OMAE 2017- 61974

DRIVEN NONLINEAR POTENTIAL FLOW WITH WAVE BREAKING AT SHALLOW-WATER BEACHES

Floriane Gidel*
mmfg@leeds.ac.uk

Onno Bokhove
o.bokhove@leeds.ac.uk

Mark Kelmanson
m.kelmanson@leeds.ac.uk

Department of Applied Mathematics
School of Mathematics
University of Leeds, UK
Project remit from the Maritime Research Institute Netherlands (MARIN):
- Mathematical and numerical modelling of water waves
Introduction

- Project remit from the Maritime Research Institute Netherlands (MARIN):
  - Mathematical and numerical modelling of water waves
    - nonlinear waves (steep waves)
    - dispersive waves (wave-wave interaction, Rogue waves etc.)
    - driven by a wavemaker (piston)
    - fast model: first guess for experiments/CFD simulations
Project remit from the Maritime Research Institute Netherlands (MARIN):

- Mathematical and numerical modelling of water waves
  - nonlinear waves (steep waves)
  - dispersive waves (wave-wave interaction, Rogue waves etc.)
  - driven by a wavemaker (piston)
  - fast model: first guess for experiments/CFD simulations

Method
Introduction

- Project remit from the Maritime Research Institute Netherlands (MARIN):
  - Mathematical and numerical modelling of water waves
    - nonlinear waves (steep waves)
    - dispersive waves (wave-wave interaction, Rogue waves etc.)
    - driven by a wavemaker (piston)
    - fast model: first guess for experiments/CFD simulations

- Method
  - Potential-flow approximation:
    - Water depth, $h(x;t)$
    - Velocity potential, $\Phi(x,z;t)$
Introduction

- Project remit from the Maritime Research Institute Netherlands (MARIN):
  - Mathematical and numerical modelling of water waves
    - nonlinear waves (steep waves)
    - dispersive waves (wave-wave interaction, Rogue waves etc.)
    - driven by a wavemaker (piston)
    - fast model: first guess for experiments/CFD simulations

- Method
  - Potential-flow approximation:
    - Water depth, $h(x,t)$
    - Velocity potential, $\Phi(x,z,t)$

  - Variational approach:
    - Gagarina et al. [1] -> robust time integrators

Introduction

- Project remit from the Maritime Research Institute Netherlands (MARIN):
  - Mathematical and numerical modelling of water waves
    - nonlinear waves (steep waves)
    - dispersive waves (wave-wave interaction, Rogue waves etc.)
    - driven by a wavemaker (piston)
    - fast model: first guess for experiments/CFD simulations

- Method
  - Potential-flow approximation:
    - Water depth, $h(x; t)$
    - Velocity potential, $\Phi(x, z; t)$
  - Variational approach:
    - Gagarina et al. [1] -> robust time integrators

Continuous equations

Variational principle:

\[ 0 = \delta \int_0^T \left\{ \int_{R(t)}^{x_e} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g (z - H(x)) \right] dz \right\} dx \right\} dt. \]

Wave generation in deep water
Variational principle:

\[ 0 = \delta \int_{0}^{T} \int_{R(t)}^{x_e} \int_{0}^{h(x,t)} \left[ \frac{\partial t \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + g (z - H(x)) \right] \, dz \, dx \, dt. \]

Wave generation in deep water
Continuous equations

Variational principle:

\[
0 = \delta \int_0^T \left\{ \int_{R(t)}^{x_e} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g (z - H(x)) \right] \, dz \, dx \right\} \, dt.
\]

Wave generation in deep water

Wave absorption at the beach
Continuous equations

**Variational principle:**

\[
0 = \delta \int_0^T \left\{ \int_{R(t)}^{x_c} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g (z - H(x)) \right] dz dx \\
+ \int_{x_c}^{L_z} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g (z - H(x)) \right] dz dx \right\} dt.
\]

Wave generation in deep water

Wave absorption at the beach
Continuous equations

Variational principle:

\[
0 = \delta \int_0^T \left\{ \int_{R(t)}^{x_e} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g(z - H(x)) \right] dz dx \right. \\
\left. + \int_{x_e}^{Lz} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g(z - H(x)) \right] dz dx \right\} dt.
\]

