Bifurcations to three-dimensional flows in a periodically driven cylindrical cavity

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Consider a Newtonian fluid of kinematic viscosity $\nu$ and density $\rho$ confined in a finite cylinder of radius $R$ and height $H$, whose sidewall oscillates harmonically in the axial direction, with period $T$ and maximum axial velocity $V_{\text{max}}$, while the top and bottom lids remain at rest. There are three non-dimensional parameters in this problem: the aspect ratio, $G = H/R$, the Reynolds number, $Re = V_{\text{max}}R/\nu$, and the Stokes number, $St = R^2/\nu T$. In the current study, the aspect ratio is fixed at $G = 2$. The flow is governed by the usual Navier-Stokes equations:

\begin{equation}
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u, \quad (1a)
\end{equation}

\begin{equation}
\nabla \cdot u = 0, \quad (1b)
\end{equation}

with no-slip velocity boundary conditions on all walls:

\begin{equation}
u(r, \theta, \pm H/2, t) = (0, 0, 0), \quad (2a)
\end{equation}

\begin{equation}
u(R, \theta, z, t) = (0, 0, V_{\text{max}} \sin(2\pi t/T)). \quad (2b)
\end{equation}

where $\nu = (u, v, w)$ is the velocity field in cylindrical coordinates $(r, \theta, z) \in [0, R] \times [0, 2\pi] \times [-H/2, H/2]$, and $p$ is the kinematic pressure. The governing equations and boundary conditions are invariant to the following spatial symmetries:

\begin{equation}
K_\theta(u, v, w)(r, \theta, z, t) = (u, -v, w)(r, -\theta, z, t), \quad (3a)
\end{equation}

\begin{equation}
R_\alpha(u, v, w)(r, \theta, z, t) = (u, v, w)(r, \theta + \alpha, z, t), \quad (3b)
\end{equation}

for any real $\alpha$. $K_\theta$ represents reflections about any meridional plane, whilst $R_\alpha$ signifies rotations about the cylinder axis. $K_\theta$ and $R_\alpha$ generate the groups $Z_2$ and $SO(2)$, but the two operators do not commute, so the symmetry group generated by $K_\theta$ and $R_\alpha$ is $O(2)$ and it acts in the periodic azimuthal $\theta$-direction. The horizontal reflection on the mid-plane $z = 0$ acts on the velocity field as:

\begin{equation}
K_z(u, v, w)(r, \theta, z, t) = (u, v, -w)(r, \theta, -z, t). \quad (4)
\end{equation}
Due to the harmonic oscillation of the sidewall, the boundary condition (2b) is not $K_z$ invariant. However, the axial velocity on the sidewall satisfies $K_z w(R, \theta, z, t) = w(R, \theta, z, t + T/2)$, and the system is invariant to the spatio-temporal symmetry consisting of a reflection about the mid-plane $z = 0$ together with a half-period evolution in time:

$$H(u, v, w)(r, \theta, z, t) = (u, v, -w)(r, \theta, -z, t + T/2).$$

The transformation $H$ generates another $Z_2$ symmetry group that commutes with $O(2)$. Hence, the complete symmetry group of the problem is $Z_2 \times O(2)$ [1]. Mention that these symmetries are analogous to those that were present in bluff-body wakes [2], and periodically forced rectangular [3] and annular [4] cavities. Therefore, the basic flow, having all the symmetries of the problem, is always axisymmetric and time-periodic, synchronous with the forcing. By increasing $Re$ beyond a critical value $Re_c(St)$, the basic state undergoes a symmetry-breaking bifurcation yielding a new 3D state. As it was predicted by Marques et al in [5], the basic state may undergo either synchronous or Neimark-Sacker bifurcations depending on the parameters. The bifurcation curves for the different modes in $(St, Re)$-space are shown in Figure 1. At low $St$, mode $B$ is the first to become critical with increasing $Re$, while at high $St$ mode $A$ is the first. At intermediate values, the quasiperiodic mode bifurcates first, in the form of modulated rotating waves $MRW$. The synchronous mode $A$ always has an azimuthal wavenumber $m = 2$ ($A_2$), the quasiperiodic mode has $m = 1$ ($MRW_1$), and the synchronous mode $B$ may have either $m = 1$ or $m = 2$ depending on $St$.

As the bifurcated solutions are no longer axisymmetric, the $O(2)$ symmetry has been broken in all cases. On the one hand, for the synchronous modes, of the continuous family of rotations

![Figure 1: Critical Reynolds number, $Re_c$, as a function of the Stokes number, $St$, for the transition from the basic state to the different three-dimensional states, $B_1, B_2, MRW_1$ and $A_2$. The crosses correspond to the four basic states depicted in [1].](image-url)
there only remains the discrete symmetry $R_2\pi/m$, a rotation of angle $2\pi/m$ around the axis, and its powers. The azimuthal wavenumber of the bifurcated solution is $m$. The continuous family of reflection symmetries about meridional planes, $K_\theta$, is also reduced to a collection of $m$ reflection planes at angles $\pi/m$ apart. The only difference between modes $A$ and $B$, is the fact that $B_1$ and $B_2$ are $H$-symmetric, meanwhile $A_2$ breaks this symmetry. However, $H$ combined with the rotation $R_\pi/m$, with $m = 2$ (half the angle of the rotational symmetry of the state), results in a space-time symmetry of the $A_2$ eigenfunction. On the other hand, the quasiperiodic mode bifurcation results in a modulated $\theta$-travelling wave, that can travel in the positive or negative $\theta$-direction; after a period of the forcing, the flow pattern repeats itself, but rotated a certain angle, $\pm \theta_0$, related to a quasiperiodic frequency. All the symmetries are broken in this case, but a new space-time symmetry (consisting of advancing one forcing period in time combined with the rotation $R_{\pm \theta_0}$) emerges.

The bifurcations to the four different states ($B_1$, $B_2$, MRW, and $A_2$) when varying the forcing frequency $St$ are separated by three codimension-two bifurcation points at which two of the states bifurcate simultaneously. An examination of the dynamics in the neighborhood of the codimension-two points where two distinct modes bifurcate simultaneously is currently being performed; these codimension-two points act as organizing centers of the dynamics, and are very likely associated with the secondary bifurcations to mixed modes and more complex dynamics. Although we have obtained some results that need to be fully analyzed, it seems that well-organised patterns might arise also in this case.

References


