ACYCLIC CLUSTER ALGEBRAS VIA RINGEL-HALL ALGEBRAS

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1. Introduction

Realise acyclic cluster algebras via Hall algebras.

Original idea was from Caldero-Chapoton, just for A,D,E case. Then Caldero-Keller gave a cluster multiplication theorem, but only in the simply-laced case.

Using the correspondence due to BMRRT between clusters and tilting modules, this provides a way of realising the cluster algebras.

These results all rely on the work of Fomin-Zelevinsky.

Here we offer a realisation that is independent of the results of FZ, and which works for all symmetrisable acyclic cluster algebras, equivalently valued graphs without vertex loops, or symmetrisable generalised Cartan matrices.

2. The Main Theorem

Let Δ be a valued graph without vertex loops and let Q be an orientation of Δ containing no oriented cycle. Let k be a finite field and Λ the basic hereditary finite dimensional k-algebra corresponding to Q. This has a primitive idempotent e_i for each vertex $i = 1, \dots, n$ of Δ. Set $\mathcal{F} := \mathbb{Q}(x_1, \dots, x_n)$ to be the function field in n variables.

We consider the tilting pairs (σ, T), where $\sigma = \{\sigma_1, \dots, \sigma_r\}$ is a subset of vertices, $\epsilon_\sigma = \epsilon_{\sigma_1} + \cdots + \epsilon_{\sigma_r}$ is the associated idempotent, $\Lambda_\sigma = \Lambda/(\epsilon_\sigma)$ the factor algebra, and $T = T_{r+1} \oplus \cdots \oplus T_n$ is a (basic) tilting module for Λ_σ.

Given such a tilting pair (σ, T) together with $1 \leq k \leq n$ we define a new tilting pair (σ', T'), called the mutation in direction k. If $k \leq r$, then up to reordering we may assume that $k = r$. We set $\sigma' := \sigma \setminus \{\sigma_r\}$. The module T is now an insincere almost complete tilting module for $\Lambda_{\sigma'}$. Thus there exists a unique complement N and we set $T' := N \oplus T$.

Otherwise, we may assume that $k = r + 1$. Let $\overline{T} = T_{r+2} \oplus \cdots \oplus T_n$. This is an almost complete tilting module for Λ_σ. If $\overline{T}$ is sincere, there exists a unique complement $N \ncong T_{r+1}$. We set $\sigma' := \sigma$ and $T' := N \oplus \overline{T}$. If $\overline{T}$ is insincere, then there exists a unique vertex v of Λ_σ not in the support of $\overline{T}$. We set $\sigma' := \sigma \cup \{v\}$ and $T' := \overline{T}$.

We can also associate to the tilting pair (σ, T) the seed $\big(X_{(\sigma,T)}, \mathrm{B}_{(\sigma,T)}\big)$. Here $X_{(\sigma,T)} = \{X_1, \dots, X_n\}$ is the cluster given by

$$X_i := x_{\sigma_i} \quad \text{for } 1 \leq i \leq r, \qquad X_j := X_{T_j} \quad \text{for } r < j \leq n,$$

and where X_M is the image of the Caldero-Chapoton map, defined in terms of Hall numbers.

The matrix $\mathrm{B}_{(\sigma,T)}$ is skew-symmetrisable and is defined entirely in terms of the Euler form of Λ as follows. Define

$$t_i := \dim \mathrm{End}(S_{\sigma_i}) \quad \text{for } 1 \leq i \leq r, \qquad t_j := \dim \mathrm{End}(T_j) \quad \text{for } r < j \leq n$$

and write $\mathrm{D}_{(\sigma,T)} := \mathrm{diag}(t_i)$. Note that by Theorem 18 the entries of $\mathrm{D}_{(\sigma,T)}$ are, up to reordering, precisely the symmetrisers of Δ. Set $\mathrm{D}_\sigma := \mathrm{diag}(t_i)_{1\leq i\leq r}$ and $\mathrm{D}_T := \mathrm{diag}(t_j)_{r<j\leq n}$.

Define the matrix

$$\mathrm{H}_{(\sigma,T)} = \begin{pmatrix} \mathrm{H}_\sigma & \mathrm{H}_{\sigma T} \\ \mathrm{H}_{T\sigma} & \mathrm{H}_T \end{pmatrix} := \begin{pmatrix} \langle P_{\sigma_i}, P_{\sigma_j}\rangle_{i\neq j} & -\langle S_{\sigma_i}, T_j\rangle \\ -\langle T_i, S_{\sigma_j}\rangle & \langle T_i, T_j\rangle_{i\neq j} \end{pmatrix}.$$

Next set

$$\mathrm{M}_{(\sigma,T)} := \begin{pmatrix} \mathrm{M}_\sigma & \mathrm{M}_{\sigma T} \\ \mathrm{M}_{T\sigma} & \mathrm{M}_T \end{pmatrix},$$

where

$$\begin{aligned}
\mathrm{M}_\sigma \mathrm{D}_\sigma &= \mathrm{H}_\sigma\big(1+\mathrm{D}_\sigma^{-1}\mathrm{H}_\sigma\big)^{-1} = \big(1+\mathrm{H}_\sigma\mathrm{D}_\sigma^{-1}\big)^{-1}\mathrm{H}_\sigma \\
\mathrm{M}_T \mathrm{D}_T &= \mathrm{H}_T\big(1+\mathrm{D}_T^{-1}\mathrm{H}_T\big)^{-1} = \big(1+\mathrm{H}_T\mathrm{D}_T^{-1}\big)^{-1}\mathrm{H}_T \\
\mathrm{M}_{\sigma T} \mathrm{D}_T &= \mathrm{H}_{\sigma T}\big(1+\mathrm{D}_T^{-1}\mathrm{H}_T\big)^{-1} \\
\mathrm{M}_{T\sigma} \mathrm{D}_\sigma &= \big(1+\mathrm{H}_T\mathrm{D}_T^{-1}\big)^{-1}\mathrm{H}_{T\sigma}.
\end{aligned}$$

Finally write

$$\mathrm{B}_{(\sigma,T)}\mathrm{D}_{(\sigma,T)} := \mathrm{M}_{(\sigma,T)}\mathrm{D}_{(\sigma,T)} - \big(\mathrm{M}_{(\sigma,T)}\mathrm{D}_{(\sigma,T)}\big)^t.$$

Clearly $\mathrm{B}_{(\sigma,T)}$ is skew-symmetrisable with respect to $\mathrm{D}_{(\sigma,T)}$.

Main Theorem. *The mutation of tilting pairs (σ, T) coincides precisely with the mutation of seeds $\big(X_{(\sigma,T)}, \mathrm{B}_{(\sigma,T)}\big)$ as defined by Fomin and Zelevinsky. Thus the subalgebra of $\mathcal{F}$ generated by the $X_{(\sigma,T)}$ equals the cluster algebra of type Δ.*

We obtain the following immediate corollaries.

Laurent phenomenon

Every cluster can be expressed as a Laurent polynomial in the variables $x_1, \ldots, x_n$. This is immediate from the definition of the Caldero-Chapoton map.

Realisation of cluster variables

The cluster variables are in bijection with the exceptional modules, together with the functions $x_1, \ldots, x_n$. Moreover, the clusters are in bijection with the pairs (σ, T), which are in turn in bijection with cluster-tilting objects in the cluster category.

The seed is determined by the cluster

This is clear, since we have given an explicit formula for computing the skew-symmetrisable matrix $\mathrm{B}_{(\sigma,T)}$.

Finite type classification

An acyclic cluster algebra is of finite type, that is has only finite many seeds, if and only if it has only finitely many cluster variables, which is if and only if the graph is of Dynkin type. This is clear, since a basic hereditary finite dimensional algebra has finitely many exceptional objects if and only if it is Dynkin.

Denominator formula
The denominator of X_M for each module M is precisely $\underline{\dim} M$. This follows directly from the Caldero-Chapoton map as in Theorem 10.

Independence of orientation
Clearly the cluster algebra depends only on the equivalence class of seeds. By considering all sections of the preprojective component of the Auslander-Reiten quiver of Λ we obtain tilting modules corresponding to any orientation of Δ.

We note that the independence of orientation, the Laurent phenomenon and the finite type classification are due to Fomin and Zelevinsky [7, 8]. The realisation of cluster variables is due to Caldero and Chapoton [4]. The denominator formula and the determination of the seed by the cluster are due to Buan, Marsh, Reiten and Todorov [3], using the cluster category [2].

The proofs in this article are independent of the combinatorics in [7, 8], and are based on the realisation of the cluster variables [4] together with the cluster multiplication theorem by Caldero and Keller [5, 6]. However, our proof of this multiplication theorem is again new and follows more closely the original approach via Ringel-Hall algebras. In particular, we use the associativity of the Hall numbers together with Green's comultiplication formula.

3. Hereditary Algebras

Let Δ be a valued graph without vertex loops and vertices $\Delta_0 = \{1, \dots, n\}$. The edges of Δ are of the form

$$i \xrightarrow{(a_{ij}, a_{ji})} j$$

for non-negative integers a_{ij} and a_{ji}, which we collect together in a matrix A. The generalised Cartan matrix corresponding to Δ is thus $\mathrm{C} = 2\mathrm{I} - \mathrm{A}$. We furthermore assume that A (equivalently C) is diagonalisable: that is AD is symmetric for some $\mathrm{D} = \mathrm{diag}(d_1, \dots, d_n)$, where the d_i are positive integers. For simplicity we assume that the greatest common divisor of the d_i is 1.

Let Q be an orientation of Δ containing no oriented cycle: that is, we replace each valued edge by a valued arrow. Define $\mathrm{R} = \mathrm{R}_Q := (r_{ij})$ via $r_{ij} := a_{ij}$ if there is an arrow $i \to j$ in Q, and 0 otherwise. Let $\mathrm{R}' = (r'_{ij}) = \mathrm{R}_{Q^{\mathrm{op}}}$, so $\mathrm{A} = \mathrm{R} + \mathrm{R}'$.

Let k be a finite field and let Λ be the k-species associated to Q, formed as follows. We take a semisimple algebra

$$\Lambda_0 := \prod_i k_i \epsilon_i, \quad k_i/k \text{ a field extension of degree } d_i$$

(so the ϵ_i give a complete set of primitive idempotents) and a k-vector space

$$\Lambda_1 := \bigoplus_{j,i} k_{ji}, \quad k_{ji}/k \text{ a field extension of degree } r'_{ji} d_i.$$

We give Λ_1 the structure of a Λ_0-bimodule by taking the natural k_j-k_i-bimodule structure on k_{ji}. The k-species Λ is the tensor algebra of Λ_1 over Λ_0.

We note that Λ is a finite dimensional basic hereditary k-algebra, and every such algebra arises in this manner. The category $\mathrm{mod}\Lambda$ of finite dimensional Λ-modules is equivalent to the category of k-representations of the species associated to Q, and a k-representation is a collection $M = (M_i, \phi_{ij})$ where M_i is a finite dimensional

k_i-vector space and ϕ_{ij} is a k_j-linear map

$$\phi_{ij} : k_{ji} \otimes_{k_i} M_i \to M_j \quad \text{for an arrow } i \to j.$$

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Z}^n$. The dimension vector of a module M is given as

$$\underline{\dim}\, M := \sum_i \dim_{k_i} M_i e_i = \sum_i (\dim_k M_i / d_i) e_i \in \mathbb{Z}^n.$$

In this way, we identify the Grothendieck group of Λ with $\mathbb{Z}^n$. The Euler form on modΛ is given by

$$\langle M, N \rangle := \dim_k \mathrm{Hom}(M, N) - \dim_k \mathrm{Ext}^1(M, N)$$

and depends only on the dimension vectors of M and N. We note that the matrix representing this form is $(\mathrm{I} - \mathrm{R})\mathrm{D}$.

We introduce the notation

$$[A, B] := \dim_k \mathrm{Hom}(A, B) \quad \text{and} \quad [A, B]^1 := \dim_k \mathrm{Ext}^1(A, B).$$

Also, unadorned Hom and Ext spaces will always refer to Λ-homomorphisms and extensions.