Wave generation in deep water

- Nonlinear potential-flow equations

\[
\begin{align*}
\nabla^2 \phi &= 0 \quad \text{in } \Omega \\
\partial_t h + \nabla h \cdot \nabla \phi - \partial_z \phi &= 0 \quad \text{at } z = h \\
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(h - b) &= 0 \quad \text{at } z = h \\
\partial_x \phi &= \partial_t R \quad \text{at } x = R
\end{align*}
\]

Wave absorption at the beach
Continuous equations

Variational principle:

\[ 0 = \delta \int_0^T \left\{ \int_{x_c}^{x_e} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g(z - H(x)) \right] dz dx \right. \]

\[ + \int_{x_c}^{x_e} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g(z - H(x)) \right] dz dx \left. \right\} dt. \]

Wave generation in deep water

- Nonlinear potential-flow equations

\[
\begin{align*}
\nabla^2 \phi &= 0 \quad \text{in } \Omega \\
\partial_t h + \nabla h \cdot \nabla \phi - \partial_z \phi &= 0 \quad \text{at } z = h \\
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(h - b) &= 0 \quad \text{at } z = h \\
\partial_x \phi &= \partial_t R \quad \text{at } x = R
\end{align*}
\]

Wave absorption at the beach
Continuous equations

Variational principle:

\[ 0 = \delta \int_0^T \left\{ \int_{R(t)} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g (z - H(x)) \right] dz \, dx \right. \]
\[ + \int_{x_c}^{L_x} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g (z - H(x)) \right] dz \, dx \right\} dt. \]

Wave generation in deep water

- Nonlinear potential-flow equations

\[
\begin{align*}
\nabla^2 \phi &= 0 & \text{in } \Omega \\
\partial_t h + \nabla h \cdot \nabla \phi - \partial_z \phi &= 0 & \text{at } z = h \\
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(h - b) &= 0 & \text{at } z = h \\
\partial_x \phi &= \partial_t R & \text{at } x = R
\end{align*}
\]

Wave absorption at the beach

\[
\begin{align*}
\bar{\phi}(x; t) &= \frac{1}{h} \int_0^h \phi(x, z; t) \, dz & \bar{h}(x; t) &= h(x; t).
\end{align*}
\]

\[
\bar{u}(x; t) = \partial_x \bar{\phi}(x; t);
\]
Continuous equations

Wave generation in deep water

- Nonlinear potential-flow equations

\[
\begin{align*}
\nabla^2 \phi &= 0 & \text{in } \Omega \\
\partial_t h + \nabla h \cdot \nabla \phi - \partial_z \phi &= 0 & \text{at } z = h \\
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(h - b) &= 0 & \text{at } z = h \\
\partial_x \phi &= \partial_t R & \text{at } x = R
\end{align*}
\]

Variational principle:

\[
0 = \delta \int_0^T \left\{ \int_{R(t)}^{x_c} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g(z - H(x)) \right] dz dx \\
+ \int_{x_c}^{L_x} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g(z - H(x)) \right] dz dx \right\} dt.
\]

Wave absorption at the beach

- Nonlinear shallow-water equations

\[
\begin{align*}
\partial_t \tilde{h} + \partial_x (\tilde{h} \tilde{u}) &= 0, & x_c \leq x \leq L_x, \\
\partial_t (\tilde{h} \tilde{u}) + \partial_x \left( \tilde{h} \tilde{u}^2 + \frac{1}{2} g \tilde{h}^2 \right) &= \frac{1}{2} g \tilde{h} \partial_x H, & x_c \leq x \leq L_x, \\
\tilde{h} \tilde{u} &= 0, & x = L_x.
\end{align*}
\]
Continuous equations

Variational principle:

\begin{align*}
0 &= \delta \int_0^T \left\{ \int_{R(x,t)} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g(z - H(x)) \right] \, dz \, dx \\
&\quad + \int_{x_c}^{L} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g(z - H(x)) \right] \, dz \, dx \right\} \, dt.
\end{align*}

Wave generation in deep water

- Nonlinear potential-flow equations

\begin{align*}
\nabla^2 \phi &= 0 & \text{in } \Omega \\
\partial_t h + \nabla h \cdot \nabla \phi - \partial_z \phi &= 0 & \text{at } z = h \\
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(h - b) &= 0 & \text{at } z = h \\
\partial_x \phi &= \partial_t R & \text{at } x = R
\end{align*}

