Thorn-forking

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Shrinking indiscernibles

Contents:

Pregeometries

(Model theoretic) algebraic closure always satisfies the axioms of an algebraic closure operator:

- $A \subseteq \mathrm{acl}\, A$.
- $A \subseteq B \quad \Rightarrow \quad \mathrm{acl}\, A \subseteq \mathrm{acl}\, B$.
- $\mathrm{acl}(\mathrm{acl}\, A) = \mathrm{acl}\, A$.
- $\mathrm{acl}\, A = \bigcup_{\text{finite } A_0 \subseteq A} \mathrm{acl}\, A_0$.

Sometimes it also satisfies the exchange law:

- $a \in \mathrm{acl}(Cb) \setminus \mathrm{acl}\, C \quad \Rightarrow \quad b \in \mathrm{acl}(Ca)$.

Algebraic closure operator + exchange law
= pregeometry.

Examples: vector spaces, alg. closed fields. In a pregeometry, maximal independent set = minimal generating set. We have a good notion of basis and dimension.

Independence by pregeometry

Exchange law:

- $a \in \mathrm{acl}(Cb) \setminus \mathrm{acl}\, C \quad \Rightarrow \quad b \in \mathrm{acl}(Ca)$.

This implies atomic points!

Without atomic points we can still have a notion of independence like linear or algebraic independence.

How can we find it?

In a pregeometry as follows:

$A \downarrow_C B \iff$

$$\text{for all } B_0 \subseteq B\text{: } \dim(B_0/C) = \dim(B_0/AC).$$

Strict independence relations

$a \mathop{\smile\!\!\!\!\!\!\mid}_{B} a \quad \Rightarrow \quad a \in \operatorname{acl} B.$ [anti-reflexivity]

$A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B \iff$ [finite character]
$A_0 \mathop{\smile\!\!\!\!\!\!\mid}_{C} B_0$ for all finite $A_0 \subseteq A$, $B_0 \subseteq B$.

For $D \subseteq C \subseteq B$: [full transitivity]
$A \mathop{\smile\!\!\!\!\!\!\mid}_{D} B \iff A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B$ and $A \mathop{\smile\!\!\!\!\!\!\mid}_{D} C$.

$A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B \iff B \mathop{\smile\!\!\!\!\!\!\mid}_{C} A$ [symmetry]

$\forall A, B, C \ \exists B' \equiv_C B$: $A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B'$. [full existence]

$A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B \subseteq \hat{B} \quad \Rightarrow$ [extension]
$\exists \hat{B}' \equiv_{ABC} \hat{B} : A \mathop{\smile\!\!\!\!\!\!\mid}_{C} \hat{B}'$.

$\forall A, B \ \exists C \subseteq B :$ [local character]
$A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B$ and $|C| \leq \kappa(|A|)$.

(We can remove full existence, as it follows from the other axioms.)

Infinite forest **4/X**

A tree or forest in which every vertex has infinite degree.

$A \mathop{\smile\!\!\!\!\!\!\,|}_C B \iff$
every path connecting A and B passes through $\operatorname{acl} C$.

($\operatorname{acl} C$ is the convex hull of C.)

- This is a strict independence relation.

- The exchange law does not hold.

- A maximal independent set need not be generating (no good notion of basis).

Can we define good candidates for strict independence relations in a general way, when there is no pregeometry?

First candidate

$A \mathop{\smile\!\!\!\!\!\!\!\;|}\nolimits^{\mathsf{a}}_C B \iff \mathsf{acl}(AC) \cap \mathsf{acl}(BC) = \mathsf{acl}\, C.$

This is a partial success. We win all modules but lose the fields.

Modules:
This is a strict independence relation. (In case of vector spaces it's the one we had before.)

Algebraically closed fields:
Full transitivity never holds, other axioms do.

Let's have a closer look at full transitivity.

Full transitivity

For $D \subseteq C \subseteq B$: [full transitivity]

$$A \mathop{\smile\!\!\!\!\!\!\,|}_D B \iff A \mathop{\smile\!\!\!\!\!\!\,|}_C B \text{ and } A \mathop{\smile\!\!\!\!\!\!\,|}_D C.$$

This is really three axioms combined into one.

