On the construction of harmonic morphisms from Euclidean spaces

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1 Introduction

A complex-valued harmonic morphism on an open subset $U$ of $\mathbb{R}^n$ is a smooth solution $z = \varphi(x^1, \ldots, x^n)$ ($x^1, \ldots, x^n \in \mathbb{R}^n$) to the system

\begin{align*}
\Delta \varphi &\equiv \sum_{i=1}^{n} \frac{\partial^2 \varphi}{\partial x^i} = 0, \\
(\nabla \varphi)^2 &\equiv \sum_{i=1}^{n} \left( \frac{\partial \varphi}{\partial x^i} \right)^2 = 0.
\end{align*}

i.e. a complex-valued null solution of Laplace’s equation. It is a fundamental problem to find all local and global solutions to this system. For $n = 3$, the local problem was solved by Collins [7] for submersive (maximal rank) solutions, then the local and global problems were completely solved by Baird and the author without assuming submersivity [2]. For $n = 4$, all submersive solutions were found by the author; it remains open whether the method finds all solutions which are not submersive everywhere. For $n > 4$, many families of solutions can be found by the methods of Baird and the author [3, 4], but it is unknown whether these give all (even submersive) solutions.

In this article we review this problem giving new simple proofs of many of the key constructions which avoid using differential geometry in any essential way. We thus hope that this paper will make the subject more accessible to Mathematicians whose primary interest is not differential geometry.
starting point is that the Cauchy-Riemann equations show that a complex-valued map which is holomorphic with respect to a Kähler structure (i.e. an orthogonal complex structure which does not vary from point to point) is a harmonic morphism. If the Kähler structure is replaced by a Hermitian structure (which may vary from point to point), the resulting map may not be a harmonic morphism but conditions can be given so that it is thus allowing the construction of new harmonic morphisms. For \( n \leq 4 \) this leads to a twistor equation which gives all submersive harmonic morphisms; we describe this construction and then discuss generalizations to \( \mathbb{R}^n \) with \( n > 4 \) and to related spaces. Note that, except when \( n = 3 \) \([2]\), it is unclear whether these methods can be extended to non-submersive harmonic morphisms.

## 2 Harmonic morphisms from \( \mathbb{R}^4 \)

Throughout this paper, let \( U \) be a non-empty connected open subset of \( \mathbb{R}^n \) \((n \in \{1, 2, \ldots \})\). Recall that a smooth map \( \varphi : U \to \mathbb{C} \) is called a harmonic morphism if it pulls back harmonic functions to harmonic functions, i.e., for all smooth harmonic functions \( f : V \to \mathbb{R} \) defined on an open subset \( V \) of the codomain \( \mathbb{C} \) with \( \varphi^{-1}(V) \) non-empty, \( f \circ \varphi \) is harmonic. Since a harmonic function is locally the real part of a holomorphic function, \( \varphi \) is a harmonic morphism if and only if it pulls back holomorphic functions to harmonic functions. Now, by the chain rule, for a holomorphic function \( f : V \to \mathbb{C} \),

\[
\Delta(f \circ \varphi) = \frac{df}{dz} \Delta \varphi + \frac{d^2 f}{dz^2} \sum_{i=1}^{n} \left( \frac{\partial \varphi}{\partial x^i} \right)^2,
\]

so that \( \varphi \) is a harmonic morphism if and only if it satisfies equations (1) and (2). Our goal is to find solutions to this system. Note that the system is invariant under isometries of the domain, i.e. Euclidean changes of coordinates, and under composition by weakly conformal (i.e. holomorphic or antiholomorphic) maps on the range. Note that, like any solution of Laplace’s equation on \( U \), a harmonic morphism \( \varphi \) is real-analytic.

Suppose that \( n \) is even, say \( n = 2m \). It is convenient to use null coordinates \((q^1, q^7, \ldots, q^m, q^m)\) where \( q^j = x^{2j-1} + i x^{2j} \), \( q^j = x^{2j-1} - i x^{2j} \) \((j = 1, \ldots, m)\) on \( \mathbb{R}^{2m} \). Then the harmonic morphism equations (1) and (2) read

\[
\frac{1}{4} \Delta \varphi = \sum_{j=1}^{m} \frac{\partial^2 \varphi}{\partial q^j \partial q^j} = 0,
\]

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\[
\frac{1}{4}(\nabla \varphi)^2 = \sum_{j=1}^{m} \frac{\partial \varphi}{\partial q^j} \frac{\partial \varphi}{\partial q^{j}} = 0. \tag{5}
\]

In particular, if \( \partial \varphi / \partial q^j \neq 0 \) for all \( j \), i.e., if \( \varphi \) is holomorphic with respect to the standard Kähler structure on \( \mathbb{R}^{2m} \) given by the identification \( \mathbb{R}^{2m} \to \mathbb{C}^m \), \( (x^1, \ldots, x^{2m}) \mapsto q = (q^1, \ldots, q^m) \), then \( \varphi \) is a harmonic morphism. To generalize this, let \( p \in \mathbb{R}^{2m} \). Recall that an almost Hermitian structure at a point \( p \) of \( \mathbb{R}^{2m} \) is an isometry \( J : T_p \mathbb{R}^{2m} \to T_p \mathbb{R}^{2m} \) of the tangent space at \( p \) which satisfies \( J^2 = -\text{Id} \). We can always find an orthonormal basis \( \{e_1, \ldots, e_{2m}\} \) of \( T_p \mathbb{R}^{2m} \) such that \( Je_{2i-1} = e_{2i} \) \( (i = 1, \ldots, m) \). Then \( J \) is called positive (resp. negative) according as \( \{e_1, \ldots, e_{2m}\} \) is positively (resp. negatively) oriented. Equip \( \mathbb{C}^{2m} \) with the inner product \( v \cdot w = v^1w^1 + \cdots + v^{2m}w^{2m} \) \( (v = (v^1, \ldots, v^{2m}), \ w = (w^1, \ldots, w^{2m}) \in \mathbb{C}^{2m}) \). Then a subspace \( V \) is called isotropic if \( v \cdot w = 0 \) for all \( v, w \in V \), it is called maximal isotropic if it is an isotropic subspace of the maximum possible dimension \( m \).