Wave absorption at the beach

- Nonlinear shallow-water equations

\begin{align*}
\partial_t \tilde{h} + \partial_x (\tilde{h} \tilde{u}) &= 0, & x_c \leq x \leq L, \\
\partial_t (\tilde{h} \tilde{u}) + \partial_x \left( \tilde{h} \tilde{u}^2 + \frac{1}{2} g \tilde{h}^2 \right) &= \frac{1}{2} g \tilde{h} \partial_x H, & x_c \leq x \leq L, \\
\tilde{h} \tilde{u} &= 0, & x = L.
\end{align*}

Continuity at the coupling interface

\begin{align*}
\left( \int_0^h \delta \phi (\partial_x \phi) \, dz \right)_{x=x_c} &= \left( \delta \phi \tilde{h} \tilde{u} \right)_{x=x_c}.
\end{align*}
Continuous equations

**Variational principle:**

\[
0 = \delta \int_0^T \left\{ \int_{R(t)} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g(z - H(x)) \right] dz dx \right. \\
+ \int_{x_c}^{L_z} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g(z - H(x)) \right] dz dx \left. \right\} dt.
\]

**Wave generation in deep water**

- Nonlinear potential-flow equations

\[
\begin{align*}
\nabla^2 \phi &= 0 & \text{in } \Omega \\
\partial_t h + \nabla h \cdot \nabla \phi - \partial_z \phi &= 0 & \text{at } z = h \\
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(h - b) &= 0 & \text{at } z = h \\
\partial_x \phi &= \partial_t R & \text{at } x = R
\end{align*}
\]

**Wave absorption at the beach**

- Nonlinear shallow-water equations

\[
\begin{align*}
\partial_t \tilde{h} + \partial_x (\tilde{h} \tilde{u}) &= 0, & x_c \leq x \leq L_x, \\
\partial_t (\tilde{h} \tilde{u}) + \partial_x \left( \tilde{h} \tilde{u}^2 + \frac{1}{2} g \tilde{h}^2 \right) &= \frac{1}{2} g \tilde{h} \partial_x H, & x_c \leq x \leq L_x, \\
\tilde{h} \tilde{u} &= 0, & x = L_x.
\end{align*}
\]

**Continuity at the coupling interface**

\[
\left( \int_0^h \delta \phi (\partial_x \phi) dz \right)_{x = x_c} = \left( \delta \tilde{\phi} \tilde{h} \tilde{u} \right)_{x = x_c}.
\]

- Boundary condition for deep water:

**Outward flux:**

\[
\int_0^h \partial_x \phi dz = \tilde{h} \tilde{u}
\]
Continuous equations

Variational principle:

\[ 0 = \delta \int_0^T \left\{ \int_{R(t)} \int_0^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g(z - H(x)) \right] \, dz \, dx \right. \]
\[ + \left. \int_{x_{c}}^{L_z} \int_{0}^{h(x,t)} \left[ \partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + g(z - H(x)) \right] \, dz \, dx \right\} \, dt. \]

Wave generation in deep water

- Nonlinear potential-flow equations

\[
\begin{align*}
\nabla^2 \phi &= 0 & \text{in } \Omega \\
\partial_t h + \nabla h \cdot \nabla \phi - \partial_z \phi &= 0 & \text{at } z = h \\
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(h - b) &= 0 & \text{at } z = h \\
\partial_x \phi &= \partial_t R & \text{at } x = R
\end{align*}
\]

Wave absorption at the beach

- Nonlinear shallow-water equations

\[
\begin{align*}
\partial_t \ddot{h} + \partial_x (\ddot{h} \dddot{u}) &= 0, & x_c \leq x \leq L_x, \\
\partial_t (\ddot{h} \dddot{u}) + \partial_x \left( \ddot{h} \dddot{u}^2 + \frac{1}{2} g \dddot{h}^2 \right) &= \frac{1}{2} g \ddot{h} \partial_x H, & x_c \leq x \leq L_x, \\
\dddot{u} &= 0, \\
\ddot{h} &= 0, & x = L_x.
\end{align*}
\]

Continuity at the coupling interface

\[
\left( \int_0^h \delta \phi (\partial_x \phi) \, dz \right)_{x = x_c} = \left( \delta \ddot{\phi} \dddot{u} \right)_{x = x_c}. 
\]

- Boundary condition for deep water:

Outward flux:

\[
\int_0^h \partial_x \phi \, dz = \ddot{h} \dddot{u}
\]