Lemma 1. *The matrices* R *and* R' *are related via* $\mathrm{R}'\mathrm{D} = \mathrm{D}\mathrm{R}^{\mathrm{tr}}$. *The Auslander-Reiten translate* τ *acts on* modΛ *and provides the isomorphism*

$$\mathrm{Ext}^1(M, N) \cong D\,\mathrm{Hom}(N, \tau M),$$

where $D = \mathrm{Hom}_k(-, k)$. *It follows that for a module* M *without projective summands,*

$$(\underline{\dim}\, M)\mathrm{R} + (\underline{\dim}\, \tau M)\mathrm{R}' = \underline{\dim}\, M + \underline{\dim}\, \tau M.$$

Moreover, for a projective module P *and an injective module* I *we have*

$$(\underline{\dim}\, P)\mathrm{R} = \underline{\dim}\, \mathrm{rad} P, \quad (\underline{\dim}\, I)\mathrm{R}' = \underline{\dim}\, I - \underline{\dim}\, \mathrm{soc} I.$$

We have an Auslander-Reiten sequence $0 \to \tau M \to E \to M \to 0$ for each non-projective indecomposable module M. The Auslander-Reiten quiver Γ_Q has as vertices the isomorphism classes of indecomposable Λ-modules and valued arrows determined by irreducible maps. We recall that if $0 \to \tau M \to E \to M \to 0$ is an Auslander-Reiten sequence, then $E = \bigoplus N^n$, where the sum runs over all arrows $[N] \xrightarrow{(n,m)} [M]$ in Γ_Q.

Let Q^* be the quiver obtained by reversing the orientation of Q and reversing the valuations of all edges. Thus Q^* has corresponding generalised Cartan matrix C^{tr}. The subquiver of the Auslander-Reiten quiver Γ_Q consisting of either the indecomposable projective modules or the indecomposable injective modules is isomorphic to Q^*.

The translation quiver $\mathbb{Z}Q^*$ has vertices $\tau^r v$ for v a vertex of Q^* and $r \in \mathbb{Z}$ and arrows

$$\tau^{r+1} v \xrightarrow{(b,a)} \tau^r u \xrightarrow{(a,b)} \tau^r v \quad \text{for } r \in \mathbb{Z} \text{ and } u \xrightarrow{(a,b)} v \text{ in } Q^*.$$

Then τ is an automorphism of $\mathbb{Z}Q^*$ given by shifting one place to the left.

Let $\mathcal{P}_Q$ and $\mathcal{I}_Q$ be the connected components of Γ_Q containing the projective and injective modules respectively. Then both $\mathcal{P}_Q$ and $\mathcal{I}_Q$ are full subquivers of the translation quiver $\mathbb{Z}Q^*$. In particular, if Δ is Dynkin, then $\Gamma_Q = \mathcal{P}_Q = \mathcal{I}_Q$ is a full subquiver of $\mathbb{Z}Q^*$.

4. Ringel-Hall Algebras

We retain the notation introduced in the previous section. In particular, Δ is a valued graph, Q an orientation of Δ and Λ the k-species associated to Q, where k is a finite field. Let A, D and R be the corresponding matrices.

For Λ-modules A, B and M we define the Hall number

$$F_{AB}^{M} := |\{U \leq M \mid U \cong B, M/U \cong A\}|$$

and write $a_M := |\mathrm{Aut}(M)|$.

Consider the set $\mathcal{E}_{AB}^{M} := \{(b,a) \mid 0 \to B \xrightarrow{b} M \xrightarrow{a} A \to 0 \text{ is exact}\}$. Sending such a pair (b,a) to $\mathrm{Im}\, b \leq M$ gives a surjection onto all submodules $U \leq M$ such that $U \cong B$ and $M/U \cong A$, and the fibres are isomorphic to $\mathrm{Aut}\, A \times \mathrm{Aut}\, B$. Thus we have the formula

$$F_{AB}^{M} = \left|\mathcal{E}_{AB}^{M}\right| / a_A a_B.$$

Proposition 2 (Riedtmann). *We have the formula*

$$F_{AB}^{M} = q^{-[A,B]} \varepsilon_{AB}^{M} a_M / a_A a_B,$$

where ε_{AB}^{M} is the size of the subset $\mathrm{Ext}^1(A,B)_M \subset \mathrm{Ext}^1(A,B)$ consisting of those classes of short exact sequences with middle term isomorphic to M.

Theorem 3 (Ringel). *If Δ is a Dynkin graph and if μ, α, β are isomorphism classes of modules, then there exists an integer polynomial $F_{\alpha\beta}^{\mu}(t) \in \mathbb{Z}[t]$ such that for a finite field k and modules M, A, B of classes μ, α, β we have $F_{AB}^{M} = F_{\alpha\alpha}^{\mu}(|k|)$.*

In fact, for an arbitrary graph Δ, the same result is true for the numbers $F_{A\beta}^{M} := \sum_B F_{AB}^{M}$ whenever A, M are preinjective modules, and dually $F_{\alpha B}^{M} := \sum_A F_{AB}^{M}$ whenever B, M are preprojective modules.

Conjecture 4. *Given a module M and a composition series $M = M_1 > M_2 > \cdots > M_n > M_{n+1} = 0$, the type of the series is the ordered sequence $(v_1, v_2, \ldots, v_n)$, where $M_i/M_{i+1} \cong S_{v_i}$, the simple module supported at vertex v_i. Let $\mathrm{comp}(M)$ denote the set of all possible types of composition series of M. Given any set $\mathcal{S}$ of types of composition series of modules of dimension vector α, define*

$$\mathrm{comp}(\mathcal{S}) := \{M \mid \mathrm{comp}(M) = \mathcal{S}\}.$$

In this way we obtain an affine $\mathbb{Z}$-scheme $\mathrm{comp}(\mathcal{S})$ whose image in the affine variety $\mathrm{Rep}(\alpha)$ is locally closed.

The conjecture now reads as follows:
Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three sets of types of composition series of modules of dimension vector α, β and $\gamma = \alpha + \beta$ respectively. Then there exists an integer polynomial $F_{\mathcal{AB}}^{\mathcal{C}}$ such that over any finite field k and any k-representation $C \in \mathrm{comp}(\mathcal{C})$,

$$F_{\mathcal{AB}}^{\mathcal{C}}(|k|) = |\{U \leq C \mid U \in \mathrm{comp}(\mathcal{B}), C/U \in \mathrm{comp}(\mathcal{A})\}|.$$

Theorem 5. *For modules M, W, X, Y, Z we have the formula*

$$\sum_X F_{AB}^{X} F_{XC}^{M} = \sum_X F_{AX}^{M} F_{BC}^{X}.$$

For a proof, see Ringel. Also Hubery for a generalisation to arbitrary exact categories.

Using this we can define the Ringel-Hall algebra $\mathcal{H}(\Lambda)$. This has a $\mathbb{Z}$-basis $u_{[M]}$ indexed by the isomorphism classes of Λ-modules and multiplication

$$u_{[N]}u_{[M]} := \sum_{[X]} F^X_{NM} u_{[X]}.$$

This is an associative unital algebra.

The Ringel-Hall algebra has a natural coassociative comultiplication

$$r(u_{[M]}) := \sum_{[A],[B]} q^{\langle A,B\rangle} F^M_{AB} \frac{a_A a_B}{a_M} u_{[A]} \otimes u_{[B]}$$

with respect to which the Ringel-Hall algebra becomes a twisted bialgebra: that is, we twist the multiplication on the tensor product $\mathcal{H}(\Lambda) \otimes \mathcal{H}(\Lambda)$ by setting

$$u_{[A]} \otimes u_{[B]} \cdot u_{[C]} \otimes u_{[D]} := q^{\langle C,B\rangle} u_{[A]}u_{[C]} \otimes u_{[B]}u_{[D]}.$$

The comultiplication then satisfies

$$r(u_{[M]}u_{[N]}) = r(u_{[M]}) \cdot r(u_{[N]}).$$

The proof of this reduces to the following theorem.

Theorem 6 (Green)**.** *For fixed modules M, N, X, Y over a k-species Λ we have the identity*

$$\sum_E F^E_{MN} F^E_{XY} \frac{a_M a_N a_X a_Y}{a_E} = \sum_{A,B,C,D} q^{-\langle A,D\rangle} F^M_{AB} F^N_{CD} F^X_{AC} F^Y_{BD} a_A a_B a_C a_D.$$

Using Riedtmann's formula, we can rewrite Green's formula as

$$\sum_E \varepsilon^E_{MN} F^E_{XY} = \sum_{A,B,C,D} q^{[M,N]-[A,C]-[B,D]-\langle A,D\rangle} F^M_{AB} F^N_{CD} \varepsilon^X_{AC} \varepsilon^Y_{BD}. \tag{1}$$

5. The Caldero-Chapoton Map

We keep the notation of the previous sections. Let $\mathcal{F} = \mathbb{Q}(x_1, \dots, x_n)$ be a transcendental extension of $\mathbb{Q}$.

For a module M we define the element

$$X_M := \sum_{\alpha,\beta} F^M_{\alpha\beta}(1) \underline{x}^{\beta\mathrm{R}+\alpha\mathrm{R}'} / \underline{x}^{\underline{\dim} M},$$

where $\underline{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Lemma 7. *Let $\eta : 0 \to N \to E \xrightarrow{\pi} M \to 0$ be a short exact sequence and let $B \leq M$ and $D \leq N$ be submodules. Set $\mathcal{U}^\eta_{BD} := \{U \leq E \mid \pi(U) = B, U \cap N = D\}$. If $\mathcal{U}^E_{BD}$ is not empty, then it is isomorphic to* $\mathrm{Hom}(B, N/D)$.

Proof. Suppose that $U_0 \in \mathcal{U}^E_{BD}$ and $f \in \mathrm{Hom}(B, N/D)$. Define

$$U_f := \{u + n \in E \mid u \in U_0, n \in N, f\pi(u) = \bar{n}\},$$

where $N \to N/D$, $n \mapsto \bar{n}$ is the canonical projection. It follows easily that $U_f \in \mathcal{U}^E_{BD}$.

Conversely, let $U \in \mathcal{U}^E_{BD}$ be another submodule. Given $b \in B$ pick $u_0 \in U_0$ and $u \in U$ mapping under π to b. Then $u - u_0 \in N$ and $f(b) := \overline{u - u_0}$ is well-defined. It is not hard to see that $U = U_f$. □

Corollary 8. (1) *For all M and N we have $X_M X_N = X_{M\oplus N}$.*

(2) *For M indecomposable non-projective we have $X_M X_{\tau M} = 1 + X_E$, where E is the middle term of the Auslander-Reiten sequence ending in M.*

Proof. We shall abuse notation and write summations over modules when we really mean isomorphism classes.

In the split case we note that, given any $B \leq M$ and $D \leq N$, there is the submodule $D \oplus B \leq N \oplus M$ and so $\mathcal{U}_{BD}^{N\oplus M}$ is never empty. The previous lemma then provides us with a bijection between the sets $\{U \leq N \oplus M \mid \underline{\dim}\, U = \eta\}$ and $\{(B, D, f) \mid B \leq M, D \leq N, f : B \to N/D, \underline{\dim}\, B + \underline{\dim}\, D = \eta\}$ for every dimension vector η. Therefore

$$\sum_{X,Y} F_{XY}^{N\oplus M} \underline{x}^{\eta \mathrm{R}+\xi \mathrm{R}'} = \sum_{A,B,C,D} q^{[B,C]} F_{AB}^{M} F_{CD}^{N} \underline{x}^{(\beta+\delta)\mathrm{R}+(\alpha+\gamma)\mathrm{R}'}.$$

Are these given by polynomials? If so, evaluating at $q = 1$ yields the required identity.