Consider the complexified tangent space \( T_p^{\mathbb{C}} \mathbb{R}^{2m} = T_p \mathbb{R}^{2m} \otimes \mathbb{C} \cong \mathbb{C}^{2m} \). Extend \( J \) to a complex-linear map on the complexified tangent space. Then it has eigenvalues \( \pm i \); let \( T_p^{(\text{1,}0)} \mathbb{R}^{2m} \) and \( T_p^{(0,\text{1})} \mathbb{R}^{2m} \) denote the corresponding eigenspaces, explicitly, \( T_p^{(\text{1,}0)} \mathbb{R}^{2m} \) has basis \( \{e_{2i-1} + ie_{2i} : i = 1, \ldots, m\} \) and \( T_p^{(0,\text{1})} \mathbb{R}^{2m} \) is its complex conjugate. Each of these is a maximal isotropic subspace of \( T_p^{\mathbb{C}} \mathbb{R}^{2m} \). Conversely, given a maximal isotropic subspace of \( T_p^{\mathbb{C}} \mathbb{R}^{2m} \), it comes from a unique almost Hermitian structure. We have a similar decomposition of the complexified cotangent space \( T_p^{\mathbb{C}} \mathbb{R}^{2m} \) so that, equivalently, an almost Hermitian structure is determined by its \((0,\text{1})\)-cotangent space.

Now let \( m = 2 \). Then given any number \( \mu \in \mathbb{C} \) consider the subspace \( V \) of \( T_p^{\mathbb{C}} \mathbb{R}^{2m} \) spanned by
\[
\frac{\partial}{\partial q^1} - \mu \frac{\partial}{\partial q^2}; \quad \frac{\partial}{\partial q^2} + \mu \frac{\partial}{\partial q^1}. \tag{6}
\]
We shall also allow \( \mu = \infty \) in which case we define \( V \) to be the subspace spanned by \( \partial / \partial q^2, \partial / \partial q^1 \). Then \( V \) is maximal isotropic so that there is a unique almost Hermitian structure \( J(\mu) = J^+(\mu) \) with \( V \) as \((0,\text{1})\)-tangent space, equivalently \( J(\mu) \) has \((1,\text{0})\)-cotangent space spanned by
\[
dq^1 - \mu dq^3; \quad dq^2 + \mu dq^3. \tag{7}
\]
Clearly \( J(\mu) \) is positive. Similarly, \( \mu \) determines a negative almost Hermitian structure \( J^-(\mu) \) by interchanging \( q^2 \) and \( q^3 \) in these formulae. More
geometrically, let $S^2$ be the unit sphere in the $(x^2, x^3, x^4)$-plane and let 
$\sigma : S^2 \to \mathbb{C} \cup \{\infty\}$ be stereographic projection from its ‘south pole’ $(-1, 0, 0)$; then it can be verified by a simple calculation that $J^+(\mu)$ (resp. $J^-(\mu)$) is the unique positive (resp. negative) almost Hermitian structure such that $J^\pm(\partial/\partial x^1) = \sigma^{-1}(i\mu)$.

Now let $\mu : U \to \mathbb{C}$ be a smooth function on an open subset of $\mathbb{R}^4$. Then it defines a smooth positive almost Hermitian structure $J = J(\mu)$ on $U$ with $(1,0)$-cotangent space spanned by (7) and $(0,1)$-tangent space by (6). Thus a smooth function $\varphi : U \to \mathbb{C}$ is holomorphic with respect to $J(\mu)$ if and only if
$$\frac{\partial \varphi}{\partial q^1} - \mu \frac{\partial \varphi}{\partial q^2}, \quad \frac{\partial \varphi}{\partial q^2} + \mu \frac{\partial \varphi}{\partial q^1} = 0.$$ (8)

Now the Lie bracket of the vectors in (6) is
$$\left[ \frac{\partial}{\partial q^1} - \mu \frac{\partial}{\partial q^2}, \frac{\partial}{\partial q^2} + \mu \frac{\partial}{\partial q^1} \right] = \left( \frac{\partial \mu}{\partial q^1} - \mu \frac{\partial \mu}{\partial q^2} \right) \frac{\partial}{\partial q^1} + \left( \frac{\partial \mu}{\partial q^2} + \mu \frac{\partial \mu}{\partial q^1} \right) \frac{\partial}{\partial q^2},$$
so that $J(\mu)$ is integrable (i.e. comes from a complex structure) on $U$ if and only if
$$\frac{\partial \mu}{\partial q^1} - \mu \frac{\partial \mu}{\partial q^2} = 0, \quad \frac{\partial \mu}{\partial q^2} + \mu \frac{\partial \mu}{\partial q^1} = 0.$$ (9)

Hence $J(\mu)$ is integrable if and only if $\mu$ is holomorphic with respect to $J(\mu)$. Note that, in this case, $w^1 = q^1 - \mu(q)q^2, \quad w^2 = q^2 + \mu(q)q^1$ are holomorphic and so define complex coordinates if their differentials $dw^1, dw^2$ are independent. (In particular, if $0 \in U$ — which we can always arrange by a shift of origin — this holds on a neighbourhood of 0.) If $\mu$ is a constant function, then $J = J(\mu)$ is the same at each point of $U$ and is called an orthogonal complex or Kähler structure.