- Boundary condition for shallow water:

Incoming flux:

\[
\dddot{u} = \int_0^h \partial_x \phi \, dz,
\]

Continuous depth:

\[
\ddot{h} = h.
\]
Spatial discretisation
Spatial discretisation

Wave generation in deep water
Spatial discretisation

Wave generation in deep water

- Dealing with moving boundaries
  - Wavemaker (prescribed)
  - Nonlinear free-surface (unknown)
Wave generation in deep water

- Dealing with moving boundaries
  - Wavemaker (prescribed)
  - Nonlinear free-surface (unknown)
- Coordinate transforms [3]

Wave generation in deep water

- Dealing with moving boundaries
  - Wavemaker (prescribed)
  - Nonlinear free-surface (unknown)
- Coordinate transforms [3]
- Fixed domain: \( \Omega = \{0 \leq \hat{x} \leq x_c; 0 \leq \hat{z} \leq H_0\} \)
Spatial discretisation

Wave generation in deep water

- Dealing with moving boundaries
  - Wavemaker (prescribed)
  - Nonlinear free-surface (unknown)
- Coordinate transforms [3]
  - Fixed domain: \( \Omega = \{0 \leq \hat{x} \leq x_c; 0 \leq \hat{z} \leq H_0\} \).
- Update the velocity in depth
Spatial discretisation

Wave generation in deep water

- Dealing with moving boundaries
  - Wavemaker (prescribed)
  - Nonlinear free-surface (unknown)
- Coordinate transforms [3]
  - Fixed domain: \( \Omega = \{0 \leq \hat{x} \leq x_c; 0 \leq \hat{z} \leq H_0\} \).
- Update the velocity in depth

\[
\begin{align*}
\nabla^2 \phi &= 0 & \text{in } \Omega \\
\partial_t h + \nabla h \cdot \nabla \phi - \partial_z \phi &= 0 & \text{at } z = h \\
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(h - b) &= 0 & \text{at } z = h \\
\partial_x \phi &= \partial_t R & \text{at } x = R
\end{align*}
\]
Spatial discretisation

Wave generation in deep water

- Dealing with moving boundaries
  - Wavemaker (prescribed)
  - Nonlinear free-surface (unknown)
- Coordinate transforms [3]
  - Fixed domain: \( \Omega = \{0 \leq \hat{x} \leq x_c; 0 \leq \hat{z} \leq H_0\} \).
- Update the velocity in depth
- Separation of variable:
  \[
  \phi_h(x, z; t) = \psi_i(x; t)\bar{\varphi}_i(z)
  \]
Wave generation in deep water

- Dealing with moving boundaries
  - Wavemaker (prescribed)
  - Nonlinear free-surface (unknown)
- Coordinate transforms [3]
  - Fixed domain: \[ \Omega = \{0 \leq \hat{x} \leq x_c; 0 \leq \hat{z} \leq H_0\}. \]
- Update the velocity in depth
- Separation of variable:
  \[ \phi_h(x, z; t) = \psi_i(x; t)\tilde{\varphi}_i(z) \]
- One element in the vertical

Spatial discretisation
Spatial discretisation

Wave generation in deep water

- Dealing with moving boundaries
  - Wavemaker (prescribed)
  - Nonlinear free-surface (unknown)
- Coordinate transforms [3]
  - Fixed domain: \( \Omega = \{0 \leq \hat{x} \leq \hat{x}_c; 0 \leq \hat{z} \leq \hat{H}_0\} \).
- Update the velocity in depth
- Separation of variable:
  \[ \phi_h(x, z; t) = \psi_i(x; t)\tilde{\varphi}_i(z) \]
- One element in the vertical
  
High order
Lagrange expansions
Spatial discretisation

Wave generation in deep water

- Dealing with moving boundaries
  - Wavemaker (prescribed)
  - Nonlinear free-surface (unknown)
- Coordinate transforms [3]
  - Fixed domain: $\Omega = \{0 \leq \hat{x} \leq x_c; 0 \leq \hat{z} \leq H_0\}$
- Update the velocity in depth
- Separation of variable:
  $$\phi_h(x, z; t) = \psi_i(x; t)\phi_i(z)$$
- One element in the vertical
- Horizontal VP, with unknowns:
  $$\begin{cases} 
  \psi_1(x; t), \ h(x; t), \\
  \psi_i(x; t), \\
  \psi_{n+1} \end{cases}$$
  at the surface
  on interior layers
Wave generation in deep water