In the almost split case, the bijections above become injections, with the only missing points those of the form $(M, 0, f)$ for $f : M \to \tau M$. Using the property that $(\underline{\dim}\, M)\mathrm{R} + (\underline{\dim}\, \tau M)\mathrm{R}' = \underline{\dim}\, E$ we have the identity

$$q^{[M,\tau M]} \underline{x}^{\underline{\dim}\, E} + \sum_{X,Y} F_{XY}^{E} \underline{x}^{\eta \mathrm{R}+\xi \mathrm{R}'} = \sum_{A,B,C,D} q^{[B,C]} F_{AB}^{M} F_{CD}^{N} \underline{x}^{(\beta+\delta)\mathrm{R}+(\alpha+\gamma)\mathrm{R}'}.$$

Now set $q = 1$ as before. □

The above result implies that we have a function $X : \Gamma_\Lambda \to \mathcal{F}$, $[M] \mapsto X_M$ which sends the mesh relations to algebraic equations: that is, the Auslander-Reiten sequence $0 \to \tau M \to \bigoplus_i B_i^{b_i} \to M \to 0$ corresponds to the equation

$$X_{\tau M} X_M = 1 + \prod_i X_{B_i}^{b_i}. \tag{2}$$

Lemma 9.

$$X_{P_i} x_i = 1 + \big(X_{\mathrm{rad} P_i}\big) \underline{x}^{e_i \mathrm{R}'} \quad \textit{and} \quad X_{I_i} x_i = 1 + \big(X_{I_i/\mathrm{soc} I_i}\big) \underline{x}^{e_i \mathrm{R}}.$$

Proof. Every submodule of P_i is either equal to P_i or is contained in $\mathrm{radP_i}$. Thus we have

$$\begin{aligned} X_{P_i} \underline{x}^{\underline{\dim}\, P_i} &= \underline{x}^{(\underline{\dim}\, P_i)\mathrm{R}} + \sum_{A,B} F_{AB}^{\mathrm{rad} P_i} \underline{x}^{\beta \mathrm{R}+(\alpha+e_i)\mathrm{R}'} \\ &= \underline{x}^{\underline{\dim}\, \mathrm{rad} P_i} + \big(X_{\mathrm{rad} P_i}\big) \underline{x}^{\underline{\dim}\, \mathrm{rad} P_i + e_i \mathrm{R}'}. \end{aligned}$$

The second statement is dual. □

Let $\Gamma_{\mathcal{D}(\Lambda)}$ be the Auslander-Reiten quiver of the bounded derived category of $\mathrm{mod}\Lambda$. This is obtained by taking copies of Γ_Λ indexed by the integers and adding arrows $[I_i]_r \xrightarrow{(a_{ij}, a_{ji})} [P_j]_{r+1}$ for $i \xrightarrow{(a_{ij}, a_{ji})} j$ an arrow in Q and $r \in \mathbb{Z}$. In particular, $\tau[P_i]_{r+1} = [I_i]_r$. We have a natural embedding of Γ_Λ in $\Gamma_{\mathcal{D}(\Lambda)}$ sending $[M]$ to $[M]_0$.

The above lemma implies that, if we extend X to the left and right of Γ_Λ inside $\Gamma_{\mathcal{D}(\Lambda)}$ via the formula (2), then $X_{\tau^- I_i} = x_i = X_{\tau P_i}$. In particular, X descends to a function on $\Gamma_{\mathcal{C}(\Lambda)}$, the quotient of $\Gamma_{\mathcal{D}(\Lambda)}$ given by identifying $\tau^- I_i$ with τP_i.

We note that $X : \Gamma_{\mathcal{C}(\Lambda)} \to \mathcal{F}$ is independent of the orientation of Q determining Λ. That is, if Q' is another orientation of Δ without oriented cycles and if Λ' is the corresponding finite dimensional algebra, then $\Gamma_{\mathcal{C}(\Lambda')} \cong \Gamma_{\mathcal{C}(\Lambda)}$ as translation quivers. Thus we can write $\Gamma_{\mathcal{C}(\Delta)}$ and the only difference in the maps is in which objects get sent to the generators $x_1, \dots, x_n$ of $\mathcal{F}$.

Remark. The translation quiver $\Gamma_{\mathcal{C}(\Lambda)}$ is precisely the Auslander-Reiten quiver of the cluster category $\mathcal{C}(\Lambda)$ associated to Λ. That is, the orbit category of the bouned derived category $\mathcal{D}(\Lambda)$ with respect to the functor $\tau^{-1}[1]$.

6. The Denominator Theorem

In this section we prove the following denominator formula directly from the Caldero-Chapoton map.

Theorem 10. *For each module M, the polynomial $X_M \underline{x}^{\underline{\dim} M}$ is not divisible by any x_i.*

We construct for each vertex l a dimension vector β satisfying the following three properties:

(1) $\beta \leq \mu$ is uniquely determined by $\theta := \beta \mathrm{R} + (\mu - \beta)\mathrm{R}'$;
(2) $\theta_l = 0$, so x_l does not divide $\underline{x}^\theta$;
(3) there is a unique submodule $B \leq M$ of dimension vector β, hence $\underline{x}^\theta$ occurs with coefficient 1 in $X_M \underline{x}^\theta$.

The theorem follows immediately.

We fix the module M and the vertex l, and assume that l is not a sink. We define subsets $\tilde{J}_0 \subset \tilde{J}_1 \subset \cdots$ of the set of vertices $\{1, \dots, n\}$ such that $\bigcup_r \tilde{J}_r = \{1, \dots, n\}$ via

$$\tilde{J}_0 := \{l\}, \quad \tilde{J}_{2r+1} := \bigcup_{j \in \tilde{J}_{2r}} \mathrm{supp} P_j, \quad \tilde{J}_{2r+2} := \bigcup_{j \in \tilde{J}_{2r+1}} \mathrm{supp} I_j.$$

That is, $\tilde{J}_{2r+1}$ is the closure under successors of $\tilde{J}_{2r}$, and $\tilde{J}_{2r+2}$ is the closure under predecessors of $\tilde{J}_{2r+1}$. This sequence determines the partition $J_r := J_r \setminus J_{r-1}$.

We note that if $j \in J_{2r+1}$ and $j \to i$, then $i \in J_{2r+1}$, whereas if $i \to j$, then $i \in J_{2r} \cup J_{2r+1} \cup J_{2r+2}$. Similarly, if $j \in J_{2r+2}$ and $i \to j$, then $i \in J_{2r+1}$, whereas if $j \to i$, then $i \in J_{2r+1} \cup J_{2r+2} \cup J_{2r+3}$.

We define β via

$$\beta_j := 0 \text{ if } j \in J_{2n}, \quad \beta_j := \mu_j \text{ if } j \in J_{2n+1}.$$

It follows from the remark above that M has a unique submodule of dimension vector β.

The corresponding monomial has exponent θ, where

$$\theta_j := \sum_{i \to j} \beta_i a_{ij} + \sum_{j \to i} (\mu_i - \beta_i) a_{ij}.$$

In particular, $\theta_l = 0$.

It remains to show Property 1.

Lemma 11. *The dimension vector $\beta \leq \mu$ is uniquely determined by $\theta := \beta \mathrm{R} + (\mu - \beta)\mathrm{R}'$.*

Proof. Suppose that $\beta' \leq \mu$ and $\beta' \mathrm{R} + (\mu - \beta')\mathrm{R}' = \theta$. Since l is not a sink there exists an arrow $l \to j$, and we may assume that there is no other path from l to j. Then $j \in J_1$ and $\theta_j = 0$. It follows that $\beta'_i = \beta_i$ for all $i \to j$ and all $j \to i$. In particular, $\beta'_l = \beta_l = 0$.

Now suppose by induction that $\beta'_i = \beta_i$ for all $i \in \tilde{J}_{2r}$. Let $v \in J_{2r+1}$ and let $j \to v$ with $j \in J_{2r} \cup J_{2r+1}$. If $j \in J_{2r}$, then our two expressions for θ_j yield $\sum_{j\to i\in J_{2r+1}}(\mu_i - \beta'_i)a_{ij} = 0$, and hence $\beta'_v = \beta_v = \mu_v$. Otherwise, if $j \in J_{2r+1}$, then by induction we may assume that $\beta'_i = \beta_i = \mu_i$ for all $j \leftarrow i \in J_{2r+1}$. We thus obtain $\sum_{j\leftarrow i\in J_{2r+2}} \beta'_i a_{ij} + \sum_{j\to i}(\mu_i - \beta'_i)a_{ij} = 0$. Thus $\beta'_v = \beta_v = \mu_v$. It follows that $\beta'_i = \beta_i = \mu_i$ for all $i \in J_{2r+1}$.

Similarly, suppose that $\beta'_i = \beta_i$ for all $i \in \tilde{J}_{2r+1}$. Let $v \in J_{2r+2}$ and let $v \to j \in J_{2r+1} \cup J_{2r+2}$. If $j \in J_{2r+1}$, then $\sum_{j\leftarrow i\in J_{2r+2}} \beta'_i a_{ij} = 0$, so $\beta'_v = \beta_v = 0$. If $j \in J_{2r+2}$, then by induction we may assume that $\beta'_i = \beta_i = 0$ for all $j \to i \in J_{2r+2}$. Therefore $\sum_{j\leftarrow i} \beta'_i a_{ij} + \sum_{j\to i\in J_{2r+3}}(\mu_i - \beta'_i)a_{ij} = 0$ and hence $\beta'_v = \beta_v = 0$.

The result follows by induction. □

If l is a sink, then we set $\tilde{J}_0 = \emptyset$ and $\tilde{J}_1 = \{l\}$, and define $\tilde{J}_r$, J_r and β exactly as before. We see again that $\theta = \beta\mathrm{R} + (\mu - \beta)\mathrm{R}'$ satisfies $\theta_l = 0$ and the same lemma as before shows that $\beta \le \mu$ is the unique dimension vector giving rise to θ. Note that, in order to show $\beta'_l = \mu_l$, we consider $\theta_j = 0$ for $j \to l$ such that there is no other path from j to l.

This completes the proof of the theorem.

7. The Caldero-Keller Cluster Multiplication Theorem

In this section we give a reformulation and new proof of the Caldero-Keller multiplication theorem (and in fact generalise their result). The new proof is more in keeping with our approach via Ringel-Hall algebras. In fact, as we shall see, the proof relies upon the associativity of Hall numbers together with Green's Theorem.

Theorem 12 (Caldero-Keller). *Let M and N be modules. Then $[M,N]^1 X_M X_N$ equals the sum*

$$\sum_{E\not\cong M\oplus N} \frac{\varepsilon^E_{MN}}{q-1}\bigg|_{q-1} X_E \quad + \sum_{A',D,I} \frac{|\mathrm{Hom}(N,\tau M)_{DA'I}|}{q-1}\bigg|_{q-1} X_A X_D X_{I[-1]}.$$

Here we have written $\mathrm{Hom}(N,\tau M)_{DA'I}$ for the set of (non-zero) homomorphisms $\theta : N \to \tau M$ with kernel D and cokernel $\tau A' \oplus I$ for some injective I and some A' without projective summands. Also, P denotes the projective summand of M and $A = P \oplus A'$.

Recall that $X_{I[-1]} = X_{\tau^- I} = \underline{x}^{\dim \mathrm{soc} I}$, and similarly $X_{P[1]} = \underline{x}^{\dim P/\mathrm{rad} P}$.

We first show that this is a generalisation of the original Caldero-Keller Theorem. Let $\mathcal{C} = \mathcal{C}(\Lambda)$ denote the cluster category of Λ and assume that $[N,M]^1 = 0$.

Consider the first sum in Theorem 12. By the assumptions on M and N we know that $\mathrm{Ext}^1_{\mathcal{C}}(M,N) \cong \mathrm{Ext}^1_\Lambda(M,N)$. Therefore we have a bijection between short exact sequences

$$\eta : 0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$$

and triangles

$$N \xrightarrow{f} E \xrightarrow{g} M \xrightarrow{\eta} N[1].$$

It follows that $\mathrm{Ext}^1_{\mathcal{C}}(M,N)_E \cong \mathrm{Ext}^1_\Lambda(M,N)_E$ for all E. Using the vector space structure on this set, we see that k^* acts freely on each $\mathrm{Ext}^1_\Lambda(M,N)_E$ for $E \not\cong M \oplus N$. Thus $\frac{\varepsilon^E_{MN}}{q-1}$ is precisely the number of points of the set $\mathbb{P}(\mathrm{Ext}^1_{\mathcal{C}}(M,N)_E)$

and taking the Euler characteristic corresponds to evaluating this polynomial at $q = 1$.

Now consider the second sum in Theorem 12. Write $M = P \oplus M'$ where P is projective and M' contains no projective summand. Then $\mathrm{Ext}^1_{\mathcal{C}}(N, M) \cong \mathrm{Hom}_\Lambda(N, \tau M')$.