**Proposition 1** [16] Let $\mu : U \to \mathbb{C}$ be a smooth function which satisfies (9); then
$$\Delta \mu = 0, \quad (\nabla \mu)^2 = 0.$$
That is, if $\mu$ represents an integrable Hermitian structure, then $\mu$ is a harmonic morphism.

**Proof** We need only note that, for any smooth function $\mu : U \to \mathbb{C}$,
$$\frac{1}{4} \Delta \mu = \frac{\partial}{\partial q^1} \left( \frac{\partial \mu}{\partial q^1} - \mu \frac{\partial \mu}{\partial q^2} \right) + \frac{\partial}{\partial q^2} \left( \frac{\partial \mu}{\partial q^2} + \mu \frac{\partial \mu}{\partial q^1} \right),$$
$$\frac{1}{4} (\nabla \mu)^2 = \frac{\partial \mu}{\partial q^1} \left( \frac{\partial \mu}{\partial q^1} - \mu \frac{\partial \mu}{\partial q^2} \right) + \frac{\partial \mu}{\partial q^2} \left( \frac{\partial \mu}{\partial q^2} + \mu \frac{\partial \mu}{\partial q^1} \right).$$

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Example 2 The function $\mu = q^1/q^7$ satisfies equations (9) and so defines a positive Hermitian structure $J(\mu)$ on $\mathbb{R}^4 \setminus \{0\}$. The map $\mu : \mathbb{R}^4 \setminus \{0\} \to \mathbb{C} \cup \{\infty\}$ can be interpreted as the composition of radial projection $\mathbb{R}^4 \setminus \{0\} \to S^3$, $x \mapsto x/|x|$, a version of the Hopf map from the 3-sphere $S^3$ to the 2-sphere $S^2$ and the conformal identification of $S^2$ with the extended complex plane $\mathbb{C} \cup \{\infty\}$ by stereographic projection.

More generally, let $\varphi : U \to \mathbb{C}$ be a smooth map which is holomorphic with respect to a smooth almost Hermitian structure $J(\mu)$ so that $\varphi$ satisfies (8) for some smooth function $\mu : U \to \mathbb{C}$. Then it quickly follows that $\varphi$ satisfies (2).

Differentiating the first equation of (8) with respect to $q^1$, the second with respect to $q^2$, and adding gives

$$\frac{1}{4} \Delta \varphi = \frac{\partial \varphi}{\partial q^1} \frac{\partial \mu}{\partial q^2} - \frac{\partial \varphi}{\partial q^2} \frac{\partial \mu}{\partial q^1}. \quad (10)$$

The right-hand side is zero if $\mu = \zeta \circ \varphi$ for some holomorphic function $\zeta$. In this case $\mu$ is also holomorphic so that $J(\mu)$ is integrable; further, the fibres of $\varphi$ are superminimal in the sense that $\mu$ is constant along each connected component of a fibre. This suggests the following result:

**Proposition 3** [16] Let $\varphi : U \to \mathbb{C}$ be a smooth submersion which is holomorphic with respect to an almost Hermitian structure $J$. Then $\varphi$ is a harmonic morphism and $J$ is integrable if and only if $\varphi$ has superminimal fibres.

**Proof** Let $p \in U$. Choose coordinates such that $J = J(\mu)$ for finite $\mu$ at $p$. Then since $d\varphi(p) \neq 0$, (8) implies that $(\partial \varphi/\partial q^1, \partial \varphi/\partial q^2) \neq (0, 0)$ at $p$. The same equation also implies that $\mu$ is automatically smooth. In null coordinates the gradients of $\varphi$ and $\mu$ are given by

$$\nabla \varphi = \left( \frac{\partial \varphi}{\partial q^1}, \frac{\partial \varphi}{\partial q^2} \right),$$

$$\nabla \mu = \left( \frac{\partial \mu}{\partial q^1}, \frac{\partial \mu}{\partial q^2} \right).$$

Suppose that $\varphi$ is a harmonic morphism. Then from (10),

$$\left( \frac{\partial \mu}{\partial q^1}, \frac{\partial \mu}{\partial q^2} \right) = \nu \left( \frac{\partial \varphi}{\partial q^1}, \frac{\partial \varphi}{\partial q^2} \right)$$

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for some $\nu \in \mathbb{C}$. Together with (8) and (9) this implies that $\nabla \mu = \nu \nabla \varphi$ so that $\mu$ is constant on the fibres of $\varphi$. The converse is similar.

Let $\varphi : U \to \mathbb{C}$ be a submersive map satisfying (2), equivalently, (5). Then there are precisely two almost Hermitian structures $J^\pm$, one positive, one negative, with respect to which $\varphi$ is holomorphic; indeed from (8), the positive one is given by $J^+] = J(\mu)$ where, writing $\varphi^I$ for $\partial \varphi/\partial q^I$ ($I = 1, 1, 2, 2$),

$$\mu = -\frac{\varphi_2}{\varphi_1} = \frac{\varphi_1}{\varphi_2},$$

(11)

Note that at least one of these fractions is well-defined (i.e. not 0/0) and that they are equal by (5). $J^-$ is defined similarly by interchanging $q^1$ and $q^2$. If, additionally, $\varphi$ satisfies (1), we can show that one of $J^\pm$ is integrable as follows:

**Theorem 4** [16] Let $\varphi : U \to \mathbb{C}$ be a submersive harmonic morphism. Then $\varphi$ is holomorphic with respect to a Hermitian structure $J$. Further, the fibres of $\varphi$ are superminimal with respect to $J$.