- Dealing with moving boundaries
  - Wavemaker (prescribed)
  - Nonlinear free-surface (unknown)
- Coordinate transforms \([3]\)
  - Fixed domain: \(\Omega = \{0 \leq \hat{x} \leq x_c; 0 \leq \hat{z} \leq H_0\}\)
- Update the velocity in depth
- Separation of variable:
  \[\phi_h(x, z; t) = \psi_i(x; t)\tilde{\varphi}_i(z)\]
- One element in the vertical
- Horizontal VP, with unknowns:
  \[
  \begin{cases}
  \psi_1(x; t), \ h(x; t), & \text{at the surface} \\
  \psi_{i'} = \psi_{i'}(\psi_1, h, t) & \text{on interior layers}
  \end{cases}
  \]
- Elimination of the interior velocity
Spatial discretisation

Wave generation in deep water

- Dealing with moving boundaries
  - Wavemaker (prescribed)
  - Nonlinear free-surface (unknown)
- Coordinate transforms [3]
  - Fixed domain: $\Omega = \{0 \leq \hat{x} \leq x_c; 0 \leq \hat{z} \leq H_0\}$.
- Update the velocity in depth
- Separation of variable:
  $$\phi_h(x, z; t) = \psi_i(x; t)\tilde{\varphi}_i(z)$$
- One element in the vertical
- Horizontal VP, with unknowns:
  $$\begin{cases} 
  \psi_1(x; t), h(x; t), \\
  \psi_i' = \psi_i'(\psi_1, h, t)
  \end{cases}$$
  - Elimination of the interior velocity
- Finite element method in the horizontal
  - 1\textsuperscript{st}-order continuous Galerkin functions
Wave generation in deep water

- Dealing with moving boundaries
  - Wavemaker (prescribed)
  - Nonlinear free-surface (unknown)
- Coordinate transforms [3]
  - Fixed domain: \( \Omega = \{0 \leq \hat{x} \leq x_c; 0 \leq \hat{z} \leq H_0\} \)
- Update the velocity in depth
- Separation of variable:
  \[
  \phi_h(x, z; t) = \psi_i(x; t) \varphi_i(z)
  \]
- One element in the vertical
- Horizontal VP, with unknowns:
  \[
  \begin{cases}
  \psi_1(x; t), h(x; t), & \text{at the surface} \\
  \psi_i = \psi_i(\psi_1, h, t) & \text{on interior layers}
  \end{cases}
  \]
- Elimination of the interior velocity
- Finite element method in the horizontal
  - 1st-order continuous Galerkin functions
- Firedrake [4]

Spatial discretisation

Wave absorption at the beach
Spatial discretisation

Wave absorption at the beach

- Shallow-water equations:

\[
\partial_t \left( \tilde{h} \tilde{u} \right) + \partial_x \left( \tilde{h} \tilde{u}^2 + \frac{1}{2} g \tilde{h}^2 \right) = \left( \frac{1}{2} g \tilde{h} \partial_x H \right).
\]
Wave breaking when $H(x) \ll \lambda$

- Shallow-water equations:

$$\partial_t \left( \tilde{h} \tilde{h} \right) + \partial_x \left( \tilde{h} \tilde{h} u + \frac{1}{2} g \tilde{h}^2 \right) = \left( \frac{1}{2} g \tilde{h} \partial_x H \right).$$
Spatial discretisation

- Wave breaking when $H(x) \ll \lambda$
  - Discontinuous: FEM not stable
    $\rightarrow$ Finite volume method

- Shallow-water equations:
  $$\partial_t \begin{pmatrix} \frac{\partial}{\partial x} 
  \frac{\partial h}{\partial x} \end{pmatrix} + \partial_x \begin{pmatrix} \frac{\partial}{\partial x} 
  h\bar{u} & \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} \bar{u} \right) \end{pmatrix} = \begin{pmatrix} 0 
  \frac{1}{2} g \bar{h} \partial_x H \end{pmatrix}.$$
Spatial discretisation

- Wave breaking when $H(x) \ll \lambda$
  - Discontinuous: FEM not stable
    $\rightarrow$ Finite volume method

- FV mesh
  - Piecewise constant functions
  - Average over the cell

- Shallow-water equations:

