

Theories controlled by formulas of VC codimension 1

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September 2008

Organisation

1. D-relations and D-minimal theories
2. VC-minimal / weakly D-minimal theories
3. Swiss cheese decomposition and dp-minimality of VC-minimal theories.

(1/8) The idea behind \bullet -minimality

T is T_0 -minimal if T extends T_0 , and for T every unary definable set (with parameters) is definable (with parameters) by a quantifier-free formula in the signature of T_0 .

In other words:

- T is a 1-minimal expansion of T_0 , i.e. for T every unary definable set is definable in the signature of T_0 .
- T_1 , the completion of T_0 that is the restriction of T to the signature of T_0 , has 1-QE, i.e. every unary definable set is definable by a quantifier-free formula.

By appropriate choice of T_0 , we get strong minimality, o-minimality, C-minimality, and p-minimality.

Simultaneous generalisation of strong minimality and o-minimality:
 T_0 the theory of partial orders \leq such that comparability is an equivalence relation.

Can we also cover C-minimality?

(2/8) D-relations

To cover C-minimality as well, we could just relax the axioms of C-relations:

$$(C1) \ C(a; bc) \Leftrightarrow C(a; cb)$$

$$(C2) \ C(a; bc) \Rightarrow \neg C(b; ac)$$

$$(C3) \ C(a; bc) \Rightarrow C(d; bc) \vee C(a; dc)$$

$$(C4) \ a \neq b \Rightarrow \exists c \neq b: C(a; bc)$$

$$(C5) \ \exists ab: a \neq b.$$

Instead of just dropping axioms (C4) and (C5), we use D-relations, which are slightly more general than (C1)–(C3), and easier to understand:

$$(D1) \ D(ab; cd) \Rightarrow D(ba; cd) \wedge D(cd; ab)$$

$$(D2) \ D(ab; cd) \Rightarrow \neg D(ac; bd)$$

$$(D3) \ D(ab; cd) \Rightarrow D(ae; cd) \vee D(ab; ce).$$

D-minimality is T_0 -minimality, for T_0 the theory of D-relations.

(3/8) D-minimal theories

The meaning of a D-relation: If a, b, c, d are nodes of a 'tree', $D(ab; cd)$ means that the 'path' from a to b and the 'path' from c to d are disjoint.

The cones: sets of the form $D(xb; cd)^M$.

Every definable unary set is a Boolean combination of cones.

The cones generate a topology (as a subbasis):

Open sets = unions of finite intersections of cones.

This topology is:

- often a definable topology;
- the correct topology in the o-minimal and C-minimal cases;
- the discrete topology in the strongly minimal case.

(4/8) VC codimension

$\Omega \subseteq \mathcal{P}(M)$ shatters $X \subseteq M$ if $\Omega \upharpoonright X = \mathcal{P}(X)$.

$$\dim_{VC} M = \max \{ |X| \mid \Omega \text{ shatters } X \}.$$

$$\dim_{VC} \varphi(\bar{x}; \bar{y}) = \dim_{VC} \{ \varphi(\bar{x}; \bar{b})^M \mid \bar{b} \in M \}.$$

$$\dim_{VC}^{op} \varphi(\bar{x}; \bar{y}) = \dim_{VC} \{ \varphi(\bar{a}; \bar{y})^M \mid \bar{a} \in M \}.$$

φ is VC-minimal if $\dim_{VC}^{op} \varphi = 1$.

$\dim_{VC}^{op} \varphi(x; \bar{y}) = 0 \iff$
every instance $\varphi(x; \bar{b})$ is trivial.

$\dim_{VC}^{op} \varphi(x; \bar{y}) \leq 1 \iff$
for all $A = \varphi(x; \bar{b}_1)^M$ and $B = \varphi(x; \bar{b}_2)^M$:
 $A \subseteq B, A \supseteq B, A \cap B = \emptyset$ or $A \cup B = M$.

(5/8) VC-minimal theories

VC codimension (and VC dimension) can also be defined for families $\{\varphi_i(x; \bar{y}_i) \mid i \in I\}$ of formulas.

VC-minimal theory:

- Has a distinguished VC-minimal family.
- Every unary definable (with parameters) set is a Boolean combination of instances of the distinguished family.

We get a D-relation:

$$D(ab; cd) \iff$$

an instance $A = \varphi(x; \bar{e})^M$ separates a and b from c and d .

Moreover, $\neg D$ is type-definable.

(6/8) Weakly D-minimal theories

Weakly D-minimal theory:

- Has a distinguished incomplete type $p(xy; zw)$ whose complement is a D-relation.
- The definable (with parameters) unary sets are Boolean combinations of definable generalised cones.

Generalised cone:

A set $A \subseteq M^1$ such that $a, b \in A, c, d \notin A \Rightarrow D(ab; cd)$.

VC-minimal = weakly D-minimal:

In a VC-minimal theory:

Instances of the VC-minimal family are generalised cones.

In a weakly D-minimal theory:

Definable generalised cones form a VC-minimal family.

Its D-relation is the distinguished one.

(7/8) Swiss cheese decomposition

In a (weakly) D-minimal theory we can distinguish a node of the 'tree' described by the D-relation, by fixing a parameter. For every generalised cone and its complement we distinguish the one not containing the root as basic. Any two basic generalised cones are comparable or disjoint.

Swiss cheese:

$A \setminus (B_1 \cup \dots \cup B_n)$, where A, B_1, \dots, B_n are definable basic generalised cones, $B_1, \dots, B_n \subseteq A$, and B_1, \dots, B_n are pairwise disjoint.

Every unary definable set is a disjoint union of Swiss cheeses.

(8/8) Dp-minimal theories

A theory is dp-minimal if the following does not exist:

- Formulas $\varphi(x; \bar{y})$ and $\psi(x; \bar{z})$.
- Parameters \bar{b}_i, \bar{c}_j (where $i, j < \omega$).
- Elements a_{ij} such that

$$\models \varphi(a_{ij}; \bar{b}_n) \iff i = n \text{ and } \models \varphi(a_{ij}; \bar{c}_n) \iff j = n.$$

Equivalently, T is dependent and the following does not exist:

- Formulas $\varphi(x; \bar{y})$ and $\psi(x; \bar{z})$.
- Parameters \bar{b}_i ($i < \omega$) such that $\langle \varphi(x; \bar{b}_i) \mid i < \omega \rangle$ is k -inconsistent for some k .
- Parameters \bar{c}_j ($j < \omega$) such that $\langle \psi(x; \bar{c}_j) \mid j < \omega \rangle$ is k -inconsistent for some k .
- Elements a_{ij} such that $\models \varphi(a_{ij}; \bar{b}_i) \wedge \psi(a_{ij}; \bar{c}_j)$.

A stable theory is dp-minimal iff every 1-type has weight ≤ 1 .

O-minimal, C-minimal and p-minimal theories are also known to be dp-minimal.