Lemma 13. *Under the bijection* $\mathrm{Hom}_\Lambda(N, \tau M') \cong \mathrm{Ext}^1_{\mathcal{C}}(N, M)$*, the morphism* $\theta : N \to \tau M'$ *is sent to the triangle*

$$M \to E \to N \xrightarrow{\theta} \tau M,$$

where $E \cong P \oplus \mathrm{Ker}\,\theta \oplus (\mathrm{Coker}\,\theta)[-1]$*. Moreover, the split triangle corresponds to the zero morphism, and* k^* *acts on non-zero morphisms just by multiplication, hence acts on each of the sets* $\mathrm{Hom}_\Lambda(N, \tau M')_{DA'I}$*.*

Proof. Let $\theta \in \mathrm{Hom}_\Lambda(N, \tau M')$ be any homomorphism. We have an exact sequence

$$0 \to D \to N \xrightarrow{\theta} \tau M' \to \tau A' \oplus I \to 0,$$

where I is injective and A' contains no projective summand. This yields the exact sequences

$$\eta : 0 \to D \xrightarrow{d} N \xrightarrow{c} C \to 0 \qquad \text{and} \qquad \eta' : 0 \to C \xrightarrow{g} \tau M' \xrightarrow{f} \tau A' \oplus I \to 0,$$

where $\theta = gc$. Thus, in the bounded derived category $\mathcal{D} = \mathcal{D}(\Lambda)$, we have the triangles

$$N \xrightarrow{c} C \xrightarrow{\eta} D[1] \xrightarrow{-d[1]} N[1] \qquad \text{and} \qquad C \xrightarrow{g} \tau M' \xrightarrow{f} \tau A' \oplus I \xrightarrow{\eta'} C[1].$$

Using the octahedral axiom we obtain triangles

$$N \xrightarrow{\theta} \tau M' \to E[1] \to N[1] \qquad \text{and} \qquad D[1] \to E[1] \to \tau A' \oplus I \xrightarrow{\eta[1] \circ \eta'} D[2].$$

Now $\eta[1] \circ \eta' \in \mathrm{Hom}_{\mathcal{D}}(\tau A' \oplus I, D[2]) = 0$ since Λ is hereditary. Therefore the second triangle is split and $E \cong \tau A'[-1] \oplus D \oplus I[-1]$. The canonical functor $\mathcal{D} \to \mathcal{C}$ is a triangle functor, and $\tau A'[-1] \cong A'$ in $\mathcal{C}$, so we obtain the following triangle in $\mathcal{C}$

$$M' \to A' \oplus D \oplus I[-1] \to N \xrightarrow{\theta} \tau M'.$$

Finally, since $\mathrm{Hom}_{\mathcal{C}}(N, P[1]) = 0$, we can add the triangle

$$P \to P \to 0 \to P[1]$$

to obtain the triangle

$$M \to A \oplus D \oplus I[-1] \to N \xrightarrow{\theta} \tau M,$$

where $A = A' \oplus P$. This construction yields every such triangle, since we have the bijection $\mathrm{Ext}^1_{\mathcal{C}}(N, M) \cong \mathrm{Hom}_\Lambda(N, \tau M')$. $\square$

We have thus shown that our theorem generalises the original Caldero-Keller Multiplication Theorem, which states that for objects M and N in $\mathcal{C}$ such that $\dim \mathrm{Ext}^1_{\mathcal{C}}(M, N) = 1$, the product $\dim \mathrm{Ext}^1_{\mathcal{C}}(M, N) \cdot X_M X_N$ equals

$$\sum_{E \ncong M \oplus N} \left. \frac{|\mathrm{Ext}^1_{\mathcal{C}}(M, N)_E|}{q-1} \right|_{q-1} X_E \quad + \sum_{E \ncong M \oplus N} \left. \frac{|\mathrm{Ext}^1_{\mathcal{C}}(N, M)_E|}{q-1} \right|_{q-1} X_E.$$

We now prove Theorem 12.

Define

$$\sigma_1 := \sum_{X,Y,E\ncong M\oplus N} \frac{\varepsilon^E_{MN}}{q-1} F^E_{XY} \underline{x}^{\eta\mathrm{R}+\xi\mathrm{R}'-(\mu+\nu)}.$$

Then $\sigma_1|_{q=1}$ is precisely the first term in the Caldero-Keller Theorem.

Lemma 14.

$$\sigma_1 = \sum_{A,B,C,D} q^{\langle B,C\rangle} \frac{q^{[M,N]^1} - q^{[B,C]^1}}{q-1} F^M_{AB} F^N_{CD} \underline{x}^{(\beta+\delta)\mathrm{R}+(\alpha+\gamma)\mathrm{R}'-(\mu+\nu)}.$$

Proof. This is just an application of Green's Theorem, or rather the reformulation given in (1). We have

$$\sum_{E,X,Y} \varepsilon^E_{MN} F^E_{XY} \underline{x}^{\eta\mathrm{R}+\xi\mathrm{R}'} = \sum_{\substack{A,B,C,D\\X,Y}} q^{[M,N]-[A,C]-[B,D]-\langle A,D\rangle} F^M_{AB} F^N_{CD} \varepsilon^X_{AC} \varepsilon^Y_{BD} \underline{x}^{\eta\mathrm{R}+\xi\mathrm{R}'}.$$

We observe that $\sum_X \varepsilon^X_{AC} = q^{[A,C]^1}$ and $\xi = \alpha+\gamma$, and similarly for Y. Using the properties of the Euler form we therefore obtain that

$$\sum_{E,X,Y} \varepsilon^E_{MN} F^E_{XY} \underline{x}^{\eta\mathrm{R}+\xi\mathrm{R}'} = \sum_{A,B,C,D} q^{[M,N]^1+\langle B,C\rangle} F^M_{AB} F^N_{CD} \underline{x}^{(\beta+\delta)\mathrm{R}+(\alpha+\gamma)\mathrm{R}'}.$$

Now consider what happens on the left hand side when $E \cong N\oplus M$. We know that $\varepsilon^{N\oplus M}_{MN} = 1$ and so Lemma 7 gives

$$\sum_{X,Y} F^{N\oplus M}_{XY} \underline{x}^{\eta\mathrm{R}+\xi\mathrm{R}'} = \sum_{A,B,C,D} q^{[B,C]} F^M_{AB} F^N_{CD} \underline{x}^{(\beta+\delta)\mathrm{R}+(\alpha+\gamma)\mathrm{R}'}.$$

Hence

$$\begin{aligned}\sigma_1 &= \sum_{X,Y,E\ncong M\oplus N} \frac{\varepsilon^E_{MN}}{q-1} F^E_{XY} \underline{x}^{\eta\mathrm{R}+\xi\mathrm{R}'-(\mu+\nu)}\\ &= \sum_{A,B,C,D} q^{\langle B,C\rangle} \frac{q^{[M,N]^1} - q^{[B,C]^1}}{q-1} F^M_{AB} F^N_{CD} \underline{x}^{(\beta+\delta)\mathrm{R}+(\alpha+\gamma)\mathrm{R}'-(\mu+\nu)},\end{aligned}$$

finishing the proof. □

Define

$$\sigma_2 := \sum_{\substack{A,D,I\\K,L,X,Y}} q^{\langle \mu-\kappa,\nu-\eta\rangle} \frac{|\mathrm{Hom}(N,\tau M)_{A'DI}|}{q-1} F^A_{KL} F^D_{XY} \underline{x}^{(\lambda+\eta)\mathrm{R}+(\kappa+\xi)\mathrm{R}'-(\alpha+\delta)+\underline{\dim}\,\mathrm{soc} I}$$

where $A = P\oplus A'$ and we consider only non-zero morphisms: that is, $D \ncong N$. Then $\sigma_2|_{q=1}$ is precisely the second term in the Caldero-Keller Theorem. The weighting $q^{\langle \mu-\kappa,\nu-\eta\rangle}$ is not essential, but will make the final step easier.

Lemma 15. *We have that*

$$|\mathrm{Hom}(N,\tau M')_{DA'I}| = \sum_{B',C} a_C F^M_{AB'} F^N_{CD} F^{\tau B'}_{IC},$$

where B' contains no projective summand and $A = A'\oplus P$.

Proof. Given such a morphism $\theta : N \to \tau M'$ we have exact sequences

$$0 \to D \xrightarrow{d} N \xrightarrow{c} C \to 0 \quad \text{and} \quad 0 \to C \xrightarrow{g} \tau M' \xrightarrow{f} \tau A' \oplus I \to 0$$

such that $gc = \theta$. Thus there is a map $\mathcal{E}^N_{CD} \times \mathcal{E}^M_{\tau A' \oplus IC}$ to $\mathrm{Hom}(N, \tau M')$ sending $\big((d,c),(g,f)\big)$ to gc. The fibres are isomorphic to $\mathrm{Aut}(\tau A' \oplus I) \times \mathrm{Aut}\, C \times \mathrm{Aut}\, D$. Hence

$$|\mathrm{Hom}(N, \tau M')_{A'DI}| = \sum_C a_C F^N_{CD} F^{\tau M'}_{\tau A' \oplus IC}.$$

Taking the pull-back of the second sequence and using the Auslander-Reiten translate yeilds the two short exact sequences

$$0 \to C \to \tau B' \to I \to 0 \quad \text{and} \quad 0 \to B' \to M' \to A' \to 0$$

where B' contains no projective summand. Since B' contains no projective summand, $[B', P] = 0$ and we obtain the sequence

$$0 \to B' \to M \to A \to 0.$$

where $A = A' \oplus P$.

The proof of the associativity of the Hall numbers, Theorem 5 uses this pull-back/push-out construction. Hence we see that

$$F^{\tau M'}_{\tau A' \oplus IC} = \sum_{B'} F^{\tau M'}_{\tau A' \tau B'} F^{\tau B'}_{IC} = \sum_{B'} F^{M'}_{AB'} F^{\tau B'}_{IC}.$$

Putting this together yields the result. □

We can now write

$$\sigma_2 := \sum_{\substack{A,B',C\neq 0,D\\ I,K,L,X,Y}} q^{\langle \mu-\kappa, \nu-\eta\rangle} \frac{a_C}{q-1} F^M_{AB'} F^N_{CD} F^{\tau B'}_{ID} F^A_{KL} F^D_{XY} \underline{x}^{(\lambda+\eta)\mathrm{R} + (\kappa+\xi)\mathrm{R}' - (\alpha+\delta) + \underline{\dim}\,\mathrm{soc} I}.$$

Lemma 16.

$$\sigma_2 = \sum_{A,B,C,D} q^{\langle B,C\rangle} \frac{q^{[B,C]^1} - 1}{q-1} F^M_{AB} F^N_{CD} \underline{x}^{(\beta+\delta)\mathrm{R} + (\alpha+\gamma)\mathrm{R}' - (\mu+\nu)}.$$

Proof. Define

$$\tilde{\sigma}_2 = \sum_{\substack{A,B',C,D\\ I,K,L,X,Y}} q^{\langle \mu-\kappa, \nu-\eta\rangle} a_C F^M_{AB'} F^N_{CD} F^{\tau B'}_{IC} F^A_{KL} F^D_{XY} \underline{x}^{(\lambda+\eta)\mathrm{R} + (\kappa+\xi)\mathrm{R}' - (\alpha+\delta) + \underline{\dim}\,\mathrm{soc} I}.$$

We begin by noting that $\alpha + \delta = (\mu + \nu) - (\beta' + \gamma)$. Also

$$\begin{aligned}\underline{\dim}\,\mathrm{soc} I &= \underline{\dim}\, I - (\underline{\dim}\, I)\mathrm{R}' = \underline{\dim}\, I + (\gamma - \tau\beta')\mathrm{R}' \\ &= \underline{\dim}\, I - \beta' - \tau\beta' + (\beta'\mathrm{R} + \gamma\mathrm{R}') = -(\beta' + \gamma) + (\beta'\mathrm{R} + \gamma\mathrm{R}').\end{aligned}$$

Hence $\underline{\dim}\,\mathrm{soc} I - (\alpha + \delta) = \beta'\mathrm{R} + \gamma\mathrm{R}' - (\mu + \nu)$ and so the exponent of $\underline{x}$ is

$$(\beta' + \lambda + \eta)\mathrm{R} + (\gamma + \kappa + \xi)\mathrm{R}' - (\mu + \nu).$$

Thus

$$\tilde{\sigma}_2 = \sum_{\substack{A,B',C,D\\ I,K,L,X,Y}} q^{\langle \mu-\kappa, \nu-\eta\rangle} a_C F^M_{AB'} F^N_{CD} F^{\tau B'}_{IC} F^A_{KL} F^D_{XY} \underline{x}^{(\beta'+\lambda+\eta)\mathrm{R} + (\gamma+\kappa+\xi)\mathrm{R}' - (\mu+\nu)}.$$

Using the associativity of the Hall numbers, Theorem 5, we obtain

$$\tilde{\sigma}_2 = \sum_{\substack{A,B',C,D,I\\K,L,X,Y}} q^{\langle A,D\rangle} a_C F^M_{KA} F^N_{DY} F^{\tau B'}_{IC} F^A_{LB'} F^D_{CX} \underline{x}^{(\beta'+\lambda+\eta)\mathrm{R}+(\gamma+\kappa+\xi)\mathrm{R}'-(\mu+\nu)}.$$

The exponent of $\underline{x}$ is now $(\alpha+\eta)\mathrm{R}+(\delta+\kappa)\mathrm{R}'-(\mu+\nu)$.