  $\partial_t \left( \bar{h} \bar{u} \right) + \partial_x \left( \bar{h}^2 \bar{u} + \frac{1}{2} g \bar{h}^2 \right) = \begin{pmatrix} 0 \\ \frac{1}{2} g \bar{h} \partial_x H \end{pmatrix}.$
Spatial discretisation

- Wave breaking when $H(x) \ll \lambda$
  - Discontinuous: FEM not stable
  → Finite volume method

- FV mesh
  - Piecewise constant functions
  - Average over the cell

- Shallow-water equations:
  
  \[
  \begin{aligned}
  \partial_t \left( \frac{\tilde{h}}{\tilde{h} \tilde{u}} \right) + \partial_x \left( \frac{\tilde{h} \tilde{u}^2 + \frac{1}{2} g \tilde{h}^2}{\tilde{h}} \right) &= \begin{pmatrix} 0 \\ \frac{1}{2} g \tilde{h} \partial_x H \end{pmatrix}.
  \end{aligned}
  \]

  Where:
  - $U$ is the state vector
  - $F(U)$ is the flux vector
  - $S$ is the topography vector

  \[
  \dot{U}_k(t) + \frac{1}{\Delta x} (F_{k+1/2}(t) - F_{k-1/2}(t)) = S_k.
  \]
- Wave breaking when \( H(x) \ll \lambda \)
  - Discontinuous: FEM not stable
    → Finite volume method
- FV mesh
  - Piecewise constant functions
  - Average over the cell
- Discontinuous flux at the interface
  → HLL flux \((F_L, F_R, \text{both})\) [5]

---

Wave breaking when $H(x) \ll \lambda$
- Discontinuous: FEM not stable
  → Finite volume method

- Shallow-water equations:

\[
\begin{align*}
\partial_t \left( \begin{array}{c} \bar{h} \\ \bar{\nu} \end{array} \right) + \partial_x \left( \begin{array}{c} \bar{h} \bar{\nu} \\ \bar{\nu}^2 + \frac{1}{2} g \bar{h}^2 \end{array} \right) &= \begin{array}{c} 0 \\ \frac{1}{2} g \bar{h} \partial_x H \end{array} \\
\end{align*}
\]

- Dry beach
  → Audusse method [6]

Temporal discretisation

$\tilde{x_c}$
Temporal discretisation

Deep water:

\[ x_c \]
Deep water:

- VP in Hamiltonian form:

\[ 0 = \delta \int_0^T p_{1k} \frac{dh_k}{dt} - H(h, p_1, t) \, dt, \]
Temporal discretisation

Deep water:

- VP in Hamiltonian form:

\[ 0 = \delta \int_0^T p_{1k} \frac{dh_k}{dt} - H(h, p_1, t) \, dt, \]

\[ \delta p_k : \quad \frac{dh_k}{dt} = \frac{\partial H(h, p_1, t)}{\partial p_1} \]

\[ \delta h_k : \quad \frac{dp_k}{dt} = \frac{\partial H(h, p_1, t)}{\partial h} \]
Deep water:

1st order Symplectic-Euler scheme

\[
\begin{align*}
\delta p_k : \quad \frac{dh_k}{dt} &= \frac{\partial H(h, p_1, t)}{\partial p_1} \\
\delta h_k : \quad \frac{dp_k}{dt} &= \frac{\partial H(h, p_1, t)}{\partial h}
\end{align*}
\]
Deep water:

- Implicit update of the depth

1\textsuperscript{st} order Symplectic-Euler scheme

$$h^{n+1} = h^n + \Delta t \frac{\partial H(p_1^n, h^{n+1}, t^n)}{\partial p_1^n}$$
Deep water:

- Implicit update of the depth
- Explicit update of the surface velocity

1\textsuperscript{st} order Symplectic-Euler scheme

\begin{align*}
  h_{n+1} &= h_n + \Delta t \frac{\partial H(p_1^n, h^{n+1}_n, t^n)}{\partial p_1^n} \\
  p_{1,n+1} &= p_{1,n} - \Delta t \frac{\partial H(p_1^n, h^{n+1}_n, t^n)}{\partial h^{n+1}_n}
\end{align*}
Temporal discretisation

