Shrinking indiscernibles

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Contents:

The independence property (1/7)

$\varphi(x;y)$ has the independence property if φ can code arbitrarily big finite sets:

$$\forall n \quad \exists a_1 \dots a_n \quad \forall S \subseteq \{1,\dots,n\}:$$
$$\{\varphi(a_i;y) \mid i \in S\} \cup \{\neg\varphi(a_i;y) \mid i \notin S\}$$
is consistent.

If φ does not have the independence property, we say 'φ is NIP'.

T is NIP if all formulas $\varphi(\bar{x};\bar{y})$ are NIP.

(By a classical result, a theory is stable iff it is NIP and does not have the strict order property. Since every stable theory is even simple, it follows that a theory is stable iff it is NIP and simple.)

Indiscernibles (2/7)

I a linear order, $J \subseteq I$.

$(a_i)_{i \in I}$ is uniform modulo J for $\varphi(x_1 \ldots x_m)$:

$$\forall u, v \in I^m \text{ s.t. } u \equiv_J^{\mathrm{qf}} v \text{ in } (I, <): \quad \varphi(a_{u_1} \ldots a_{u_m}) \iff \varphi(a_{v_1} \ldots a_{v_m}).$$

$(a_i)_{i \in I}$ is indiscernible modulo J over B:

> $(a_i)_{i \in I}$ is uniform modulo J for all formulas over B.

We drop J if $J = \emptyset$.
We drop B if $B = \emptyset$.

(Note: The tuples in this definition are not required to be strictly descending. In order to simplify the rest of the talk we will assume that they are.)

Poizat's classical result (3/7)

« T a la propriété d'indépendance si et seulement si il existe une suite indicernable dans l'ordre (sur $\emptyset$) sécable. »

Here is a free translation (of one direction).

Theorem (Poizat 1981)
Suppose T is NIP. Given:

- I, a complete linear order;
- $(a_i)_{i \in I}$ indiscernible;
- $\varphi(x; b)$.

Then $\exists$ finite $J \subseteq I$ s.t.
$(a_i)_{i \in I}$ is uniform modulo J for $\varphi(x; b)$.

Proof
Suppose not.
Then $\forall n$ $\exists$ an indiscernible sequence $(a_i)_{i<\omega}$ such that $\models \varphi(a_i, b)$ iff i is even.
For all $S \subseteq \{1, \ldots, n\}$,
$\{\varphi(a_i; y) \mid i \in S\} \cup \{\neg\varphi(a_i; y) \mid i \notin S\}$
is consistent, so φ is not NIP.

The Baldwin-Benedikt result (4/7)

"If M lacks IP and I is order-indiscernible with order type a complete dense linear order, then for every L-formula $\varphi(\bar{x}, \bar{y})$ there is a quantifier-free $<$-formula $\psi(\bar{w}, \bar{y})$ such that for every $\bar{m}$ there is a $\bar{c}_{\bar{m}} \in I$ such that $\forall \bar{y} \in P[\psi(\bar{c}_{\bar{m}}, \bar{y}) \equiv \varphi(\bar{m}, \bar{y})]$."

The following is a free translation.

Theorem (Baldwin-Benedikt 2000)
Suppose T is NIP. Given:

- I, a complete linear order;
- $(a_i)_{i \in I}$ indiscernible;
- $\varphi(x_1 \ldots x_m; b)$.

Then $\exists$ finite $J \subseteq I$ s.t.

$(a_i)_{i \in I}$ is uniform modulo J for $\varphi(x_1 \ldots x_m; b)$.

Note: The case $m = 1$ is precisely Poizat's classical result.

Proof **(5/7)**

We may assume that I is also dense.
$J = \{j \in I \mid j \text{ critical}\}$.
j is <u>critical</u> if there is $\bar{u} = u_1 \dots u_m$, $j = u_k$, s.t. *(U denotes an open interval in I)*
$\forall U \ni j\ \exists j' \in U$: $\quad \models \varphi(a_{\bar{u}}; b) \leftrightarrow \neg\varphi(a_{\bar{u}[j'/j]}; b)$.

- $(a_i)_{i \in I}$ is uniform modulo J for $\varphi(x_1 \dots x_m; b)$.

Show $\bar{u} \equiv_J^{\mathrm{qf}} \bar{v}$ implies $\models \varphi(a_{\bar{u}}; b) \leftrightarrow \varphi(a_{\bar{v}}; b)$ by induction on the last k s.t. $u_k \neq v_k$.

- J is finite.

Otherwise there are infinitely many critical points with the same k.
Consider $(M, I, <, f)$, where $M \models T$ contains $(a_i)_{i \in I}$ and $f(i) = a_i$. Work in a monster model extending this.

There is an indiscernible sequence of $\bar{u}^n$ witnessing that u_k^n is critical; with distinct u_k^n. Using density and criticality, we can shift every second element a bit, preserving indiscernibility, but changing the truth value of $\varphi(f(\bar{u}^n); b)$. Contradiction to case $m = 1$.

Shrinking indiscernibles (6/7)

Corollary

Suppose T is NIP.

Given:

- B, a finite set of parameters;
- I, a complete linear order;
- $(a_i)_{i \in I}$ indiscernible.

Then $\exists\ J \subseteq I$, of size $|J| < |T|^+$, s.t.
$(a_i)_{i \in I}$ is indiscernible modulo J over B.

Corollary

Suppose T is NIP.

Given a finite set B,

every indiscernible sequence $(a_i)_{i<|T|^+}$
has an end piece indiscernible over B.

Definition

If in the second corollary we can replace $|T|^+$ by $\aleph_0$, then T is called <u>strongly NIP</u>.

If the same is true even for the first corollary, then T is called <u>strongly$^+$ NIP</u>.

Shelah's conjecture on NIP fields (7/7)

Theorem (Shelah, Sh783)
Every superstable or o-minimal theory is strongly$^+$ NIP.

Theorem (Shelah/Hrushovski, Sh783+Sh863)
The theory of a p-adic field is strongly NIP but not strongly$^+$ NIP.

Conjecture (Shelah, Sh863)
Every strongly$^+$ NIP field is

- algebraically closed or
- real closed.

Conjecture (Shelah, Sh863)
Every strongly NIP field is

- algebraically closed or
- real closed or
- a valuation field
  (similar to the p-adic fields).

References

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