The previous lemma implies

$$\sum_{L,X,I}\sum_{B',C} a_C F^A_{LB'} F^D_{CX} F^{\tau B'}_{IC} = q^{[D,\tau A]} = q^{[A,D]^1},$$

and so

$$\begin{aligned}\tilde{\sigma}_2 &= \sum_{A,D,K,Y} q^{[A,D]} F^M_{KA} F^N_{DY} \underline{x}^{(\alpha+\eta)\mathrm{R}+(\delta+\kappa)\mathrm{R}'-(\mu+\nu)}\\ &= \sum_{A,B,C,D} q^{[B,C]} F^M_{AB} F^N_{CD} \underline{x}^{(\beta+\delta)\mathrm{R}+(\alpha+\gamma)\mathrm{R}'-(\mu+\nu)}.\end{aligned}$$

Finally, consider the zero morphism $N \to \tau M'$, or equivalently the case $C = 0$. This satisfies $A = M$, $D = N$, $B' = I = 0$, and so we have the contribution to $\tilde{\sigma}_2$ of

$$\sum_{K,L,X,Y} q^{\langle L,X\rangle} F^M_{KL} F^N_{XY} \underline{x}^{(\lambda+\eta)\mathrm{R}+(\kappa+\xi)\mathrm{R}'-(\mu+\nu)}.$$

Hence

$$\sigma_2 = \sum_{A,B,C,D} q^{\langle B,C\rangle} \frac{q^{[B,C]^1}-1}{q-1} F^M_{AB} F^N_{CD} \underline{x}^{(\beta+\delta)\mathrm{R}+(\alpha+\gamma)\mathrm{R}'-(\mu+\nu)},$$

as was to be shown. □

We obtain that

$$\sigma_1+\sigma_2 = \frac{q^{[M,N]^1}-1}{q-1} \sum_{A,B,C,D} q^{\langle B,C\rangle} F^M_{AB} F^N_{CD} \underline{x}^{(\beta+\delta)\mathrm{R}+(\alpha+\gamma)\mathrm{R}'-(\mu+\nu)}.$$

Evaluating at $q = 1$ completes the proof of Theorem 12.

7.1. **Cluster Multiplication Theorem: The Second Case.** We let M and P be modules with P projective and consider the objects M and $P[1]$ in the cluster category. Set $I = \nu(P)$ where $D = \mathrm{Hom}_k(-,k)$ is the standard duality and $\nu = D\,\mathrm{Hom}(-,\Lambda)$ is the Nakayama functor. Thus I is the injective module such that $\mathrm{soc} I \cong P/\mathrm{rad}P$. We know that

$$\mathrm{Ext}^1_{\mathcal{C}}(M,P[1]) \cong \mathrm{Ext}^1(M,I) \quad\text{and}\quad \mathrm{Ext}^1_{\mathcal{C}}(P[1],M) \cong \mathrm{Hom}(P,M).$$

Theorem 17. *The product* $[P,M]X_M\underline{x}^{\dim P/\mathrm{rad}P}$ *equals*

$$\sum_{B,I'} \frac{|\mathrm{Hom}(M,I)_{BI'}|}{q-1}\bigg|_{q-1} X_B \underline{x}^{\dim \mathrm{soc} I'} + \sum_{A,P'} \frac{|\mathrm{Hom}(P,M)_{P'A}|}{q-1}\bigg|_{q-1} X_A \underline{x}^{\dim P'/\mathrm{rad}P'},$$

where $\mathrm{Hom}(M,I)_{BI'}$ *is the set of (non-zero) morphisms* $M \to I$ *with kernel* B *and cokernel* I'*, and dually for* $\mathrm{Hom}(P,M)_{P'A}$.

As before, the morphism $\theta : M \to I$ correponds to the triangle in $\mathcal{C}$ given by

$$I[-1] \to E \to M \xrightarrow{\theta} I, \quad \text{where } E \cong \operatorname{Ker}\theta \oplus (\operatorname{Coker}\theta)[-1].$$

We note that from θ we obtain the two short exact sequences

$$0 \to B \to M \to A \to 0, \quad 0 \to A \to I \to I' \to 0,$$

where I' is again injective. Now, as in Lemma 15, we know that

$$|\operatorname{Hom}(M,I)_{BI'}| = \sum_A a_A F^M_{AB} F^I_{I'A}.$$

Define

$$\sigma_1 := \sum_{A,B,I',X,Y} \frac{a_A}{q-1} F^M_{AB} F^I_{I'A} F^B_{XY} \underline{x}^{\eta\mathrm{R}+\xi\mathrm{R}'-\beta+\underline{\dim}\,\mathrm{soc} I'},$$

so that $\sigma_2|_{q=1}$ equals the first term in the theorem. Now, as in Lemma 16, we can show that

$$\sigma_1 = \sum_{A,B} \frac{q^{[B,I]}-1}{q-1} F^M_{AB} \underline{x}^{\beta\mathrm{R}+\alpha\mathrm{R}'-\mu+\underline{\dim}\,\mathrm{soc} I}.$$

Similarly, the morphism $\theta : P \to M$ corresponds to the triangle

$$M \to E \to P[1] \xrightarrow{\theta[1]} M[1], \quad \text{where } E \cong (\operatorname{Ker}\theta)[1] \oplus \operatorname{Coker}\theta.$$

From θ we obtain the exact sequences

$$0 \to P' \to P \to B \to 0, \quad 0 \to B \to M \to A \to 0,$$

where P' is again projective. Dually to the above we set

$$\sigma_2 := \sum_{A,B,P',X,Y} q^{\langle \mu-\xi, \underline{\dim}\, I\rangle} \frac{a_B}{q-1} F^M_{AB} F^P_{BP'} F^A_{XY} \underline{x}^{\eta\mathrm{R}+\xi\mathrm{R}'-\alpha+\underline{\dim}\, P'/\mathrm{rad} P'},$$

so that $\sigma_1|_{q=1}$ equals the second term. Again we have introduced a weighting to simplify the final step. Then

$$\sigma_2 = \sum_{A,B} q^{[B,I]} \frac{q^{[P,A]}-1}{q-1} F^M_{AB} \underline{x}^{\beta\mathrm{R}+\alpha\mathrm{R}'-\mu+\underline{\dim}\, P/\mathrm{rad} P}.$$

Since $[P,M] = [P,A] + [B,I]$ we obtain that

$$\sigma_1 + \sigma_2 = \frac{q^{[P,M]}-1}{q-1} \sum_{A,B} F^M_{AB} \underline{x}^{\beta\mathrm{R}+\alpha\mathrm{R}'-\mu+\underline{\dim}\, P/\mathrm{rad} P}.$$

Evaluating at $q=1$ finishes the proof.

8. Tilting modules

A Λ-module is called basic, or square-free, if it is a direct sum of pairwise non-isomorphic indecomposable modules. A module is rigid if it has no self-extensions, and exceptional if it is both rigid and indecomposable.

A module is sincere if every simple module occurs as a composition factor. Note that if a module M is not sincere, then there exists a vertex $i \notin \operatorname{supp}(M)$. Thus M is a module for the algebra $\Lambda/\Lambda\epsilon_i\Lambda$ corresponding to the quiver obtained by deleting the vertex i from Q. We also know (for example Kerner) that every sincere exceptional module is faithful.

A tilting module for Λ is a basic rigid module T such that there exists a coresolution

$$0 \to \Lambda \to T^0 \to T^1 \to 0, \quad T^i \in \mathrm{add}(T).$$

The following construction of tilting modules is due to Bongartz. Let T be a rigid module and consider the universal extension

$$0 \to \Lambda \to L \to T^r \to 0, \qquad \text{where } r = [T,\Lambda]^1/[T,T]. \tag{3}$$

Then there exists a summand of $T \oplus L$ which is a tilting module.

We recall some results from Happel-Ringel.

Theorem 18 (Happel-Ringel)**.** (1) *If X and Y are indecomposable modules with $[X,Y]^1 = 0$, then any non-zero map $Y \to X$ is either a monomorphism or an epimorphism.*

(2) *The dimension vectors of the indecomposable summands of a basic rigid module are linearly independent in the Grothendieck group of Λ.*

(3) *A basic rigid module T is a tilting module if and only if the number of indecomposable summands of T equals the number n of simple Λ-modules.*

In particular, let $T = \bigoplus_{i=1}^n T_i$ be a tilting module. Then $\mathrm{End}(T_i)$ is always a field, and we write $t_i := \dim \mathrm{End}(T_i)$. Also, we can order the summands of T such that $\mathrm{Hom}(T_i,T_j) \neq 0$ implies $i \leq j$.

A basic rigid module T with $n-1$ indecomposable summands is called an almost complete tilting module. A complement to T is an exceptional module M such that $T \oplus M$ is a tilting module.

Theorem 19 (Happel-Unger)**.** *Let T be an almost complete tilting module.*

(1) *If T is sincere, then there exist precisely two complements to T, otherwise there exists precisely one complement.*

(2) *Suppose T has two complements M and N such that $\mathrm{Ext}^1(M,N) \neq 0$. Then there is a unique non-split sequence $0 \to N \to E \to M \to 0$ and $E \in \mathrm{add}(T)$. Moreover, $[M,M] = [N,N] = [M,N]^1$.*

We offer a streamlined proof making explicit use of the Bongartz complement.

Proof. Let T be an almost complete tilting module and consider the construction by Bongartz. This yields a rigid module $T \oplus L$ containing a tilting module as a direct summand. Let N be an indecomposable summand of L not contained in $\mathrm{add}(T)$. Then $T \oplus N$ is a basic rigid module with n summands, and hence is a tilting module by Theorem 18. Furthermore, $L \in \mathrm{add}(T \oplus N)$. In particular, every almost complete tilting module (in fact, every basic rigid module) has a complement.

Let $M \ncong N$ be another complement to T. Applying $\mathrm{Hom}(-,M)$ to (3) shows that $[N,M]^1 = 0$. Thus $[M,N]^1 \neq 0$, so consider the universal extension

$$0 \to N^r \to E \to M \to 0, \qquad \text{where } r = [M,N]^1/t_N \text{ and } t_N := [N,N].$$

We first show that $T \oplus N \oplus E$ is rigid, so that $E \in \mathrm{add}(T \oplus N)$.

By construction, $[E,N]^1 = [N^r,N]^1 = 0$. Also, $[E,T]^1 = 0 = [T,E]^1$. Applying $\mathrm{Hom}(-,E)$ to (3) now gives that $[N,E]^1 = 0$. Next apply $\mathrm{Hom}(-,M)$ and $\mathrm{Hom}(E,-)$ to the above sequence to obtain that $[E,M]^1 = 0 = [E,E]^1$, as required.

Now, N is not a summand of E. For, if we had $E = N \oplus E'$, then using that $\mathrm{End}(N)$ is a field and $N \ncong M$ we obtain the universal sequence $0 \to N^{r-1} \to$

$E' \to M \to 0$ as a direct summand of our original sequence, a contradiction. Thus $E \in \operatorname{add}(T)$.

It follows that $[M,E]^1 = 0$. Therefore $\operatorname{End}(M)$ surjects onto $\operatorname{Ext}^1(M,N^r)$, which implies that $r=1$ and $[M,E]=0=[M,N]$. Since $\operatorname{Hom}(N^r,N)$ surjects onto $\operatorname{Ext}^1(M,N)$ and $r=1$, we have $[N,N]=[M,N]^1=[M,M]$.