**Deep water:**
- Implicit update of the depth
- Explicit update of the surface velocity

**Shallow water:**

1st order Symplectic-Euler scheme

\[
\begin{align*}
    h^{n+1} &= h^n + \Delta t \frac{\partial H}{\partial p_1^n} \\
    p_1^{n+1} &= p_1^n - \Delta t \frac{\partial H}{\partial h^{n+1}}
\end{align*}
\]
Temporal discretisation

Deep water:

- Implicit update of the depth
- Explicit update of the surface velocity

1\textsuperscript{st} order Symplectic-Euler scheme

\[
\begin{align*}
    h^{n+1} &= h^n + \Delta t \frac{\partial H (p_1^n, h^{n+1}, t^n)}{\partial p_1^n} \\
    p_1^{n+1} &= p_1^n - \Delta t \frac{\partial H (p_1^n, h^{n+1}, t^n)}{\partial h^{n+1}}
\end{align*}
\]

Shallow water:

- Space-discrete equation:

\[
\dot{U}_k(t) + \frac{1}{\Delta x} (F_{k+1/2}(t) - F_{k-1/2}(t)) = S_k.
\]
Temporal discretisation

Deep water:

- Implicit update of the depth
- Explicit update of the surface velocity

Shallow water:

- Space-discrete equation:

\[
\dot{U}_k(t) + \frac{1}{\Delta x} \left( F_{k+1/2}(t) - F_{k-1/2}(t) \right) = S_k.
\]

\[
U_{k+1} = U_k - \frac{\Delta t}{\Delta x} \left( F_{k+1/2} - F_{k-1/2} - S_k^n \right) dt.
\]

1st order Symplectic-Euler scheme

\[
h_{n+1} = h^n + \Delta t \frac{\partial H}{\partial p^n_1} \left( p^n_1, h^{n+1}, t^n \right)
\]

\[
p^{n+1}_1 = p^n_1 - \Delta t \frac{\partial H}{\partial h^{n+1}} \left( p^n_1, h^{n+1}, t^n \right)
\]

Godunov scheme (forward Euler)
Temporal discretisation

Deep water:
- Implicit update of the depth
- Explicit update of the surface velocity

Shallow water:
- Space-discrete equation:
  - Fully explicit

1\textsuperscript{st} order Symplectic-Euler scheme
\[ h^{n+1} = h^n + \Delta t \frac{\partial H(p_1^n, h^{n+1}, t^n)}{\partial p_1^n} \]
\[ p_1^{n+1} = p_1^n - \Delta t \frac{\partial H(p_1^n, h^{n+1}, t^n)}{\partial h^{n+1}} \]

Godunov scheme (forward Euler)
\[ U_{k}^{n+1} = U_{k}^{n} - \frac{\Delta t}{\Delta x} \left( F_{k+1/2}^{n} - F_{k-1/2}^{n} - S_{k}^{n} \right) dt. \]
- Averaged flux:
\[ F_{k+1/2}^{n} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} F_{k+1/2}(t) dt \]
Algorithm

Initialization:

\[ h^0 = H(x); \quad \phi^0 = 0; \]
\[ \ddot{h}^0 = H(x); \quad \ddot{h}u^0 = 0; \]
Initialization:

\[
\begin{align*}
    h^0 &= H(x); \quad \phi^0 = 0; \\
    \tilde{h}^0 &= H(x); \quad \tilde{h}\tilde{u}^0 = 0;
\end{align*}
\]

Flux at \(x = x_c\) in SW

\[ (\tilde{h}\tilde{u})^{n} \]
Algorithm

Initialization:

\[ h^0 = H(x); \quad \phi^0 = 0; \]
\[ \tilde{h}^0 = H(x); \quad \tilde{h} \phi^0 = 0; \]

Flux at \( x=x_c \) in SW

\[ (\tilde{h} \phi)^n \]

Compute DW BCs

\[ x_C \]

\[ F_{-1/2} \]
Algorithm

Initialization:
\[ h^0 = H(x); \quad \phi^0 = 0; \]
\[ \tilde{h}^0 = H(x); \quad \tilde{h}\tilde{\nu}^0 = 0; \]

Flux at \( x = x_c \) in SW
\[ (\tilde{h}\tilde{\nu})^\tau \]

Compute DW BCs

Solve the DW equations
Algorithm

Initialization:

\[ h^0 = H(x); \quad \phi^0 = 0; \]
\[ \tilde{h}^0 = H(x); \quad \tilde{h}\tilde{u}^0 = 0; \]