**Proof** Writing $\varphi_{IJ}$ for $\partial^2 \varphi/\partial q^I \partial q^J$ etc., the harmonic morphism equations read

$$\varphi_{11} + \varphi_{22} = 0,$$

(12)

$$\varphi_1 \varphi_T + \varphi_2 \varphi_T = 0.$$  

(13)

Let $p \in U$. Choose coordinates such that $\partial/\partial x^3, \partial/\partial x^4$, or, equivalently, $\partial/\partial q^1, \partial/\partial q^2$ span the kernel of $d\varphi(p)$, thus $\varphi_2(p) = \varphi_T(p) = 0$. Then from (13) we have $\varphi_1(p) \varphi_T(p) = 0$. Without loss of generality we can assume that

$$\varphi_1(p) \neq 0$$

(14)

so that $\varphi_T(p) = 0$. Now differentiating (13) with respect to anything shows that all first derivatives of $\varphi_T$ are zero at $p$. Differentiating (13) with respect to $q^1$ then $q^T$, evaluating at $p$ and using (14) gives $\varphi_{111}(p) = 0$ then (12) gives $\varphi_{222}(p) = 0$. Then differentiating (13) with respect to $q^2$ then $q^T$ and evaluating at $p$ gives $\varphi_{22}(p) \varphi_{22}(p) = 0$ so that

$$\varphi_{22}(p) = 0 \quad \text{or} \quad \varphi_{22}(p) = 0.$$  

Suppose that $\varphi_{22}(p) = 0$. Then, at $p$, the function $\mu = -\varphi_2/\varphi_1$ satisfies

$$\frac{\partial \mu}{\partial q^T} - \mu \frac{\partial \mu}{\partial q^1} = \frac{\partial}{\partial q^T} \left( -\frac{\varphi_T}{\varphi_1} \right) = 0,$$

$$\frac{\partial \mu}{\partial q^2} + \mu \frac{\partial \mu}{\partial q^1} = \frac{\partial}{\partial q^2} \left( -\frac{\varphi_T}{\varphi_1} \right) = 0.$$  


showing that $\mu$ satisfies (9) so that the positive almost Hermitian structure $J^+ = J(\mu)$ is integrable at $p$. Further $\partial \mu / \partial q^2 = (\partial / \partial q^2) (-\varphi_2 / \varphi_1) = 0$ at $p$, confirming that the fibre of $\varphi$ is superminimal at $p$ with respect to $J(\mu)$.

If, instead, $\varphi_{22}(p) = 0$, interchanging $q^2$ and $q^3$ shows that the corresponding negative almost Hermitian structure $J^-$ is integrable at $p$.

Thus, at each point, one of $J^+, J^-$ is integrable. Now, either $J^-$ is integrable on the whole of $U$, or there is a point $p$ of $U$ where $J^-$ is not integrable. In the latter case, by continuity, there is an open neighbourhood of $p$ on which $J^-$ is not integrable; then, by the above, $J^+$ must be integrable on that neighbourhood. Now, since $\varphi$ is real-analytic, the formula (11) for $\mu$ shows that it is also real analytic. Thus if the integrability condition (9) holds on an open set, it holds everywhere; it follows that $J^+$ is integrable on the whole of $U$. Thus one of $J^+, J^-$ is integrable on the whole of $U$.

Thus finding (submersive) harmonic morphisms $\varphi : U \to \mathbb{C}$ is equivalent to finding Hermitian structures on $U$. To do this, recall that if $\mu : U \to \mathbb{C}$, $q \mapsto \mu(q)$ represents a Hermitian structure, then $w^1 = q^1 - \mu(q)q^2$, $w^2 = q^2 + \mu(q)q^3$ are holomorphic with respect to $J(\mu)$. Although $\{dw^1, dw^2\}$ might be linearly dependent, $\text{span}\{dw^1, dw^2, d\mu\} = \text{span}\{dq^1 - \mu dq^2, dq^2 + \mu dq^1\}$ always has dimension 2. Hence there is a holomorphic function of three variables $\psi(w^1, w^2, \mu)$ with $d\psi \neq 0$ such that $\mu = \mu(q)$ satisfies

$$\Psi(q, \mu) \equiv \psi(q^1 - \mu q^2, q^2 + \mu q^3, \mu) = 0.$$  \hspace{1cm} (15)

Further, $\partial \Psi / \partial \mu \neq 0$ at points $(q, \mu(q))$, otherwise, by the chain rule we would have $\partial \Psi / \partial q^I = 0$ for all $I$ and so $d\psi = 0$. This establishes the last statement of the following:

**Proposition 5** Let $\psi(w^1, w^2, \mu)$ be a holomorphic function of three variables. Set

$$\Psi(q, \mu) = \psi(q^1 - \mu q^2, q^2 + \mu q^3, \mu).$$

Suppose that $\partial \Psi / \partial \mu \neq 0$ at a point $(q_0, \mu_0)$. Then the local solution $\mu = \mu(q)$ to the equation

$$\Psi(q, \mu) = 0$$  \hspace{1cm} (16)

through $(q_0, \mu_0)$ represents a Hermitian structure. All Hermitian structures (resp. positive Hermitian structures) on open subsets of $\mathbb{R}^4$ are given this way locally up to isometries (resp. orientation preserving isometries) of $\mathbb{R}^4$. 