Moreover, if T is insincere, then $[P,T]=0$ for some non-zero projective module P. Applying $\operatorname{Hom}(P,-)$ to the above sequence yields that $[P,M]=0$, and hence $T \oplus M$ is an insincere tilting module, a contradiction. Hence there exists a unique complement in $\operatorname{mod}\Lambda$.

Now suppose that $M' \ncong N$ is also a complement to T. As above, we obtain a sequence

$$0 \to N \to E' \to M' \to 0, \qquad E' \in \operatorname{add}(T).$$

Applying $\operatorname{Hom}(-,M)$ to this sequence and $\operatorname{Hom}(-,M')$ to the sequence for M yields $[M',M]^1 = 0 = [M,M']^1$. Thus $T \oplus M \oplus M'$ is rigid, so $M \cong M'$ and there are at most two complements to T.

It remains to show that T has two complements when it is sincere. Let N be the Bongartz complement and consider the natural morphism $f : N \to F = \bigoplus X^{[N,T]}$. In particular, every morphism $N \to Y$ with $Y \in \operatorname{add}(T)$ factors through f. Then f is injective precisely when T is sincere, and in this case the cokernel contains a second complement to T as a direct summand.

Let A be the image of f and consider the induced exact sequences

$$0 \to K \to N \to A \to 0 \quad \text{and} \quad 0 \to A \to F \to G \to 0.$$

We know that $[A,T]^1 = 0 = [T,A]^1$ and $[K,T]^1 = 0$. Also, $[N,T]=[A,T]$ and hence $[K,T]=0$. Applying $\operatorname{Hom}(K,-)$ to (3) shows that $[K,\Lambda]=[K,L]=[K,N^s]$ for some $s \geq 1$. Thus if $K \neq 0$, then K is projective and T is insincere. We deduce that f is injective and obtain the exact sequence

$$0 \to N \to F \to G \to 0.$$

Now, $[T,G]^1 = 0$ and since $\operatorname{Hom}(F,T) \to \operatorname{Hom}(N,T)$ is surjective, $[G,T]^1 = 0$. In particular, $[G,F]^1 = 0$. Applying $\operatorname{Hom}(G,-)$ yields $[G,G]^1 = 0$, so that $T \oplus G$ is rigid. We know that $G \notin \operatorname{add}(T)$, otherwise the sequence would split. Thus G has as direct summand a second complement to T. □

9. Seeds and Mutation

For $\sigma \subset \Delta_0 = \{1,\ldots,n\}$ define the idempotent $\epsilon_\sigma = \sum_{i\in\sigma} \epsilon_i$ and set $\Lambda_\sigma := \Lambda/\Lambda\epsilon_\sigma\Lambda$.

We consider the pairs (σ,T), where $\sigma = \{\sigma_1,\ldots,\sigma_{r-1}\} \subset \Delta_0$ and $T = \bigoplus_{i=r}^n T_i$ is a tilting module for Λ_σ. Such a pair will be called a tilting pair. The corresponding cluster is given as

$$X_{(\sigma,T)} := \{X_1,\ldots,X_n\} := \{x_{\sigma_1},\ldots,x_{\sigma_{r-1}},X_{T_r},\ldots,X_{T_n}\}.$$

Recall that $\mathrm{D} = \operatorname{diag}(d_i)$ is the diagonalising matrix for A. Define $\mathrm{D}_{(\sigma,T)} := \operatorname{diag}(t_i)$, where $t_i := \dim \operatorname{End}(S_{\sigma_i}) = d_{\sigma_i}$ for $1 \leq i < r$ and $t_j := \dim \operatorname{End}(T_j)$ for $r \leq j \leq n$. Write $\mathrm{D}_\sigma := \operatorname{diag}(t_i)_{1\leq i<r}$ and $\mathrm{D}_T := \operatorname{diag}(t_j)_{r\leq j\leq n}$. Note that Theorem 18 implies that, up to reordering, $\mathrm{D}_{(\sigma,T)}$ coincides with D.

Define the matrix

$$\mathrm{H}_{(\sigma,T)} = \begin{pmatrix} \mathrm{H}_\sigma & \mathrm{H}_{\sigma T} \\ \mathrm{H}_{T\sigma} & \mathrm{H}_T \end{pmatrix} := \begin{pmatrix} \langle P_{\sigma_i}, P_{\sigma_j}\rangle_{i\neq j} & -\langle S_{\sigma_i}, T_j\rangle \\ -\langle T_i, S_{\sigma_j}\rangle & \langle T_i, T_j\rangle_{i\neq j} \end{pmatrix}.$$

Next set

$$\mathrm{M}_{(\sigma,T)} := \begin{pmatrix} \mathrm{M}_\sigma & \mathrm{M}_{\sigma T} \\ \mathrm{M}_{T\sigma} & \mathrm{M}_T \end{pmatrix},$$

where

$$\begin{aligned} \mathrm{M}_\sigma \mathrm{D}_\sigma &= \mathrm{H}_\sigma\big(1+\mathrm{D}_\sigma^{-1}\mathrm{H}_\sigma\big)^{-1} = \big(1+\mathrm{H}_\sigma\mathrm{D}_\sigma^{-1}\big)^{-1}\mathrm{H}_\sigma \\ \mathrm{M}_T \mathrm{D}_T &= \mathrm{H}_T\big(1+\mathrm{D}_T^{-1}\mathrm{H}_T\big)^{-1} = \big(1+\mathrm{H}_T\mathrm{D}_T^{-1}\big)^{-1}\mathrm{H}_T \\ \mathrm{M}_{\sigma T}\mathrm{D}_T &= \mathrm{H}_{\sigma T}\big(1+\mathrm{D}_T^{-1}\mathrm{H}_T\big)^{-1} \\ \mathrm{M}_{T\sigma}\mathrm{D}_\sigma &= \big(1+\mathrm{H}_T\mathrm{D}_T^{-1}\big)^{-1}\mathrm{H}_{T\sigma}. \end{aligned}$$

Finally write

$$\mathrm{B}_{(\sigma,T)}\mathrm{D}_{(\sigma,T)} := \mathrm{M}_{(\sigma,T)}\mathrm{D}_{(\sigma,T)} - \big(\mathrm{M}_{(\sigma,T)}\mathrm{D}_{(\sigma,T)}\big)^t.$$

Clearly $\mathrm{B}_{(\sigma,T)}$ is skew-symmetrisable with respect to $\mathrm{D}_{(\sigma,T)}$. We call the pair $\big(X_{(\sigma,T)}, \mathrm{B}_{(\sigma,T)}\big)$ a seed.

Remark 20. *Note that that $\mathrm{M}_{(\sigma,T)}$ is determined by the following procedure.*

Let $r \leq i$. Then there exist unique elements ρ_i and χ_i in the Grothendieck group of Λ_σ such that

$$\begin{aligned} \langle \rho_i, T_j\rangle &= \langle T_i, T_j\rangle \quad \text{for } r<j\neq i, && \langle \rho_i, T_i\rangle = 0 \\ \langle T_j, \chi_i\rangle &= \langle T_j, T_i\rangle \quad \text{for } r<j\neq i, && \langle T_i, \chi_i\rangle = 0. \end{aligned}$$

It follows that $\rho_i := \sum_j m_{ij}\,\underline{\dim}\,T_j$ and $\chi_i := \sum_j m_{ji}t_i/t_j\,\underline{\dim}\,T_j$. Similarly for the matrix M_σ.

Let $i<r$. Then there exist unique elements ρ_i and χ_i such that

$$\langle \rho_i, T_j\rangle = -\langle S_{\sigma_i}, T_j\rangle \quad \text{and} \quad \langle T_j, \chi_i\rangle = -\langle T_j, S_{\sigma_i}\rangle \quad \text{for all } r<j.$$

We again have that $\rho_i := \sum_j m_{ij}\,\underline{\dim}\,T_j$ and $\chi_i := \sum_j m_{ji}t_i/t_j\,\underline{\dim}\,T_j$.

Alternatively, for $i<r$ let $\Lambda_i := \Lambda_{\sigma\setminus\{\sigma_i\}}$ and write $\overline{P}_{\sigma_i}$ for the Λ_i-projective corresponding to vertex σ_i. Then $\mathrm{rad}\overline{P}_{\sigma_i}$ is a Λ_σ-projective, and hence has a coresolution

$$0 \to \mathrm{rad}\overline{P}_{\sigma_i} \to X_0 \to X_1 \to 0 \qquad \text{with } X_0, X_1 \in \mathrm{add}(T).$$

Since $\langle \overline{P}_{\sigma_i}, T\rangle = 0$, it follows that $\rho_i = \underline{\dim}\,X_0 - \underline{\dim}\,X_1$. Dually, the Λ_i-injective $\overline{I}_{\sigma_i}$ yields the resolution

$$0 \to Y_1 \to Y_0 \to \overline{I}_{\sigma_i}/S_{\sigma_i} \to 0 \qquad \text{with } Y_0, Y_1 \in \mathrm{add}(T)$$

and $\chi_i = \underline{\dim}\,Y_0 - \underline{\dim}\,Y_1$.

We now describe how to mutate tilting pairs. Given a tilting pair (σ,T) together with $1\leq k\leq n$ we define a new tilting pair $\mu_k(\sigma,T) = (\sigma',T')$ called the mutation in direction k.

If $k<r$, then after reordering we may assume that $k=r-1$. We set $\sigma' = \sigma\setminus\{\sigma_{r-1}\}$. The module T is now an insincere almost complete tilting module for $\Lambda_{\sigma'}$. Hence there exists a unique complement N and we set $T' := N\oplus T$.

Otherwise we may assume that $k=r$, so consider $\overline{T} := \bigoplus_{i=r+1}^n T_i$. This is an almost complete tilting module for Λ_σ. If $\overline{T}$ is sincere, then there exists a unique complement $N \not\cong T_r$. Set $\sigma' := \sigma$ and $T' := \overline{N} \oplus T$. If $\overline{T}$ is insincere, then there exists a unique vertex v of Λ_σ not in the support of $\overline{T}$. In this case, set $\sigma' := \sigma \cup \{v\}$ and $T' := \overline{T}$.

Main Theorem. *The mutation of tilting pairs* (σ, T) *coincides precisely with the Fomin-Zelevinsky mutation of seeds* $\big(X_{(\sigma,T)}, \mathrm{B}_{(\sigma,T)}\big)$*. Thus the subalgebra of* $\mathcal{F}$ *generated by the* $X_{(\sigma,T)}$ *equals the cluster algebra of type* Δ.

For the proof it is enough to consider mutation at $k = r$, and we have the two cases depending on whether $\overline{T} = \bigoplus_{i=r+1}^n T_i$ is sincere or not. In both cases we use the Cluster Multiplication Theorem to show that the matrix B has the correct values on its r-th row.

10. The Sincere Case

Let $\sigma = \{1, \dots, r-1\}$ (after relabelling vertices) and let $T = \bigoplus_{i=r+1}^n$ be a sincere almost complete tilting module for Λ_σ. Let M and N such that $[N, M]^1 = 0$ be the two complements of T in modΛ_σ. Let $\mathrm{M}^M = (m_{ij}^M) = \mathrm{M}_{(\sigma, M \oplus T)}$ be the matrix corresponding to $(\sigma, M \oplus T)$, and similarly let $\mathrm{M}^N = \mathrm{M}_{(\sigma, N \oplus T)}$ be the matrix corresponding to $(\sigma, N \oplus T)$. Note that $t_r = [M, M] = [N, N] = [M, N]^1$.

Recall that a left add(T)-approximation of N is a morphism $N \to E$ with $E \in$ add(T) such that the induced map $\mathrm{Hom}(E, X) \to \mathrm{Hom}(N, X)$ is an epimorphism for all $X \in$ add(T) . Dually a right add(T)-approximation is a morphism $D \to N$ such that $\mathrm{Hom}(D, X) \to \mathrm{Hom}(N, X)$ is an epimorphism for all $X \in$ add(T).

