Understanding forking and thorn-forking

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Barcelona
10 May 2006

Overview

- 3 examples of independence relations (3 slides)
- Axioms of independence (5 slides)
- Tuning the axioms (7 slides)
- Re-inventing forking and thorn-forking (5 slides)

Example: vector space

A, B, C sets of vectors.
$\langle AB \rangle =$ span of $A \cup B$.

$$A \underset{C}{\perp} B \iff \langle AC \rangle \cap \langle BC \rangle = \langle C \rangle$$

$$\iff \begin{cases} \bar{a} \in A \text{ lin. indep. } / \langle C \rangle \\ \Rightarrow \quad \bar{a} \text{ lin. indep. } / \langle BC \rangle \end{cases}$$

Example: algebraically closed field

A, B, C sets of field elements.
$\mathrm{acl}(AB) =$ smallest algebraically closed field $\supseteq A \cup B$.

$$A \underset{C}{\downarrow} B \iff \begin{cases} \bar{a} \in A \text{ alg. indep. } / \ \mathrm{acl}(C) \\ \Rightarrow \quad \bar{a} \text{ alg. indep. } / \ \mathrm{acl}(BC) \end{cases}$$

$$\Longrightarrow \ \mathrm{acl}(AC) \cap \mathrm{acl}(BC) = \mathrm{acl}(C)$$

Example: forest

Consider the model companion of the theory of unordered forests (signature: one binary relation). Models are precisely the disjoint unions of trees in which every vertex has infinite degree.
A, B, C sets of vertices.
$\mathrm{conv}(AB) =$ convex hull of $A \cup B$.

$$A \underset{C}{\downarrow} B \iff \text{every path from } A \text{ to } B \text{ meets } \mathrm{conv}(C)$$
$$\implies \mathrm{conv}(AC) \cap \mathrm{conv}(BC) = \mathrm{conv}(C)$$

Axioms of independence

We work in a big saturated model of a complete first-order theory.

- Forking was introduced by Saharon Shelah to study independence in stable theories. In stable theories forking gives rise to an 'independence relation'.
- Thorn-forking was introduced by Thomas Scanlon and Alf Onshuus to describe independence in a wider class of theories. In rosy theories thorn-forking gives rise to an 'independence relation'.
- All stable theories are rosy. All o-minimal theories are rosy.
- Forking=thorn-forking in stable theories.

There are some problems:

- Definitions not very intuitive.
- Proofs of above facts surprisingly complicated.

We will see how to solve these problems.

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Axioms of independence

- invariance
- finite character
- anti-reflexivity

- symmetry
- existence
- extension
- local character

- full transitivity

Axioms of independence

invariance

$$\left.\begin{array}{l} A \underset{C}{\perp} B \\ (A', B', C') \equiv (A, B, C) \end{array}\right\} \implies A' \underset{C'}{\perp} B'$$

finite character

$$A \underset{C}{\not\perp} B \iff \left\{\begin{array}{l} \exists\, \bar{a} \in A,\ \bar{b} \in B,\ \bar{c} \in C,\ \varphi : \\ \quad \models \varphi(\bar{a}, \bar{b}, \bar{c}) \\ \quad \models \varphi(\bar{a}', \bar{b}, \bar{c}) \implies \bar{a}' \underset{C}{\not\perp} \bar{b} \end{array}\right.$$

anti-reflexivity

$$a \underset{B}{\perp} a \implies a \in \operatorname{acl} B.$$

Axioms of independence

symmetry	$A \underset{C}{\downarrow} B \iff B \underset{C}{\downarrow} A$
existence	$\forall A, B, C \quad \exists A' \equiv_C A: \quad A' \underset{C}{\downarrow} B.$
extension	$A \underset{C}{\downarrow} B, \quad \hat{B} \supseteq B \implies \exists A' \equiv_{BC} A: \quad A' \underset{C}{\downarrow} \hat{B}$
local character	$\forall A, B \quad \exists C \subseteq B: \quad A \underset{C}{\downarrow} B, \quad \lvert C \rvert < \kappa(\lvert A \rvert)$

Axioms of independence

full transitivity For $D \subseteq C \subseteq B$:

$$A \underset{D}{\downarrow} B \iff A \underset{D}{\downarrow} C \text{ and } A \underset{C}{\downarrow} B$$

Tuning the axioms

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base monotonicity For $D \subseteq C \subseteq B$:

$$A \mathop{\smile\!\!\!\!\!\!|}_{D} B \implies A \mathop{\smile\!\!\!\!\!\!|}_{C} B$$

monotonicity For $D \subseteq C \subseteq B$:

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transitivity For $D \subseteq C \subseteq B$:

$$A \mathop{\smile\!\!\!\!\!\!|}_{D} B \impliedby A \mathop{\smile\!\!\!\!\!\!|}_{D} C \text{ and } A \mathop{\smile\!\!\!\!\!\!|}_{C} B$$