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Proof  By the chain rule,
\[ \frac{\partial \psi}{\partial q^1} - \mu \frac{\partial \psi}{\partial q^2} = \mu \frac{\partial \psi}{\partial w^1} - \mu \frac{\partial \psi}{\partial w^2} = 0, \quad \frac{\partial \psi}{\partial q^2} + \mu \frac{\partial \psi}{\partial q^1} = -\mu \frac{\partial \psi}{\partial w^1} + \mu \frac{\partial \psi}{\partial w^1} = 0. \]
Further, if \( q \mapsto \mu(q) \) is a local solution, then we have
\[ \frac{\partial \psi}{\partial q^I} + \frac{\partial \psi}{\partial \mu} \frac{\partial \mu}{\partial q^I} = 0 \quad (I = 1, 2, \overline{2}) \]
so that
\[ 0 = \frac{\partial \psi}{\partial q^I} - \mu \frac{\partial \psi}{\partial q^2} = \frac{\partial \psi}{\partial \mu} \left( \frac{\partial \mu}{\partial q^I} - \mu \frac{\partial \mu}{\partial q^2} \right), \quad 0 = \frac{\partial \psi}{\partial q^2} + \mu \frac{\partial \psi}{\partial q^1} = \frac{\partial \psi}{\partial \mu} \left( \frac{\partial \mu}{\partial q^2} + \mu \frac{\partial \mu}{\partial q^1} \right). \]

It follows that \( \mu \) is holomorphic with respect to \( J(\mu) \) so that \( J(\mu) \) is a Hermitian structure.

Note that (i) the inverse function theorem guarantees the existence of a unique smooth solution through \( (q_0, \mu_0) \) on a small enough neighbourhood of \( q_0 \); (ii) that solution is submersive provided that one of the partial derivatives \( \partial \psi / \partial q^I \) is non-zero; (iii) the isometry of \( \mathbb{R}^4 \) is only required to make sure that \( J = J(\mu) \) for some finite \( \mu \). Similar remarks apply to the other results of this type below.

Example 6 Set \( \psi(w^1, w^2, \mu) = w^1 \); then (16) has the solution \( \mu = q^1 / q^2 \) discussed in Example 2.

To find all harmonic morphisms \( \varphi : U \to \mathbb{C}, \ z = \varphi(q) \), note that, by Theorem 4, after a Euclidean change of coordinates, \( \varphi \) is holomorphic with respect to a Hermitian structure \( J(\mu) \) and that, locally, \( \mu \) depends only on the value \( z \) of \( \varphi \); we are thus led to the following:

Theorem 7 Let \( \mu(z) \) be a given holomorphic function of one variable and \( \psi(w^1, w^2, z) \) a given holomorphic function of three variables. Set
\[ \Psi(q, z) = \psi(q^1 - \mu(z)q^2, q^2 + \mu(z)q^1, z). \]
Suppose that, at a point \( (q_0, z_0) \) satisfying \( \Psi(q_0, z_0) = 0 \), we have \( \partial \psi / \partial z \neq 0 \). Then the local solution \( z = \varphi(q) \) through that point to the equation
\[ \Psi(q, z) = 0 \quad (17) \]
defined on some neighbourhood $U$ of $q_0$ is a harmonic morphism. It is holomorphic and has superminimal fibres with respect to the Hermitian structure $J$ whose value at a point $q \in U$ is $J(\mu(\varphi(q)))$. All submersive harmonic morphisms from open subsets of $\mathbb{R}^4$ to $\mathbb{C}$ are given this way locally up to isometries of $\mathbb{R}^4$.

**Proof** That a local solution $\varphi$ is holomorphic with respect to the almost Hermitian structure $J$ follows in a similar way to the proof of Proposition 5; that $J$ is integrable then follows since $\mu \circ \varphi$ is holomorphic with respect to $J$. That $\varphi$ is a harmonic morphism follows from Proposition 3 (or just from (10)). That local solutions give all submersive harmonic morphisms follows from combining Theorem 4 with Proposition 5.

**Remarks** 1. The idea of a complex-valued harmonic morphism and its characterization as a solution of the system (1,2) is essentially due to Jacobi in 1848 [15] who studied the case $n = 3$. The characterization generalizes to harmonic morphisms between any Riemannian [8, 14] or semi-Riemannian [9] manifolds.

2. Theorem 4 generalizes to harmonic morphisms from any (orientable) Einstein 4-dimensional manifold to any Riemann surface, see [16]. For a unified treatment of space forms, see [1]. For further discussion of this result and comparison with the ‘dual’ case of branched minimal immersions of Riemann surfaces in 4-manifolds, see [17]. If $\varphi$ fails to be submersive at a point, it is an open problem whether the Hermitian structure of Theorem 4 extends over that point, see [16].

3. There is a version of the material in this section for maps from Minkowski space in which Hermitian structure is replaced by shear-free ray congruence [5]. Hermitian structures and shear-free ray congruences both complexify to holomorphic foliations of $\mathbb{C}^4$ by null planes giving a complex version of Theorem 4, see [5, 6].
3 Harmonic morphisms from higher dimensional Euclidean spaces

Given \( \mu = (\mu_1, \ldots, \mu_m) \in \mathbb{C}^{m(m-1)/2} \), let \((M^2_j(\mu))\) be the skew symmetric matrix

\[
(M^2_j(\mu)) = \begin{pmatrix}
0 & \mu_1 & \mu_2 & \cdots & \mu_{m-1} \\
-\mu_1 & 0 & \mu_m & \cdots & \mu_{2m-3} \\
-\mu_2 & -\mu_m & 0 & \cdots & \mu_{3m-6} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\mu_{m-1} & -\mu_{2m-3} & -\mu_{3m-6} & \cdots & 0
\end{pmatrix}.
\] (18)

For any \( p \in \mathbb{R}^{2m} \), this defines a positive almost Hermitian structure \( J(\mu) \), namely that with \((1,0)\)-cotangent space spanned by

\[ dq^i - M^2_j(\mu) dq^j \quad (i = 1, \ldots, m), \]
equivalently, with \((0,1)\)-tangent space spanned by

\[ \frac{\partial}{\partial q^i} + M^2_j(\mu) \frac{\partial}{\partial q^j} \quad (i = 1, \ldots, m). \]

Here, and elsewhere, we use the double summation convention: sum over any repeated subscript and superscript. Note that when \( m = 2 \) these bases reduce to (7) and (6) respectively. After an orientation preserving Euclidean tranformation, any positive almost Hermitian structure at a point can be written as \( J(\mu) \) — the Euclidean tranformation only being necessary to avoid one or more of the \( \mu_i \) being infinite.