Lemma 21. *For the extension* $0 \to N \to E \to M \to 0$*, the map* $N \to E$ *is a minimal left* add(T)*-approximations of* N*, and dually the map* $E \to M$ *is a minimal right approximation. We write*

$$\underline{\dim}\, E = \sum_j h_{Nj}\, \underline{\dim}\, T_j = \sum_j h_{jM}\, \underline{\dim}\, T_j.$$

Proof. Clearly $\mathrm{Hom}(E, X) \to \mathrm{Hom}(N, X)$ is surjective for all $X \in$ add(T), thus E is a left approximation. Conversely, given any other approximation $N \to F$, this must be injective since $N \to E$ is injective. Let G be the cokernel. Then $[T, G]^1 = 0 = [G, T]^1$. In particular, $[G, F]^1 = 0$, so $[G, G]^1 = 0$. Thus $T \oplus G$ is exceptional, hence $G \in$ add$(T \oplus M$. Since $[T, N]^1 = 0$, the sequence $0 \to N \to E \to M \to 0$ is a summand of $0 \to N \to F \to G \to 0$, proving minimality. The result for $E \to M$ is dual. □

The first term in the Cluster Multiplication Theorem thus reads $\prod_i X_{T_i}^{h_{Ni}}$.

For the second term we need to understand the unique non-zero map $N \to \tau M$. Given a rigid module T we define the right perpendicular category (see Geigle-Lenzing) via

$$T^\perp := \{X : [T, X] = 0, [T, X]^1 = 0\}.$$

This category is isomorphic to modΛ' for some finite dimensional hereditary k-algebra Λ'. Thus if T is an almost complete tilting module, then Λ' has a unique simple module and so equals add(C) for some exceptional module C. Dually we can define the left perpendicular category ${}^\perp T$.

Proposition 22. *The essentially unique non-zero morphism* $N \to \tau M$ *yields the short exact sequences*

$$0 \to D \to N \to C \to 0, \quad 0 \to B \to M \to A \to 0 \quad \textit{and} \quad 0 \to C \to \tau B \to I \to 0,$$

where

(1) $M \to A$ *is the minimal left* $\mathrm{add}(T)$*-approximation of* M*;*
(2) $D \to N$ *is the minimal right* $\mathrm{add}(T)$*-approximation of* N*;*
(3) B *is indecomposable and generates* ${}^{\perp}T$ *inside* $\mathrm{mod}\Lambda_\sigma$*;*
(4) C *is indecomposable and generates* $T^{\perp}$ *inside* $\mathrm{mod}\Lambda_\sigma$*;*
(5) I *is injective and* $\mathrm{soc}I$ *has support inside* σ*.*

Moreover $\mathrm{End}(B) \cong \mathrm{End}(C) \cong \mathrm{End}(N) \cong \mathrm{End}(M)$*. We write*

$$\underline{\dim}\, A := \sum_j h_{Mj}\, \underline{\dim}\, T_j \quad \textit{and} \quad \underline{\dim}\, D := \sum_j h_{jN}\, \underline{\dim}\, T_j$$

and set $\alpha := \underline{\dim}\,\mathrm{soc}I$*.*

Proof. The image C of $N \to \tau M$ must be indecomposable since the morphism is unique up to an automorphism of C. Also, as in the proof of the Caldero-Keller Multiplication Theorem, we have three induced sequences

$$0 \to D \to N \to C \to 0, \quad 0 \to B \to M \to A \to 0, \quad 0 \to C \to \tau B \to I \to 0,$$

where B contains no projective summand and I is injective. Note that the first two are sequences of Λ_σ modules.

We first apply the functors $\mathrm{Hom}(T,-)$ and $\mathrm{Hom}(-,T)$ to the sequence $0 \to D \to N \to C \to 0$ to deduce that $[T,C]^1 = 0 = [D,T]^1$ and that there is an exact sequence

$$0 \to \mathrm{Hom}(T,D) \to \mathrm{Hom}(T,N) \to \mathrm{Hom}(T,C) \to \mathrm{Ext}^1(T,D) \to 0.$$

Similarly, using the sequence $0 \to B \to M \to A \to 0$, we deduce that $[T,A]^1 = 0 = [B,T]^1$ and that there is an exact sequence

$$0 \to \mathrm{Hom}(A,T) \to \mathrm{Hom}(M,T) \to \mathrm{Hom}(B,T) \to \mathrm{Ext}^1(A,T) \to 0.$$

Next, applying $\mathrm{Hom}(T,-)$ to the sequence $0 \to C \to \tau B \to I \to 0$ gives $[T,C] = 0 = [T,I]$ and $[B,T] = 0$. This implies that $[T,D]^1 = 0 = [A,T]^1$. Furthermore, $T^{\perp} = \mathrm{add}(C)$ and ${}^{\perp}T = \mathrm{add}(B)$.

Finally we see that both A and D are rigid. For, we have the sequence $0 \to N \to E \to M \to 0$ with $E \in \mathrm{add}(T)$, so $[T,D]^1 = 0$ implies $[N,D]^1 = 0$. Applying $\mathrm{Hom}(-,D)$ to the sequence $0 \to D \to N \to C \to 0$ yields $[D,D]^1 = 0$. Dually we have $[A,M]^1 = 0$, hence $[A,A]^1 = 0$.

Thus $A, D \in \mathrm{add}(T)$, and $D \to N$ and $M \to A$ are right and left $\mathrm{add}(T)$-approximations respectively. If $F \to N$ is another approximation, then since $D \to N$ is injective, the induced maps $D \to F \to D$ give the identity on D. Hence D is a direct summand of F, so $D \to N$ is minimal. Dually for the epimorphism $M \to A$.

Next we note that $[C,C] = [N,C]$ since $[D,C] = 0$. Also Theorem 18 tells us that $[N,D] = 0$, so $[N,N] = [N,C]$. We shall also need that $[N,I] = 0$. To see this, we observe that $[N,C]^1 = 0$ and that $[N,\tau B] = [B,N]^1 = [M,N]^1$, which equals $[N,N]$ by Theorem 19. Thus $[N,I] = [N,\tau B] - [N,C] = 0$.

Let $\alpha := \underline{\dim}\,\mathrm{soc}I$, so $\alpha_i d_i := [S_i, I] = \langle S_i, I\rangle$. We know that the dimension vectors of the summands of $T \oplus N$ form a basis for the Grothendieck group of Λ_σ. Therefore, since $\langle -, I\rangle$ vanishes on each of these summands, we see that $\mathrm{supp}\alpha \subset \sigma$. □

Using this result, we see that the Cluster Multiplication Theorem has second term $\prod_i X_{T_i}^{h_{Mi}+h_{iN}} \underline{x}^{\alpha}$.

Thus, in this case, the Main Theorem reduces to the following proposition.

Proposition 23. *Let* $\mathrm{B}_{(\sigma,N\oplus T)}=(b^N_{ij})$. *Then*

$$b^N_{rj}=\begin{cases}-\alpha_j & \text{if } j<r;\\ h_{Nj}-h_{jN}-h_{Mj} & \text{if } r<j\end{cases}\qquad \text{and}\qquad \mathrm{B}_{(\sigma,M\oplus T)}\overset{\mu_n}{\longleftrightarrow}\mathrm{B}_{(\sigma,N\oplus T)}.$$

10.1. **Proof of Proposition 23.**

Lemma 24. *Let* $r<i,j$. *Then* $m^M_{ij}=m^N_{ij}+(h_{iN}h_{Nj}-h_{iM}h_{Mj})t_i/t_r$. *Also,* $m^M_{rj}=h_{Mj}$ *and* $m^M_{rj}+m^N_{rj}=h_{jM}$, *and similarly* $m^N_{ir}=h_{iN}t_i/t_r$ *and* $m^M_{ir}+m^N_{ir}=h_{Ni}t_i/t_r$.

Proof. It is clear from the definitions that $\rho^M_r=\underline{\dim}\,A$ and $\chi^N_r=\underline{\dim}\,D$. Thus $m^M_{rj}=h_{Mj}$ and $m^N_{ir}=h_{iN}t_i/t_r$. Now consider

$$\rho^N_i=m^N_{ir}\,\underline{\dim}\,N+\sum_j m^N_{ij}\,\underline{\dim}\,T_j=-m^N_{ir}\,\underline{\dim}\,M+\sum_j(m^N_{ij}+m^N_{ir}h_{Nj})\,\underline{\dim}\,T_j.$$

We have $\langle\rho^N_i,M\rangle=\langle\rho^N_i,E\rangle-[T_i,N]=[T_i,M]-h_{iM}t_i$, so that

$$\langle\rho^M_i-\rho^N_i,T_j\rangle=0 \text{ for all } j\quad\text{and}\quad\langle\rho^M_i-\rho^N_i,M\rangle=h^N_{iM}t_i.$$

It follows that

$$\rho^M_i-\rho^N_i=h_{iM}t_i/t_r(\underline{\dim}\,M-\underline{\dim}\,A)=h_{iM}t_i/t_r\,\underline{\dim}\,M-\sum_j h_{iM}h_{Mj}t_i/t_r\,\underline{\dim}\,T_j.$$

Equating coefficients yields $m^M_{ir}+m^N_{ir}=h_{iM}t_i/t_r$ and

$$m^M_{ij}-m^N_{ij}=m^N_{ir}h_{Nj}-h_{iM}h_{Mj}t_i/t_r=\big(h_{iN}h_{Nj}-h_{iM}h_{Mj}\big)t_i/t_r.$$

Finally, since $\langle M,\chi^N_r\rangle=\langle E,\chi^N_r\rangle=0$, we have $\chi^M_r+\chi^N_r=\underline{\dim}\,E$. □

Lemma 25. *Let* $i<r<j$. *Then* $m^M_{ir}=-m^N_{ir}=\langle S_i,C\rangle/t_r$ *and* $m^M_{ij}=m^N_{ij}+m^N_{ir}h_{Nj}$. *Similarly* $m^N_{ri}=-m^M_{ri}=\langle B,S_i\rangle/t_i$ *and* $m^M_{ji}=m^N_{ji}+h_{Nj}m^N_{ri}t_j/t_r$.

Proof. It is clear that $\rho^M_i-\rho^N_i=0$, since it vanishes on each T_j as well as on M. Writing as before

$$\rho^N_i=m^N_{ir}\,\underline{\dim}\,N+\sum_j m^N_{ij}\,\underline{\dim}\,T_j=-m^N_{ir}\,\underline{\dim}\,M+\sum_j(m^N_{ij}+m^N_{ir}h_{Nj})\,\underline{\dim}\,T_j,$$

we obtain $m^M_{ir}+m^N_{ir}=0$ and $m^M_{ij}=m^N_{ij}+m^N_{ir}h_{Nj}$ as required. Now consider

$$\langle\rho^N_i,N\rangle=m^N_{ir}t_r+\sum_j m_{ij}m^N_{ij}[T_j,N]=m^N_{ir}t_r+\sum_j m^N_{ij}[T_j,D]=m^N_{ir}t_r+\langle\rho^N_i,D\rangle.$$

Thus

$$m^N_{ir}t_r=\langle\rho^N_i,N\rangle-\langle\rho^N_i,D\rangle=-\langle S_i,N\rangle+\langle S_i,D\rangle=-\langle S_i,C\rangle.$$

The results for χ_i follow analogously. □

We can now complete the proof of Proposition 23.

Recall that $b^N_{ij}=m^N_{ij}-m^N_{ji}t_i/t_j$. Clearly $b^M_{rj}+b^N_{rj}=0$ for $j<r$ whereas for $r<j$ we have

$$b^M_{rj}+b^N_{rj}=h_{jM}-h_{Nj}=0.$$

Moreover, $b^N_{rj}=h_{Nj}-h_{Mj}-h_{jN}$ for $r<j$ and for $j<r$ we have

$$b^N_{rj}t_j=\langle B,S_j\rangle+\langle S_j,C\rangle=\langle\underline{\dim}\,S_j,\underline{\dim}\,C-\underline{\dim}\,\tau B\rangle.$$

Using the sequence $0 \to C \to \tau B \to I \to 0$ we obtain that $b_{rj}^N = -\langle S_j, I\rangle/t_j = -\alpha_j$.

Consider $r < i, j$. From the lemmas we see that

$$b_{ij}^M = b_{ij}^N + \big((h_{iN} + h_{Mi})h_{Nj} - h_{Ni}(h_{Mj} + h_{jN})\big)t_i/t_r.$$

Set $\delta_{ij}^N := \big(\left|b_{ir}^N\right| b_{rj}^N + b_{ir}^N \left|b_{rj}^N\right|\big)/2$ and note that $\delta_{ij}^N = 0$ unless both b_{ir}^N and b_{rj}^N have the same sign.