$A \mathop{\smile\!\!\!\!\!\!\,|}_D B \quad \Rightarrow \quad A \mathop{\smile\!\!\!\!\!\!\,|}_C B.$ [base monoton.]

$A \mathop{\smile\!\!\!\!\!\!\,|}_D B \quad \Rightarrow \quad A \mathop{\smile\!\!\!\!\!\!\,|}_D C.$ [monotonicity]

$A \mathop{\smile\!\!\!\!\!\!\,|}_C B$ and $A \mathop{\smile\!\!\!\!\!\!\,|}_D C$ [partial trans.]
$\Rightarrow \quad A \mathop{\smile\!\!\!\!\!\!\,|}_D B.$

Problematic for $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{a}}$: only base monotonicity. It holds iff the lattice of algebraically closed sets is modular.

What can we do?

Second candidate

If base monotonicity is broken, let's fix it!

$$A \mathop{\smile\!\!\!\!\!\!\;|}_{D}^{\mathsf{M}} B \iff A \mathop{\smile\!\!\!\!\!\!\;|}_{C}^{\mathsf{a}} B \text{ for all } C \text{ s.t. } D \subseteq C \subseteq \mathrm{acl}(BD).$$

• Now we can only prove partial transitivity on the other side.

• Really problematic: symmetry, extension, local character.

Theorem If $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathsf{M}}$ is symmetric, then it's a strict independence relation.

Proof Straightforward but long. Describe $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathsf{M}}$ in terms of local forking and redo large parts of Byunghan Kim's thesis.

$\mathop{\smile\!\!\!\!\!\!\;|}^{\mathsf{M}}$ is good whenever $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathsf{a}}$ is good (e.g. modules). $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathsf{M}}$ is also good whenever we have a pregeometry.

But not for the infinite forest.

Third candidate

For the infinite forest $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.3em}}^{\mathsf{M}}$ is no good because symmetry and extension are broken. Let's fix extension!

$$A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.3em}}^{\text{þ}}_{C} B \iff \forall \hat{B} \supseteq B \;\; \exists \hat{B}' \equiv_{ABC} \hat{B} : A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.3em}}^{\mathsf{M}}_{C} \hat{B}.$$

This does it for the infinite forest. How about the general case?

Theorem
1. $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.3em}}^{\text{þ}}$ always satisfies all axioms except perhaps symmetry and local character.
2. For $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.3em}}^{\text{þ}}$, symmetry and local character are equivalent.
3. If $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.3em}}^{\text{þ}}$ is not a strict independence relation, then there is none. If it is one, it is the weakest.

Proof
1. Check that most axioms are preserved when passing from $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.3em}}^{\mathsf{M}}$ to $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.3em}}^{\text{þ}}$.
2. As for forking in Byunghan Kim's thesis.
3. Easy.

Theorem (Harnik-Harrington)
T is stable iff there is a (necessarily unique) strict independence relation s.t.

$$A \equiv_M A',\ A \mathop{\smile\!\!\!\!\!\!\mid}_M B,\ A' \mathop{\smile\!\!\!\!\!\!\mid}_M B \quad \Rightarrow \quad A \equiv_{MB} A'.$$

How can we show it must be $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$?
(I.e. forking = thorn-forking in stable T?)

Show that forking and thorn-forking have the same Morley sequences. A Morley sequence $(a_i)_{i\in\omega+\omega^*}$ over B is B-indiscernible and satisfies $\operatorname{acl}(B(a_i)_{i\in\omega}) \cap \operatorname{acl}(B(a_i)_{i\in\omega^*}) = \operatorname{acl} B$.

For forking in stable T this condition is sufficient because of canonical bases. So forking has at least as many Morley sequences as thorn-forking. Therefore forking is weaker than thorn-forking, hence equal.

(Proof also works for simple T with elimination of hyperimaginaries.)

The lattice connection

A certain part of stability theory can be described nicely in terms of the lattice of algebraically closed sets and its (elementary) automorphisms.

T trivial 1-based rosy $\iff$ lattice is distributive.

T 1-based rosy $\iff$ lattice is modular.

$\smile\!\!\!\!\!\!\mid^{\mathsf{M}}$ is symmetric $\iff$ lattice is M-symmetric.

T superrosy $\iff$ lattice is arithmetic and T rosy with finite coding.