The influence of bottom topography on the linear stability of baroclinic waves in the thermally driven rotating annulus

Thomas von Larcher\textsuperscript{1}, Alexandre Fournier\textsuperscript{2}, and Rainer Hollerbach\textsuperscript{3}

\textsuperscript{1}Institute for Mathematics, Freie Universität Berlin, Berlin, Germany,
\textsuperscript{2}Institut de Physique du Globe de Paris, Paris, France,
\textsuperscript{3}Department of Applied Mathematics, University of Leeds, Leeds, LS2 9JT, UK

Summary

We present results of a linear stability analysis of non-axisymmetric thermally driven flows in the classical model of the rotating cylindrical gap of fluid with a lateral temperature gradient and a sloping bottom endwall configuration. For comparison, results of a flat bottom endwall configuration are also discussed. In both cases, the model set-up has a free top surface.

With respect to the model geometry, we make use of a Fourier-spectral element method, that raise considerable simplifications in mesh design and in implementation of the method.

Motivation

Our model set-up (fig. 1) is conform with the classical cylindrical gap of fluid, which rotates uniformly around its vertical axis of symmetry, with a radial temperature difference between its inner and outer vertical sidewalls. A sloping bottom endwall configuration allows one to mimic the $\beta$-effect, which with respect to terrestrial conditions characterizes the variation of the Coriolis parameter with latitude due to the earth’s spherical shape (cf. [1]). In the sloping bottom endwall case, the bottom has an angle $\gamma = 35^\circ$ to the horizontal, with $\frac{\partial d}{\partial s} > 0$, where $s$ is the radial coordinate.

Fully 3D wavy flow patterns evolve from the release of potential energy through baroclinic instability that occur due to the temperature gradient and rotation. A vertically and horizontally sheared mean flow develops from which regular and complex wave flows of different wave number can then emerge. The flow regimes are determined mainly by two forcing parameters, i.e. the temperature difference $\Delta T$ and the angular velocity $\Omega$. The Prandtl number is also a key parameter (e.g., [2]), but is kept fixed in our study.

Our study presented here is concerned with the influence of the bottom endwall configuration on the onset of linear instability of baroclinic waves. We therefore compute the instability curve which denotes the transition where small perturbations of the axisymmetric basic flow can growth and baroclinic waves of different wave number can then develop. Recent studies focus
Figure 1: Sketch of the model set-up with $a$ as the inner radius, $b$ as the outer radius, $d$ as the fluid depth and $\gamma$ as the angle of inclination of the sloping bottom endwall. An exemplary drawing of a baroclinic wave pattern is also shown, i.e. the meandering jet with ridges at the outer sidewall and troughs at the inner sidewall, with wave number $m = 4$. The flow pattern propagates prograde at the free top surface.

It is worth to note, that our numerical model is based on a laboratory set-up, described in detail in [3], that is one of only few exclusive reference experiments within the german priority program 'Multiple Scales in Fluid Mechanics and Meteorology' (MetStröm) that focus on the development of appropriate spatiotemporal, multiple scales numerical model concepts (see http://metstroem.mi.fu-berlin.de).

**Governing equations, parameters, and boundary conditions**

In their nondimensional form, the governing equations to be solved are

\[
\frac{\partial \tilde{U}}{\partial t} + \tilde{U} \cdot \nabla \tilde{U} + \sqrt{Ta} \hat{e}_z \times \tilde{U} = -\nabla p + \nabla^2 \tilde{U} + Ra \frac{1}{Pr} \Theta \hat{e}_z + \Delta T \frac{Ta}{4} \hat{e}_s \quad (1)
\]

\[
\frac{\partial \Theta}{\partial t} + \tilde{U} \cdot \nabla \Theta = \frac{1}{Pr} \nabla^2 \Theta \quad (2)
\]
together with the incompressibility condition \( \nabla \vec{U} = 0 \). The Navier-Stokes equation has been written in the Boussinesq approximation, in which variations in density are included only in the buoyancy term. Length has been scaled by the gap width \((b-a)\), time by the viscous diffusion time \((b-a)^2/\nu\), \(U\) by \((\nu/(b-a))\), and \(T\) by \(\Delta T\).

The nondimensional parameters are the Taylor number \(Ta\), measuring the rotation, the Rayleigh number \(Ra\), measuring the thermal forcing, and the Prandtl number \(Pr\):

\[
Ta = \frac{4 \Omega^2 (b-a)^4}{\nu^2}, \quad Ra = \frac{\alpha \Delta T g (b-a)^3}{\nu \kappa}, \quad Pr = \frac{\nu}{\kappa},
\]

measuring material properties of the fluid, specifically the ratio of the viscosity \(\nu\) to the thermal diffusivity \(\kappa\) \((g\) is the acceleration due to gravity, and \(\alpha\) is the volumetric expansion coefficient of the fluid). We will choose the Prandtl number to \(Pr = 7.16\), as in the experiment of von Larcher et al (cf. [3]).

The boundary conditions associated with Equation 1 and 2 are

\[
(\vec{U} = 0, \Theta = 0), \quad (\vec{U} = 0, \Theta = 1)
\]

at the inner and outer sidewalls, respectively.

The boundary conditions at the top are potentially more complicated, and depend first of all on whether the top is free or a rigid lid. We will take the top to be free, as in [3]. The appropriate boundary conditions are then

\[
U_z = 0, \quad \partial_z U_s = 0, \quad \partial_z U_\Phi = 0, \quad \partial_z \Theta = 0.
\]

Finally, the boundary conditions on the – either flat or sloping – bottom boundary are

\[
\vec{U} = 0, \quad \partial_n \Theta = 0
\]

where \(n\) is the direction normal to the boundary (and thus depends on the bottom angle \(\gamma\)).

These equations and associated boundary conditions are solved using the finite-spectral-element code developed by Fournier et al ([4], [5]), designed to cope with any axisymmetric container, as our annulus here is. The axisymmetry of the container is first used to decompose the problem into Fourier modes in \(\phi\); in the meridional \((s,z)\) plane each Fourier mode is then decomposed into finite-spectral-elements.

The Fourier decomposition in \(\phi\) is particularly convenient for problems of the type considered here, where one has an axisymmetric basic state and then wishes to consider the onset of non-axisymmetric instabilities. With a Fourier decomposition in \(\phi\) as part of the code already, one can then avoid fully 3D calculations completely, and instead only has a series of 2D calculations, first for the axisymmetric basic state, and then to test the stability of single non-axisymmetric modes at a time.
Results

We find significant differences between the instability curve for the sloping and the flat bottom endwall configuration. In case of a sloping bottom endwall, the wave flow regime is extended to lower rotation rates, i.e. the transition curve is shifted systematically to lower Taylor numbers. Moreover, the upper part of the transition curve becomes almost horizontal in the flat bottom endwall case, where it is then almost independent of the rotation rate. Instead, a keen reversement of the instability curve is found in the sloping bottom endwall case.

References