Suppose b_{rj}^N is non-negative, so equals h_{Nj}. Then $\delta_{ij}^N = 0$ unless $b_{ir}^N = (h_{Mi} + h_{iN})t_i/t_r$ is also non-negative, in which case $\delta_{ij}^N = (h_{iN} + h_{Mi})h_{Nj}t_i/t_r$. If $b_{rj} = -(h_{Mj} + h_{jN})$ is non-positive, then $\delta_{ij}^N = 0$ unless $b_{ir}^N = -h_{Ni}t_i/t_r$, in which case $\delta_{ij}^N = -h_{Ni}(h_{Mj}+h_{jN})t_i/t_r$. We thus obtain that $b_{ij}^M = b_{ij}^N + \big(\left|b_{ir}^N\right| b_{rj}^N + b_{ir}^N \left|b_{rj}^N\right|\big)/2$ for all $r < i, j$.

Now suppose that $i < r < j$. Then $b_{ij}^M = b_{ij}^N + h_{Nj}(m_{ir}^N - m_{ri}^N t_i/t_r) = b_{ij}^N + b_{ir}^N h_{Nj}$. Now $b_{ir}^N = \alpha_i t_i/t_r$ is never negative, so $b_{ij}^M = b_{ij}^N + \big(\left|b_{ir}^N\right| b_{rj}^N + b_{ir}^N \left|b_{rj}^N\right|\big)/2$ holds for all $i < r < j$.

Finally it is clear that $m_{ij}^M = m_{ij}^N$ for all $i, j < r$, and hence $b_{ij}^M = b_{ij}^N$. Since $b_{ir}^N \geq 0 \geq b_{rj}^N$ the result follows.

This completes the proof of Proposition 23.

11. The insincere case

Let $\sigma = \{1, \ldots, r-1\}$ (after relabelling vertices) and let $T = \bigoplus_{i=r+1}^n$ be an insincere almost complete tilting module for Λ_σ. Let N be the unique complement of T in modΛ_σ and let $v \notin \sigma$ be the unique vertex not in the support of T.

In order to use the Cluster Multiplication Theorem, we need to understand the morphisms $M \to I_v$ and $P_v \to M$.

Proposition 26. *There is a unique non-zero morphism $P_v \to N$. The cokernel $N \to C$ is the minimal left* add(T)*-approximation of N and the image is the Λ_σ-projective $\overline{P}_v$ corresponding to v. Let P be the kernel and set $\underline{\dim}\, C = \sum_j h_{Nj}\, \underline{\dim}\, T_j$ and $\alpha := \underline{\dim}\, P/\mathrm{rad} P$.*

Dually there is a unique non-zero morphism $N \to I_v$. This has kernel $D \to N$ the minimal right add(T)*-approximation and image the Λ_σ-injective $\overline{I}_v$ corresponding to v. Let I be the cokernel and set $\underline{\dim}\, D = \sum_j h_{jN}\, \underline{\dim}\, T_j$ and $\beta := \underline{\dim}\, \mathrm{soc} I$.*

Moreover, $[P_v, P_v] = [N, N] = [P_v, N]$ and $\overline{P}_v$ is a generator for ${}^\perp T$ inside modΛ_σ. *Dually for I_v.*

Note that the supports of α and β are disjoint since Q has no oriented cycles.

Proof. Since $N \oplus T$ is a tilting module and since $[\overline{P}_v, T] = 0$ we have a coresolution

$$0 \to \overline{P}_v \to N^s \to C \to 0 \quad \text{with } C \in \mathrm{add}(T)$$

for some $s \geq 1$. Applying Hom$(\overline{P}_v, -)$ shows that $s = 1$ and $[\overline{P}_v, N] = [\overline{P}_v, \overline{P}_v]$. Furthermore, since $[C, N] = 0$ by Theorem 18, we have $[N, N] = [\overline{P}_v, N]$. Clearly $\overline{P}_v$ is a generator for ${}^\perp T$.

Clearly $N \to C$ is a left add(T)-approximation of N, and it is minimal since $N \to C$ is surjective (c.f. the proof of Proposition 22). Finally, since S_v occurs only once as a composition factor of N, $[P_v, P_v] = [P_v, N] = [\overline{P}_v, N] = [N, N]$.

The result for I_v is dual. □

The Caldero-Keller Multiplication Theorem thus reads as

$$X_N x_v = \prod_i X_{T_i}^{h_{Ni}} \underline{x}^{\alpha} + \prod_i X_{T_i}^{h_{iN}} \underline{x}^{\beta},$$

and the Main Theorem reduces to the following proposition.

Proposition 27. *Let* $\mathrm{B}_{(\sigma,N\oplus T)} = (b_{ij}^N)$. *Then*

$$b_{rj}^N = \begin{cases} \alpha_j - \beta_j & \text{if } j < r; \\ h_{Nj} - h_{jN} & \text{if } r < j \end{cases} \qquad \text{and} \qquad \mathrm{B}_{(\sigma,N\oplus T)} \overset{\mu_r}{\longleftrightarrow} \mathrm{B}_{(\sigma\cup\{v\},T)}.$$

11.1. **Proof of Proposition 27.**

Lemma 28. (1) *Let* $r < i,j$. *Then* $m_{ij}^v = m_{ij}^N + h_{iN}h_{Nj}t_i/t_r$. *Also* $m_{ir}^N = h_{iN}t_i/t_r$ *and* $m_{rj}^N = h_{Nj}$.

(2) *Let* $i < r < j$. *Then* $m_{ij}^v = m_{ij}^N + \beta_i h_{Nj}t_i/t_r$ *and* $m_{ir}^N = \beta_i t_i/t_r$. *Also* $m_{ji}^v = m_{ji}^N + h_{jN}\alpha_i t_j/t_r$ *and* $m_{ri}^N = \alpha_i$.

(3) *Let* $i,j < r$. *Then* $m_{ij}^N = m_{ij}^v + m_{ir}^v m_{rj}^v$. *Also* $m_{ir}^v = \alpha_i t_i/t_r$ *and* $m_{rj}^v = \beta_j$.

Proof. It is clear that $\rho_r^N = \underline{\dim}\, C$, hence $m_{rj}^N = h_{Nj}$. Now consider $\rho_i^v - \rho_i^N = -m_{ir}^N \underline{\dim}\, N + \sum_j (m_{ij}^v - m_{ij}^N)\underline{\dim}\, T_j$ for $r < i$. This vanishes on each T_j whereas

$$\langle \rho_i^v, N\rangle = \sum_j m_{ij}^v [T_j, N] = \sum_j m_{ij}^v [T_j, D] = \langle \rho_i^v, D\rangle = [T_i, D] - h_{iN}t_i.$$

Thus $\langle \rho_i^v - \rho_i^N\rangle = [T_i, D] - h_{iN}t_i - [N, i] = -h_{iN}t_i$. We deduce that

$$\rho_i^v - \rho_i^N = h_{iN}t_i/t_r(\underline{\dim}\, C - \underline{\dim}\, N) = -h_{iN}t_i/t_r\,\underline{\dim}\, N + \sum_j h_{iN}h_{Nj}t_i/t_r\,\underline{\dim}\, T_j$$

so $m_{ir}^N = h_{iN}t_i/t_r$ and $m_{ij}^v = m_{ij}^N + h_{iN}h_{Nj}t_i/t_r$.

For $i < r$ the difference $\rho_i^v - \rho_i^N$ again vanishes on all T_j and

$$\langle \rho_i^v - \rho_i^N, N\rangle = \langle \rho_i^v, D\rangle + \langle S_i, N\rangle = \langle S_i, I_v\rangle - \langle S_i, I\rangle = -\beta_i t_i.$$

Thus $\rho_i^v - \rho_i^N = \beta_i t_i/t_r(\underline{\dim}\, C - \underline{\dim}\, N)$, hence $m_{ir}^N = \beta_i t_i/t_r$ and $m_{ij}^v = m_{ij}^N + \beta_i h_{Nj}t_i/t_r$. The result for χ_i is similar, using that

$$\langle N, \chi_i^v - \chi_i^N\rangle = -\langle P, S_i\rangle = -\alpha_i t_i,$$

hence that $\chi_i^v - \chi_i^N = \alpha_i t_i/t_r(\underline{\dim}\, D - \underline{\dim}\, C)$.

Finally, the result for $i,j < r$ is entirely analogous to the first case, using the dimension vectors

$$\bar\rho_i^v = m_{ir}^v \underline{\dim}\, P_v + \sum_j m_{ij}^v \underline{\dim}\, P_j \quad \text{and} \quad \bar\rho_i^N = \sum_j m_{ij}^N \underline{\dim}\, P_j,$$

which satisfy $\langle \bar\rho_i^v, P_j\rangle = \langle \bar\rho_i^N, P_j\rangle = [P_i, P_j]$ if $j \neq i$ and equals 0 if $i = j$, and $\langle \bar\rho_i^v, P_v\rangle = [P_i, P_v]$. □

Lemma 29. *For* $r < j$ *we have* $b_{rj}^v = h_{jN} - h_{Nj}$.

Proof. Recall that $\overline{P}_v =$ is the Λ_σ-projective corresponding to vertex v. We consider a coresolution

$$\mathrm{rad}\overline{P}_v \to X_0 \to X_1 \to 0, \quad X_0, X_1 \in \mathrm{add}(T),$$

so that $\rho_r^v = \underline{\dim}\, X_0 - \underline{\dim}\, X_1$.

Similarly for the Λ_σ-injective $\overline{I}_v/S_v$ we have

$$0 \to Y_1 \to Y_0 \to \overline{I}_v/S_v \to 0, \quad Y_0, Y_1 \in \mathrm{add}(T)$$

and $\chi_r^v = \underline{\dim}\, Y_0 - \underline{\dim}\, Y_1$.

Note that $\rho_r^v - \chi_r^v = \sum_j (m_{rj}^v - m_{jr}t_r/t_j)\,\underline{\dim}\, T_j = \sum_j b_{rj}^v\,\underline{\dim}\, T_j$.

Consider $\langle \overline{I}_v, T_j\rangle$. On the one hand this equals

$$\langle N, T_j\rangle - \langle D, T_j\rangle = \langle \underline{\dim}\, C - \underline{\dim}\, D, \underline{\dim}\, T_j\rangle,$$

whereas on the other hand it equals

$$\langle S_v, T_j\rangle + \langle \overline{I}_v/S_v, T_j\rangle = \langle -\rho_r^v + \chi_r^v, T_j\rangle.$$

Thus

$$\rho_r^v - \chi_r^v = \underline{\dim}\, D - \underline{\dim}\, C = \sum_j (h_{jN} - h_{Nj})\,\underline{\dim}\, T_j.$$

Hence $b_{rj}^v = h_{jN} - h_{Nj}$ as required. □

The can now complete the proof of Proposition 27.

It is clear from the lemmas that b_{rj}^N equals $\alpha_j - \beta_j$ if $j < r$, and equals $h_{Nj} - h_{jN}$ if $r < j$. Also $b_{rj}^N + b_{rj}^v = 0$ for all j.

We next observe that for $r < i, j$, $b_{ij}^v = b_{ij}^N + (h_{iN}h_{Nj} - h_{jN}h_{Ni})t_i/t_r$. By considering the two cases $b_{rj} = h_{jN}$ and $b_{rj} = -h_{Nj}$ separately, as in the proof of Proposition 23, we deduce that $b_{ij}^v = b_{ij}^N + \big(\left|b_{ir}^N\right| b_{rj}^N + b_{ir}^N \left|b_{rj}^N\right|\big)/2$ for all $r < i, j$.

Next suppose that $i < r < j$. Then $b_{ij}^v = b_{ij}^N + (\beta_i h_{Nj} - \alpha_i h_{jN})t_i/t_r$. Since the supports of α and β are disjoint, it follows that $b_{ij}^v = b_{ij}^N + \big(\left|b_{ir}^N\right| b_{rj}^N + b_{ir}^N \left|b_{rj}^N\right|\big)/2$ for all $i < r < j$.

Finally, if $i, j < r$ then $b_{ij}^N = b_{ij}^v + \big(\left|b_{ir}^v\right| b_{rj}^v + b_{ir}^v \left|b_{rj}^v\right|\big)/2$ follows analogously to the case $r < i, j$.

This completes the proof of Proposition 28, and hence also the proof of the Main Theorem.

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