of $\mathcal{A}$.
- $X \cong \bigoplus_{n\in\mathbb{Z}}(H^n X)[-n]$ for all $X \in \mathcal{D}$.
- We identify $\mathcal{A}$ with the stalk complexes of degree 0 in $\mathcal{D}$.

We have the Hall algebra $\mathcal{H}(\mathcal{A})$ with structure constants F_{NM}^X.

We have the derived Hall algebra $\mathcal{H}(\mathcal{D})$ with structure constants g_{NM}^X.

The Hereditary Case, II

Theorem

The derived Hall algebra $\mathcal{H}(\mathcal{D})$ has generators $u_{X[n]}$ for $X \in \mathcal{A}$ and $n \in \mathbb{Z}$, and relations

- $u_{N[m]}u_{M[m]} = \sum_{[X]} F^X_{NM} u_{X[m]}$.
- $u_{N[m+1]}u_{M[m]} = \sum_{[K],[C]} q^{\langle C,K\rangle} H^{K,C}_{M,N} u_{C[m]}u_{K[m+1]}$.
- $u_{N[n]}u_{M[m]} = q^{(-1)^{n-m}\langle M,N\rangle} u_{M[m]}u_{N[n]}$ *if* $n > m+1$.

where

$$H^{KC}_{MN} := \frac{|\operatorname{Ex}(K, M, N, C)|}{|\operatorname{Aut}(M) \times \operatorname{Aut}(N)|}$$

and

$$\langle M, N\rangle := \dim \operatorname{Hom}(M, N) - \dim \operatorname{Ext}^1(M, N).$$

Hence $\mathcal{H}(\mathcal{A})$ is naturally a subalgebra of $\mathcal{H}(\mathcal{D})$.

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