Now let \( U \) be an open subset of \( \mathbb{R}^{2m} \) and let \( \mu : U \rightarrow \mathbb{C}^{m(m-1)/2} \) be a smooth function. Then \( \mu \) defines a smooth positive almost Hermitian structure \( J(\mu) \) on \( U \). Let \( \varphi : U \rightarrow \mathbb{C} \) be holomorphic with respect to \( J(\mu) \). Then,

\[ \frac{\partial \varphi}{\partial q^i} + M^2_j(\mu) \frac{\partial \varphi}{\partial q^j} = 0 \quad (i = 1, \ldots, m). \] (19)

Writing \( M^2_j \) for \( M^2_j \circ \mu \), differentiating (19) with respect to \( q^i \) and summing over \( i = 1, \ldots, m \) gives

\[ \frac{1}{4} \Delta \varphi \equiv \sum_{i=1}^m \frac{\partial^2 \varphi}{\partial q^i \partial q^i} = - \sum_{i,j=1}^m \frac{\partial M^2_j}{\partial q^i} \frac{\partial \varphi}{\partial q^j}. \] (20)
the remaining term vanishing by the antisymmetry of the matrix \((M^j_i)\).
Now suppose that \(\mu = \zeta \circ \varphi\) for some holomorphic map \(\zeta\); then the chain rule applied to (20) shows that \(\Delta \varphi = 0\). Thus one half of Proposition 3 generalizes to:

**Proposition 8** Let \(\varphi : U \to \mathbb{C}\) be submersive and holomorphic with respect to an almost Hermitian structure \(J\). If \(\varphi\) has superminimal fibres, then \(J\) is integrable and \(\varphi\) is a harmonic morphism.

However the converse no longer holds for \(m > 2\).

Since the functions \(w^i = q^i - M^j_i(q)q^j\) are holomorphic with respect to \(J(\mu)\), similar reasoning to that in the four-dimensional case yields:

**Theorem 9** [4] Let \(\mu_1(z), \ldots, \mu_{m(m-1)/2}(z)\) be given holomorphic functions of one complex variable and \(\psi(w^1, \ldots, w^m, z)\) a given holomorphic function of \(m + 1\) complex variables. Set
\[
\Psi(q, z) = \psi \left( q^1 - M^j_1(z)q^j, q^2 - M^j_2(z)q^j, \ldots, q^m - M^j_m(z)q^j, z \right).
\]
Suppose that, at a point \((q_0, z_0)\) satisfying \(\Psi(q_0, z_0) = 0\), we have \(\partial \Psi / \partial z \neq 0\). Then the local solution \(z = \varphi(q)\) through that point to the equation
\[
\Psi(q, z) = 0
\]
defined on a neighbourhood \(U\) of \(q_0\) is a harmonic morphism. It is holomorphic and has superminimal fibres with respect to the Hermitian structure whose value at a point \(q \in U\) is \(J(\mu(\varphi(q)))\). All submersive harmonic morphisms from open subsets of \(\mathbb{R}^{2m}\) to \(\mathbb{C}\) which are holomorphic and have superminimal fibres with respect to a Hermitian structure are given this way locally up to isometries of \(\mathbb{R}^{2m}\).

When \(m = 2\) this reduces to Theorem 7 where it gives all submersive harmonic morphisms locally. It does not give all submersive harmonic morphisms when \(m > 2\), see Example 15 below.

**Example 10** Let \(m = 3\) and set \(\psi(w^1, w^2, w^3, z) = w^1w^2 - w^3\), \(\mu_1 = \mu_2 = z\), \(\mu_3 = 0\). Then (22) becomes the quadratic equation
\[
z^2q^1(q_2^2 + q_3^2) + z\{q^2(q_2^2 + q_3^2) - |q^1|^2\} + (q^3 - q^1q^2) = 0.
\]
By Theorem 9, any local solution \(z = \varphi(q)\) is a harmonic morphism holomorphic with respect to the Hermitian structure \(J(\mu(\varphi(q)))\) and with superminimal fibres.

11
**Example 11** Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be $m$ holomorphic functions of a single complex variable and consider the holomorphic function

$$
\psi(w^1, \ldots, w^m, z) = \alpha_1(z)w^1 + \alpha_2(z)w^2 + \cdots + \alpha_m(z)w^m - 1.
$$

Then in Euclidean coordinates equation (22) reads

$$(\alpha_1 - M_k^k)\alpha_k x^1 + i(\alpha_1 + M_k^k)\alpha_k x^2 + \cdots + (\alpha_m - M_m^k)\alpha_k x^{2m-1} + i(\alpha_m + M_m^k)\alpha_k x^{2m} = 1.$$ 

This is of the form

$$
\sum_{i=1}^{n} \beta_i(z) x^i = 1 \quad (23)
$$

where $n = 2m$ and the $\beta_i$ are holomorphic and satisfy $\sum_{i=1}^{n} (\beta_i(z))^2 = 0$. Local solutions $z = \varphi(q)$ are submersive complex-valued harmonic morphisms on open subsets of $\mathbb{R}^{2m}$ with fibres which are planes of codimension 2. All such harmonic morphisms are given this way locally.

