The nature of the transition from axisymmetric, toroidal Taylor vortices to wavy vortex flow is much less clear than the transition from stable flow to Taylor vortices. The traditional picture of the transition to wavy vortex flow involves the toroidal Taylor vortices acquiring a travelling azimuthal waviness, evident as a periodic deformation of the vortices in the axial direction, with all vortices in the annulus having the same phase. But this view is not quite correct. While the axisymmetric cellular structure of non-wavy Taylor vortices consists of independent streamtubes, wavy vortices have substantial fluid transport between vortices that results in streams of axial flow that extend axially through the entire stack of vortices [1-3].

Details of wavy vortex flow vary depending on the experimental setup or the numerical approach used. The number of azimuthal waves is typically 3 to 7 [4], depending on the conditions by which the transition is approached. The waves travel at a speed that is similar to the azimuthal velocity at the center of a vortex [3], which is 30 to 50% of the inner cylinder surface [1, 2, 4-7]. The speed decreases as the Taylor number increases [2, 4, 6], increases as the radius ratio ($\eta = R_i/R_o$) increases [5, 6], and is nearly independent of the number of waves [6, 8, 9]. The Taylor number for transition to wavy vortices is not firmly established. For a radius ratio of $\eta=0.85$ and infinitely long cylinders, transition is predicted to occur at a reduced rotating Reynolds number, $\varepsilon = Re/Re_c - 1 \approx 0.13$, where $Re_c$ is the critical rotating Reynolds number for transition to Taylor vortex flow [5, 10]. For finite-length cylinders, experiments and simulations indicate higher critical values for $\varepsilon$ [9, 11, 12].

While nonlinear theory has been used to successfully predict the onset of waviness, the physical mechanism by which the flow becomes wavy and the axial velocity field develops has been the subject of surprisingly little inquiry. Marcus suggested that a local, inviscid centrifugal instability of the radial outflow jet is responsible for the azimuthal waviness [1]. However, it has been noted that the radial outflow results in strong azimuthal flows in the outflow region as it transports high azimuthal momentum fluid outward [5, 13]. Akonur and Lueptow went further to show experimentally that a shear layer comes about as the vortical motion carries high azimuthal velocity fluid outward from near the inner cylinder and carries low azimuthal velocity fluid inward from near the outer cylinder [3]. The fluid at the top of one vortex has high azimuthal velocity, while fluid at the bottom of that vortex has low azimuthal velocity, with the next adjacent vortex having exactly the reverse. A typical azimuthal velocity profile in figure 1 shows the nature of the resulting shear layers.
However, it has been argued that this shear instability should not occur in Taylor-Couette flow because the shear layers are too weak [14].

Figure 1. Azimuthal velocity in a frame rotating the traveling wave at a cylindrical surface midway across the annular gap for $\varepsilon = 1.53$. The axial coordinate is $\zeta = z/d$, and the horizontal coordinate is $\gamma = [(R_i + R_o)/2]0/d$, corresponding to the dimensionless azimuthal distance along the cylindrical surface. $\ast =$ Vortex centers; $\circ =$ inflow or outflow regions. Solid curves are least-squares fits. Reproduced from figure 7 of [3].

Nevertheless, the inflections in these azimuthal shear layers could result in a Kelvin-Helmholtz type instability. Two key aspects are important. First, the inviscid analysis indicates that all wavelengths are unstable, so the preferred wavelength, if there is one, is determined by viscous or geometric effects [4]. In this case, the wavelength is necessarily an integer fraction of the circumference of the Taylor-Couette cell. Second, the wave travels at a speed equal to the average of the two velocities generating the shear layer, $V_1$ and $V_2$, or $0.5(V_1 + V_2)$. The speed of the waves can be addressed directly from experimental data. Estimating $V_1$ and $V_2$ from the azimuthal velocity (figure 7 of [3]) and comparing this to the measured wave speed [2], $c$, results in a reasonably good match for different values of $\varepsilon$, as shown in the table. Furthermore, the estimated speed decreases as the reduced Reynolds number increases, consistent with the trend for the measured speed.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$V_1$</th>
<th>$V_2$</th>
<th>$0.5(V_1 + V_2)$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.28</td>
<td>0.25</td>
<td>0.61</td>
<td>0.43</td>
<td>0.44</td>
</tr>
<tr>
<td>1.48</td>
<td>0.24</td>
<td>0.54</td>
<td>0.39</td>
<td>0.36</td>
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<td>0.23</td>
<td>0.47</td>
<td>0.35</td>
<td>0.30</td>
</tr>
</tbody>
</table>

To further explore this, we use two approaches based on a secondary stability analysis building on a weakly nonlinear saturation of the Taylor vortices. In one case, we remove from the secondary
stability problem the shear due to the azimuthal component of the Taylor vortices, leaving the centrifugal terms (the mechanism proposed by Marcus). In the other, we remove the centrifugal terms from the secondary stability analysis, leaving the azimuthal shear (the mechanism suggested by Davey and Jones). In both cases, we compare the growth rates for the secondary instability as a function of the axial and azimuthal wave numbers and Taylor number to the full stability problem. Preliminary results indicate that the growth rate is governed by the centrifugal terms for Reynolds numbers that are slightly supercritical but tends to be eventually governed by the azimuthal shear as the Reynolds number increases. Moreover, unlike centrifugal forces, the azimuthal shear tends to select non-zero azimuthal wave numbers, suggesting that the wavy vortices are promoted by azimuthal shear rather than centrifugal forces. We are continuing to explore the physics of the transition further analytically and numerically.

References