Flux at \( x = x_c \) in SW

Compute DW BCs

Solutions at \( x = x_c \) in DW

\[ h^{n+1}, \psi^{n+1}_1, \psi^{n+1}_i, \psi^{n+1}_i \]

Solve the DW equations

\[ x_C \]

\[ F_{-1/2} \]
Algorithm

 Initialization:

\[ h^0 = H(x); \quad \phi^0 = 0; \]
\[ \ddot{h}^0 = H(x); \quad \ddot{\phi}^0 = 0; \]

Solution at \( x = x_c \) in SW

\[ \left( \ddot{h} \ddot{\phi} \right)_{n} \]

Compute DW BCs

Solve the DW equations

Solutions at \( x = x_c \) in DW

\[ h^{n+1}, \psi_1^{n+1}, \psi_i^*, \psi_i^{n+1} \]

Compute SW BCs

\[ \mathcal{F}_{-1/2} \]
Algorithm

Initialization:

$F = F_1 / 2$

$x_c$

$h_n, \psi_n, \phi_n$

$h_{n+1}, \psi_{n+1}, \phi_{n+1}$

Solutions at $x = x_c$ in SW

Compute SW BCs

Solve the SW equations

Compute DW BCs

Solve the DW equations

Flux at $x = x_c$ in SW

$h_0 = H(x)$; $\phi_0 = 0$; $\overline{h_0} = 0$
Algorithm

Initialization:
\[ h^0 = H(x); \quad \phi^0 = 0; \]
\[ \bar{h}^0 = H(x); \quad \bar{h}\bar{u}^0 = 0; \]

Solve the SW equations

Flux at \( x = x_c \) in SW
\( (\bar{h}\bar{u})^n \)

Compute DW BCs

Compute SW BCs

Solve the DW equations

Solutions at \( x = x_c \) in DW
\[ h^{n+1}, \psi_1^{n+1}, \psi_i^*, \psi_i^{n+1} \]
Results

Water depth

$\lambda = 2.0 \, \text{m} \, ; \, H_0 = 1.0 \, \text{m}$
Results

Water depth

H(x) \geq \frac{\lambda}{2}

\lambda = 2.0 \text{ m} ; H_0 = 1.0 \text{ m}
Results

Water depth

\[ H(x) \geq \frac{\lambda}{2} \quad \text{and} \quad H(x) \leq \frac{\lambda}{20} \]

\[ \lambda = 2.0 \text{ m} ; H_0 = 1.0 \text{ m} \]
Results

Water depth

\[ H(x) \geq \frac{\lambda}{2} \quad \text{H(x) \leq \frac{\lambda}{20}} \]

Coupling point: \[ \frac{\lambda}{6} \]

\[ \lambda = 2.0 \, \text{m} \, ; \, H_0 = 1.0 \, \text{m} \]
Results

Velocity

x-velocity X

z-velocity
• Deep water model validated against experiments at the Maritime Research Institute Netherlands (MARIN)
Summary

- Deep water model validated against experiments at the Maritime Research Institute Netherlands (MARIN)

- Coupled model:
  - ✓ Transparent boundary condition (absorbed waves in SW)
  - ✓ Low computational time ($O(h)$ on one core)
  - ✓ Possibility to couple the model to wave-structure interactions

Salwa (SURFs-UP project)
Summary

- Deep water model validated against experiments at the Maritime Research Institute Netherlands (MARIN)

- Coupled model:
  - ✓ Transparent boundary condition (absorbed waves in SW)
  - ✓ Low computational time ($O(h)$ on one core)
  - ✓ Possibility to couple the model to wave-structure interactions

- Future improvements:
  - o Higher resolution and parallel computations
  - o Optimized location of the coupling point
  - o 3D (already implemented for deep water)
  - o Validation against measurements at MARIN (August)
Summary

- Deep water model validated against experiments at the Maritime Research Institute Netherlands (MARIN)

- Coupled model:
  - ✓ Transparent boundary condition (absorbed waves in SW)
  - ✓ Low computational time (O(h) on one core)
  - ✓ Possibility to couple the model to wave-structure interactions

- Future improvements:
  - o Higher resolution and parallel computations
  - o Optimized location of the coupling point
  - o 3D (already implemented for deep water)
  - o Validation against measurements at MARIN (August)

Thank you!