If a harmonic morphism $\varphi : \mathbb{R}^{2m} \supset U \rightarrow \mathbb{C}$, $z = \varphi(q)$ is independent of the first coordinate $x^1$, then is factors to give a harmonic morphism from an open subset of $\mathbb{R}^{2m-1}$. Simple chain rule arguments show that local solutions to (22) are independent of $x^1$ if $\Psi(q, z)$ is independent of $x^1$ and this holds if and only if $\psi$ is of the form

$$
\psi(w^1, \ldots, w^m, z) = \overset{\sim}{\psi}(w^2 - \mu_1(z)w^1, \ldots, w^m - \mu_m(z)w^1, z) \quad (24)
$$

for some holomorphic function $\overset{\sim}{\psi}(\overset{\sim}{w}, z) = \overset{\sim}{\psi}(\overset{\sim}{w}^1, \ldots, \overset{\sim}{w}^{m-1}, z)$.

**Example 12** When $m = 2$, with $\psi$ of the form (24), equation (22) reads

$$
\overset{\sim}{\psi}(q^2 - 2\mu(z)x^2 - \mu(z)^2 q^2, z) = 0
$$

or, in Euclidean coordinates $(x^1, x^2, x^3, x^4)$,

$$
\overset{\sim}{\psi}((1 - \mu(z)^2)x^3 + i(1 + \mu(z)^2)x^4 - 2i\mu(z)x^2, z) = 0.
$$

Local solutions to this factor to harmonic morphisms on open subsets of $\mathbb{R}^3$ whose fibres are straight lines. By Theorem 7, all submersive harmonic morphisms on open subsets of $\mathbb{R}^3$ are given this way locally. All other harmonic morphisms can then be obtained by composing with weakly conformal maps, see [2].
Example 13 Let $\alpha_1(z), \alpha_2(z), \ldots, \alpha_{m-1}(z)$ be holomorphic functions of a single complex variable and set
\[
\tilde{\psi}(\tilde{w}^1, \ldots, \tilde{w}^{m-1}, z) = \alpha_1(z)\tilde{w}^1 + \alpha_2(z)\tilde{w}^2 + \cdots + \alpha_{m-1}(z)\tilde{w}^{m-1} - 1.
\]
Then equation (22) is of the form (23) with $n = 2m - 1$. Local solutions factor to submersive complex-valued harmonic morphisms on open subsets of $\mathbb{R}^{2m-1}$ with fibres which are planes of codimension 2. All such harmonic morphisms are given this way.

If a harmonic morphism $\varphi : \mathbb{R}^{2m} \supset U \to \mathbb{C}$ satisfies $\varphi(\nu q) = \varphi(q)$ whenever $\nu \in (0, \infty)$ and $q, \nu q \in U$ then it factors to a harmonic morphism on an open subset of $S^{2m-1}$. Any local solution to (22) will be of this form if $\psi(w, z)$ is homogeneous in $w$.

Example 14 Let
\[
\psi(w^1, w^2, w^3, z) = (w^1)^2 - z^2 ((w^2)^2 + (w^3)^2)
\]
and let $\mu_1(z), \mu_2(z), \mu_3(z)$ be arbitrary holomorphic functions of $z$. Then (22) reads
\[
\left( q^1 - \mu_1 q^7 - \mu_2 q^3 \right)^2 = z^2 \left( \left( q^2 + \mu_1 q^7 - \mu_3 q^3 \right)^2 + \left( q^3 + \mu_2 q^7 + \mu_3 q^3 \right)^2 \right).
\]
Local solutions $z = \varphi(q)$ clearly factor to $S^5$ (and, in fact, to $\mathbb{R}P^5$).

In a similar vein, we can construct examples on even-dimensional spheres by choosing the function $\tilde{\psi}(\tilde{w}, z)$ in the Weierstrass representation (22) to be homogeneous in $\tilde{w}$, for example we may take $m = 4$ and $\tilde{\psi}(\tilde{w}^1, \tilde{w}^2, \tilde{w}^3, z) = (\tilde{w}^1)^2 + \tilde{w}^2 \tilde{w}^3$. With $\mu_1 = \mu_2 = \mu_3 = z, \mu_4 = \mu_5 = \mu_6 = z^2$ this gives a quartic equation in $z$ defining harmonic morphisms from open subsets of $S^6$ to $\mathbb{C}$ (see [4]).

Remarks 1. Another way of seeing that local solutions to (22) are harmonic morphisms is to note that, for fixed $q$, the function $z \mapsto \Psi(q, z)$ is holomorphic and, for fixed $z$, the function $q \mapsto \Psi(q, z)$ is a harmonic morphism (since it is holomorphic with respect to the Kähler structure $J(\mu(z))$). Whenever $\Psi$ has these two properties, local solutions are harmonic morphisms, see [12].