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$$B \mathop{\smile\!\!\!\!\!\!|}_{D} A \Longleftarrow C \mathop{\smile\!\!\!\!\!\!|}_{D} A \text{ and } B \mathop{\smile\!\!\!\!\!\!|}_{C} A$$

transitivity For $D \subseteq C \subseteq B$:

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Tuning the axioms

symmetry

$$A \underset{C}{\perp} B \iff B \underset{C}{\perp} A$$

existence

$$\forall A, B, C \quad \exists A' \equiv_C A : \quad A' \underset{C}{\perp} B.$$

extension

$$A \underset{C}{\perp} B, \quad \hat{B} \supseteq B \implies \exists A' \equiv_{BC} A : \quad A' \underset{C}{\perp} \hat{B}$$

local character

$$\forall A, B \quad \exists C \subseteq B : \quad A \underset{C}{\perp} B, \quad |C| < \kappa(|A|)$$

Tuning the axioms

symmetry

$$A \underset{C}{\downfree} B \iff B \underset{C}{\downfree} A$$

existence

$$\forall A, B, C \quad \exists A' \equiv_C A: \quad A' \underset{C}{\downfree} B.$$

extension

$$A \underset{C}{\downfree} B, \quad \hat{B} \supseteq B \implies \exists A' \equiv_{BC} A: \quad A' \underset{C}{\downfree} \hat{B}$$

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$$\forall A, B \quad \exists C \subseteq B: \quad A \underset{C}{\downfree} B, \quad |C| < \kappa(|A|)$$

normality

$$A \underset{C}{\downfree} B \implies AC \underset{C}{\downfree} B$$

Tuning the axioms

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$$A \underset{C}{\downarrow} B \iff B \underset{C}{\downarrow} A$$

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$$\forall A, B, C \quad \exists A' \equiv_C A: \quad A' \underset{C}{\downarrow} B.$$

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normality
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Tuning the axioms

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$$A \mathop{\downarrow}_{C} B \iff B \mathop{\downarrow}_{C} A$$

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Morley sequences and symmetry

Morley sequence over C:
C-indiscernible sequence $(\bar{a}_i)_{i<\kappa}$ such that $\bar{a}_{<\lambda} \mathop{\smile\!\!\!\!\!\!|}_C \bar{a}_\lambda$ for all $\lambda < \kappa$.

Lemma I
If $\bar{a}_0 \mathop{\smile\!\!\!\!\!\!|}_C B$,
then there is a BC-indiscernible Morley sequence $(\bar{a}_i)_{i<\omega}$ over C.

Lemma II
If there is a BC-indiscernible Morley sequence $(\bar{a}_i)_{i<\omega}$ over C,
then $B \mathop{\smile\!\!\!\!\!\!|}_C \bar{a}_0$.

Lemmas I and II can be proved without using symmetry.

Corollary
Symmetry follows from the other axioms.

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Proof of Lemma I

Lemma I
If $\bar{a}_0 \mathop{\smile\!\!\!\!\!|}_C B$,
then there is a BC-indiscernible Morley sequence $(\bar{a}_i)_{i<\omega}$ over C.

Proof.

Using existence, we can create arbitrarily long sequences $(\bar{a}_i)_{i<\omega}$ such that $\bar{a}_{<\lambda} \mathop{\smile\!\!\!\!\!|}_C \bar{a}_\lambda$ for all $\lambda < \kappa$ and $\bar{a}_i \equiv_{BC} \bar{a}$.
Using Erdős-Rado, we can 'extract' a sequence $(\bar{a}_i)_{i<\omega}$ which is indiscernible over BC. □

Proof of Lemma II

Lemma II

If there is a BC-indiscernible Morley sequence $(\bar{a}_i)_{i<\omega}$ over C, then $B \mathop{\smile\!\!\!\!\!\!|}_C \bar{a}_0$.

Proof.

Let $\kappa \geq \kappa(|B|)$ be regular. Let $(\bar{a}_i)_{i<\kappa}$ be a BC-indiscernible Morley sequence over C.
By local character there is $D \subseteq C\{\bar{a}_i \mid i < \kappa\}$ such that $B \mathop{\smile\!\!\!\!\!\!|}_D C\{\bar{a}_i \mid i < \kappa\}$.
By regularity, $D \subseteq C\{\bar{a}_i \mid i < \lambda\}$ for some $\lambda < \kappa$.
Hence $B \mathop{\smile\!\!\!\!\!\!|}_{C\{\bar{a}_i \mid i<\lambda\}} C\{\bar{a}_i \mid i < \kappa\}$ by base monotonicity.
Hence $B \mathop{\smile\!\!\!\!\!\!|}_{C\{\bar{a}_i \mid i<\lambda\}} \bar{a}_\lambda$.
Hence $B \mathop{\smile\!\!\!\!\!\!|}_C \bar{a}_\lambda$ by transitivity.
Hence $B \mathop{\smile\!\!\!\!\!\!|}_C \bar{a}$ by invariance. □

The tuned axioms of independence

- invariance
- finite character
- anti-reflexivity

- extension
- local character
- normality

- base monotonicity
- transitivity*

Why are these axioms better?

