The lattice of algebraically closed sets

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Contents:

Independence relations **(1/9)**

Independence relation
= invariant relation satisfying:

(symmetry)
$A \mathop{\smile\!\!\!\!\!\!|}_C B \iff B \mathop{\smile\!\!\!\!\!\!|}_C A$

(monotonicity/finite character)
$A \mathop{\smile\!\!\!\!\!\!|}_C B \iff A_0 \mathop{\smile\!\!\!\!\!\!|}_C B_0$
$\forall$ finite $A_0 \subseteq A$, $B_0 \subseteq B$

(full transitivity)
$A \mathop{\smile\!\!\!\!\!\!|}_D B \iff A \mathop{\smile\!\!\!\!\!\!|}_D C$ and $A \mathop{\smile\!\!\!\!\!\!|}_C B$,
provided that $D \subseteq C \subseteq B$

(extension)
$A \mathop{\smile\!\!\!\!\!\!|}_D C$, $C \subseteq B \quad \Rightarrow \quad \exists A' \equiv_{C \cup D} A : A' \mathop{\smile\!\!\!\!\!\!|}_D B$

(local character)
$\exists \kappa \quad \forall B$, finite $A_0 \quad \exists C \subseteq B$: $\quad A_0 \mathop{\smile\!\!\!\!\!\!|}_C B$
and $|C| < \kappa$.

Not required: boundedness / stationarity over models; independence theorem / amalgamation property / chain condition.

Strict independence relations (2/9)

Easy consequence of the axioms:
$A \mathop{\smile\!\!\!\!\!\!\;|}_C B \iff \mathrm{acl}(A \cup C) \mathop{\smile\!\!\!\!\!\!\;|}_{\mathrm{acl}\, C} \mathrm{acl}(B \cup C)$

$\mathop{\smile\!\!\!\!\!\!\;|}$ is strict
$\iff$ $a \mathop{\smile\!\!\!\!\!\!\;|}_C a$ implies $a \in \mathrm{acl}\, C$
$\iff$ $\mathrm{acl}(A \cup C) \cap \mathrm{acl}(B \cup C) = \mathrm{acl}\, C$
whenever $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$.

Acl = lattice of algebraically closed sets
The axioms can be stated in terms of Acl and its automorphisms.

Examples:
- forking (Shelah, Kim)
- o-minimal independence (Pillay, Steinhorn)
- þ-forking (Scanlon, Onshuus)

Objects of this talk (3/9)

1. Re-invent the wheel:
   Find a strict independence relation that is the correct one for vector spaces and algebraically closed fields (ACF).

2. Name-dropping:
   Mention 4 important mathematicians who were involved in the short boom of lattice theory in the 1930s.

3. Understand thorn-forking.

Modularity **(4/9)**

Definition (J. v. Neumann)
$A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_{C} B \iff \mathrm{acl}(A \cup C) \cap \mathrm{acl}(B \cup C) = \mathrm{acl}\, C.$

Theorem
$\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}$ is a strict independence relation
$\iff$ Acl is modular.

Proof sketch
$\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}$ is almost an independence relation and certainly strict.
Only problem:
$A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_{D} B$ and $D \subseteq C \subseteq B \quad \Rightarrow \quad A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_{C} B$.
This holds $\iff$ Acl is modular.

This works for 1-based theories such as vector spaces. But not for ACF: not modular.

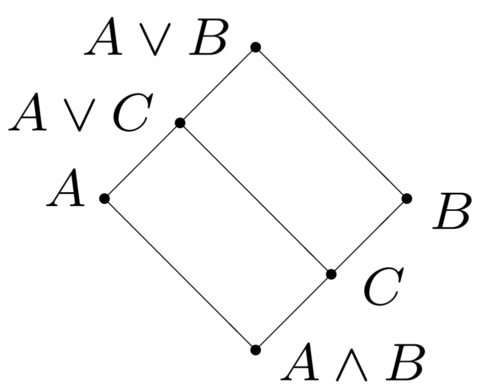
Generalising modularity (5/9)

An active field in the 1930s.
Several attempts:

- Birkhoff's condition (Birkhoff 1935)
- semimodularity (Birkhoff 1935)
- Mac Lane's condition (Mac Lane 1938)
- M-symmetry (Wilcox 1938)

All conditions agree for finite lattices.
All conditions differ in general.

$M(A, B)$ if A and B form a modular pair:



$$C = (A \vee C) \wedge B$$

M-symmetry: $M(A, B) \iff M(B, A)$.

M-dividing (6/9)

Definition (Wilcox 1938)
$A \perp B \iff A \wedge B = 0$ and $M(A, B)$.

Definition (Scheuermann 1996)
$$A \mathop{\smile\!\!\!\!\!\!\!\;|}^{\mathsf{M}}_{C} B \iff \mathrm{acl}(A \cup C) \cap \mathrm{acl}(B \cup C) = \mathrm{acl}\, C \text{ and } M(\mathrm{acl}(A \cup C), \mathrm{acl}(B \cup C)).$$

Theorem (Adler 2005)
$\mathop{\smile\!\!\!\!\!\!\!\;|}^{\mathsf{M}}$ is a strict independence relation
$\iff$ Acl is M-symmetric.

Proof idea
Byunghan Kim: symmetry of dividing implies simplicity. Generalise his proof.

M-forking = þ-forking

$\mathop{\smile}\limits^{|}{}^{\mathsf{M}}$ is the correct notion for 1-based theories (like vector spaces) and for ACF.

But not preserved under taking reducts!
Saharon Shelah:

$$A/C \text{ does not fork over } D \iff \forall B \supseteq C \quad \exists A' \equiv_{C \cup D} A : \quad A'/B \text{ does not divide over } D.$$

Define M-forking $\mathop{\smile}\limits^{|}{}^{*}$ from M-dividing $\mathop{\smile}\limits^{|}{}^{\mathsf{M}}$:

$$A \mathop{\smile}\limits^{|}{}^{*}_{D} C \iff \forall B \supseteq C \quad \exists A' \equiv_{C \cup D} A : \quad A' \mathop{\smile}\limits^{|}{}^{\mathsf{M}}_{D} B.$$

Theorem
M-forking = þ-forking: $\mathop{\smile}\limits^{|}{}^{*} = \mathop{\smile}\limits^{|}{}^{\text{þ}}$.

Some consequences (8/9)

Acl is modular $\iff$ T is 1-based rosy
$\iff$ $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}} = \mathop{\smile\!\!\!\!\!\!|}^{\text{a}}$.

Acl is M-symmetric
$\iff$ T rosy and $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}} = \mathop{\smile\!\!\!\!\!\!|}^{\text{M}}$.

T is rosy
$\iff$ $\exists$ a strict independence relation
$\Rightarrow$ $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ is **the** coarsest.

Hence T simple $\Rightarrow$ $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ is coarser than forking independence, $A \mathop{\smile\!\!\!\!\!\!|}^{\text{f}}_{C} B \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{C} B$.

Theorem (Scheuermann 1996)
A strict independence relation with (weak) canonical bases is **the** coarsest for its theory.

Hence: Forking = þ-forking
for simple theories with EHI.
(first proof by Clifton Ealy)

Related results (9/9)

T trivial 1-based rosy $\iff$ Acl is distributive.

T superrosy $\iff$
T is finitely coded and Acl is arithmetic.

Definitions:

T finitely coded $\iff$
T rosy and every **global** type is free over a finite set. (Much weaker than superrosy!)

Acl arithmetic
$\iff$ compact elements form a sublattice
$\iff$ they form an ideal
$\iff$ every algebraically closed subset of a finitely generated set is finitely generated.