2. For interpretations and generalizations of (20) to general almost Hermitian manifolds, see [13].
4 Further developments

There are several techniques to obtain further complex-valued harmonic morphisms on Euclidean and related spaces:

1. Let $\varphi^1, \ldots, \varphi^\ell$ be complex-valued harmonic morphisms on an open subset $U$ of $\mathbb{R}^n$ which are holomorphic with respect to the same almost Hermitian structure. Write $\varphi = (\varphi^1, \ldots, \varphi^\ell) : U \rightarrow \mathbb{C}^\ell$ and let $\zeta : V \rightarrow \mathbb{C}$, $\zeta = \zeta(z^1, \ldots, z^\ell)$ be holomorphic on an open subset of $\varphi(U)$. Then $\zeta \circ \varphi$ is a harmonic morphism. Indeed, it is holomorphic with respect to $J$ and

$$\Delta(\zeta \circ \varphi) = \frac{\partial \zeta}{\partial z^\alpha} \Delta \varphi^\alpha + \frac{\partial^2 \zeta}{\partial z^\alpha \partial z^\beta} \nabla \varphi^\alpha \cdot \nabla \varphi^\beta.$$

Since the $\varphi^\alpha$ are holomorphic with respect to the same almost Hermitian structure, their gradients $\nabla \varphi^\alpha$ lie in an isotropic subspace so that

$$\nabla \varphi^\alpha \cdot \nabla \varphi^\beta = 0 \quad (\alpha, \beta = 1, \ldots, \ell), \quad (25)$$

hence $\Delta(\zeta \circ \varphi) = 0$ as required.

2. More generally, call a family $\{\varphi^1, \ldots, \varphi^\ell\}$ of harmonic functions $U \rightarrow \mathbb{C}$ orthogonal if (25) holds; in particular, each $\varphi^i$ is a harmonic morphism. Then the same proof shows that $\zeta \circ (\varphi^1, \ldots, \varphi^\ell)$ is a harmonic morphism for any holomorphic map $\zeta$.

Example 15 It can be quickly checked that the functions $\varphi^1 = q^1 - q^mq_7$, $\varphi^2 = q^2 + q^mq_7^T$, $\varphi^3 = q^3, \ldots, \varphi^m = q^m$ form an orthogonal family. Hence setting $\varphi = (\varphi^1, \ldots, \varphi^m)$, the composition $\zeta \circ \varphi$ is a harmonic morphism for any holomorphic function $\zeta$. In particular taking

$$\zeta = z^1 \cdots z^{m-1} + z^m$$

we obtain the globally defined harmonic morphism $\varphi : \mathbb{R}^{2m} \rightarrow \mathbb{C}$ given in null coordinates by

$$\varphi(q) = (q^1 - q^mq_7^T)(q^2 + q^mq_7^T)q^3 \cdots q^m.$$

It is easily checked that this is submersive and does not have superminimal fibres with respect to any almost Hermitian structure, so, in particular, is not holomorphic with respect to any Kähler structure. It is full in the sense that it does not factor through the orthogonal projection onto any proper subspace. No such example can occur for $m = 2$; indeed, any complex-valued
submersive harmonic morphism globally defined on $\mathbb{R}^4$ is holomorphic with respect to a Kähler structure [16].

Taking, instead,

$$\zeta = (z^1 z^m - z^2) + (z^3)^2 + \ldots + (z^{m-2})^2 + z^{m-1}$$

gives a submersive complex-valued harmonic morphism globally defined on $\mathbb{R}^{2m}$; again this does not have superminimal fibres with respect to any almost Hermitian structure. However, this time, it is not full since it is independent of the first Euclidean coordinate and so factors to a harmonic morphism on $\mathbb{R}^{2m-1}$; this harmonic morphism is full. No such example can occur for $m = 2$, indeed, any complex-valued harmonic morphism globally defined on $\mathbb{R}^3$ is an orthogonal projection followed by a weakly conformal map [2].

These two examples provide full harmonic morphisms $\mathbb{R}^n \to \mathbb{C}$ for each $n > 4$ which are not, and are not the reduction of, maps holomorphic with respect to Kähler structures. See [3] for more information and a different treatment of these examples.

**Example 16** [10] The functions $\mathbb{R}^8 = \mathbb{C}^4 \to \mathbb{C}$ given by $\varphi^1(q) = q^1, \varphi^2(q) = \overline{q^2}, \varphi^3(q) = q^3, \varphi^4(q) = \overline{q^4}$ are all holomorphic with respect to the Kähler structure with $(1,0)$-cotangent space spanned by $dq^1, dq^2, dq^3, dq^4$ and so form an orthogonal family of harmonic functions. Hence, for example,

$$\varphi(q) = (q^5 q^7)/(q^1 q^3)$$

is a complex-valued harmonic morphism on a dense open subset of $\mathbb{C}^4$. This example factors to a complex-valued harmonic morphism on a dense open subset of $\mathbb{C}P^3$. This construction can be generalized to provide complex-valued harmonic morphisms on Riemannian symmetric spaces of rank one, giving globally defined harmonic morphisms on real, complex and quaternionic hyperbolic spaces [11].

3. We can replace the complex variable $z \in \mathbb{C}$ in (22) by $z = (z^1, \ldots, z^k)$. Then local solutions $z = \varphi(q)$ to the equation (22) are holomorphic with respect to the Hermitian structure $J(\mu(\varphi(q)))$, however the components of $\varphi$ may not be harmonic. Nevertheless, the Laplacian of each component can be computed in terms of the holomorphic data $\mu, \psi$, and judicious choice of these data gives new families of harmonic morphisms, see [3]. This procedure theoretically gives all submersive holomorphic harmonic morphisms though to find all possible values of the data would seem to be a difficult problem.
4. We end by remarking that, when \( m \geq 3 \), it is unknown, even in the submersive case, whether every complex-valued harmonic morphism defined on an open subset of \( \mathbb{R}^{2m} \) is holomorphic with respect to a Hermitian structure; it is thus unclear whether the above methods give all complex-valued submersive harmonic morphisms on open subsets of Euclidean spaces of dimensions more than 4. Thus there remains much research to do to find all solutions to the system (1,2) when \( n > 4 \).

References


[15] Jacobi, C.G.J. Über eine Lösung der partiellen Differentialgleichung \( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \), J. Reine Angew. Math. 36 (1848), 113–134.