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Because it is very straightforward to prove, for arbitrary T:

- Forking independence satisfies all axioms except local character.
- Thorn-forking independence satisfies all axioms except local character.
- Dividing independence satisfies all axioms except local character and extension.
- Thorn-dividing independence satisfies all axioms except local character and extension.

We will now re-invent forking and thorn-forking.

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Re-inventing forking

T need not have an independence relation. Or there may be several.

Problem

Define a relation $\mathop{\smile\hskip-0.9em^{|}\ }^{\mathrm{d}}$ such that $A \mathop{\smile\hskip-0.9em^{|}\ }^{\mathrm{d}}_{C} B$ implies $A \mathop{\smile\hskip-0.9em^{|}\ }_{C} B$ for *every* independence relation $\mathop{\smile\hskip-0.9em^{|}\ }$.

Consequence of Lemmas I+II

$B \mathop{\smile\hskip-0.9em^{|}\ }_{C} \bar{a}_0$ iff one of the Morley sequences $(\bar{a}_i)_{i<\omega}$ over C is BC-indiscernible.

We do not know which sequences are Morley sequences for an independence relation $\mathop{\smile\hskip-0.9em^{|}\ }$. So we define:

$$B \mathop{\smile\hskip-0.9em^{|}\ }^{\mathrm{d}}_{C} A$$
$$\Longleftrightarrow$$

for every indiscernible sequence $(\bar{a}_i)_{i<\omega}$ such that $\bar{a}_0 \in A$ there is $B' \equiv_C B$ such that the sequence is BC-indiscernible.

This is Shelah's dividing independence.

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Re-inventing forking

Proposition

$\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}$ satisfies all axioms except extension and local character.

Proof: Straightforward checking.

The new axioms are remarkably robust.

$$A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_C B \iff \text{for every superset } \hat{B} \supset B \text{ there is } A' \equiv_{BC} A \text{ such that } A' \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_C \hat{B}.$$

Proposition

$\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}$ satisfies all axioms except local character.

Proof: Extension holds by definition. The other axioms are preserved.

$\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}$ is Shelah's forking independence. T is *simple* if $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}$ satisfies local character (hence $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}$ is an independence relation).

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Re-inventing thorn-forking

Problem

Define a relation $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{M}}$ such that $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$ for some independence relation $\mathop{\smile\!\!\!\!\!\!\,|}$ implies $A \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{M}}_C B$.

Recall some axioms:

anti-reflexivity

$$a \mathop{\smile\!\!\!\!\!\!\,|}_B a \implies a \in \operatorname{acl} B.$$

base monotonicity For $D \subseteq C \subseteq B$: $A \mathop{\smile\!\!\!\!\!\!\,|}_D B \implies A \mathop{\smile\!\!\!\!\!\!\,|}_C B$.

$$A \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{M}}_D B$$

$$\iff$$

for all C such that $D \subseteq C \subseteq \operatorname{acl}(BD)$,
$\operatorname{acl}(AC) \cap \operatorname{acl}(BD) = \operatorname{acl} C$.

Re-inventing thorn-forking

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for all C such that $D \subseteq C \subseteq \operatorname{acl}(BD)$,
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Re-inventing thorn-forking

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$$\Longleftrightarrow$$

for all C such that $D \subseteq C \subseteq \operatorname{acl}(BD)$,

$$\operatorname{acl}(AC) \cap \operatorname{acl}(BD) = \operatorname{acl} C.$$

Re-inventing thorn-forking

$\mathop{\smile\!\!\!\!\!\!\mid}^{\mathbb{M}}$ is not Alf Onshuus' thorn-dividing independence, but we do have:

Proposition

$\mathop{\smile\!\!\!\!\!\!\mid}^{\mathbb{M}}$ satisfies all axioms except extension and local character.

Proof: Straightforward checking.

The axioms are robust, so we can apply the same trick as before:

$$A \mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}_{C} B$$
$$\iff$$
$$\text{for every superset } \hat{B} \supset B \text{ there is } A' \equiv_{BC} A \text{ such that } A' \mathop{\smile\!\!\!\!\!\!\mid}^{\mathbb{M}}_{C} \hat{B}.$$

Proposition

$\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}$ satisfies all axioms except local character.

$\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$ is precisely Alf Onshuus' thorn-forking independence. T is *rosy* if $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$ satisfies local character (hence $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$ is an independence relation).

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Outlook

- Local theory for forking and thorn-forking.
- Canonical bases.
- O-minimal theories and other pregeometric theories.
- Hyperimaginaries?