New results on twistor lifts of harmonic maps

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Harmonic maps

Let $\varphi : (M, g) \to (N, h)$ be a smooth map between compact smooth Riemannian manifolds. The **energy** or **Dirichlet integral** of φ is

$$E(\varphi) = \int_M e(\varphi)\,\omega_g = \int_M \tfrac{1}{2}|\mathrm{d}\varphi|^2\,\omega_g$$

where ω_g = volume measure and, for any $p \in M$,

$$|\mathrm{d}\varphi_p|^2 = \text{Hilbert–Schmidt norm}^2 \text{ of } \mathrm{d}\varphi_p = g^{ij} h_{\alpha\beta}\,\varphi_i^\alpha \varphi_j^\beta\,.$$

The map φ is called **harmonic** if the first variation of E (for variations φ_t of the map φ) vanishes at φ, i.e., $\frac{\mathrm{d}}{\mathrm{d}t} E(\varphi_t)\big|_{t=0} = 0$.
We compute

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\varphi_t)\big|_{t=0} = -\int_M \big\langle \tau(\varphi), v \big\rangle\,\omega_g$$

where $v = \partial\varphi_t/\partial t|_{t=0}$ is the **variation vector field** of (φ_t), and $\tau(\varphi) = \operatorname{div} \mathrm{d}\varphi$ is the **tension field of** φ given as follows:

Tension field of a map φ

$$\begin{aligned}\tau(\varphi) &= \operatorname{div}\, \mathrm{d}\varphi = -\mathrm{d}^*_\nabla \mathrm{d}\varphi = \operatorname{Trace} \nabla \mathrm{d}\varphi \\ &= \sum_{i=1}^{m} \nabla \mathrm{d}\varphi(e_i, e_i) = \sum_{i=1}^{m} \{ \nabla^\varphi_{e_i} (\mathrm{d}\varphi(e_i)) - \mathrm{d}\varphi(\nabla^M_{e_i} e_i) \} ;\end{aligned}$$

in local coordinates on N this reads

$$\begin{aligned}\tau(\varphi)^\gamma &= g^{ij} \left(\frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial \varphi^\gamma}{\partial x^k} + L^\gamma_{\alpha\beta} \frac{\partial \varphi^\alpha}{\partial x^i} \frac{\partial \varphi^\beta}{\partial x^j} \right) \\ &= \Delta^M \varphi^\gamma + g(\operatorname{grad}\, \varphi^\alpha, \operatorname{grad}\, \varphi^\beta)\, L^\gamma_{\alpha\beta} .\end{aligned}$$

where Γ^k_{ij} (resp. $L^\gamma_{\alpha\beta}$ denotes the Christoffel symbols of M (resp. N) and Δ^M = **Laplacian on functions** $f : M \to \mathbb{R}$ given by

$$\Delta^M f = \operatorname{div}\, \operatorname{grad}\, f = \operatorname{div} \mathrm{d}f = -\mathrm{d}^* \mathrm{d}f = \operatorname{Trace} \nabla \mathrm{d}f$$

$$= \sum_{i=1}^{m} \{ e_i(e_i(f)) - (\nabla^M_{e_i} e_i) f \} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|}\, g^{ij} \frac{\partial f}{\partial x^j} \right).$$

Examples of harmonic maps

$\varphi : M \to N$ is harmonic $\iff$

$$\tau(\varphi) \equiv \mathrm{Trace}\, \nabla \mathrm{d}\varphi = 0 \quad \textit{(Harmonic equation)}$$

1. $\varphi : \mathbb{R}^m \supseteq U \to \mathbb{R}^n$ is harmonic iff $\Delta\varphi = 0$

   ($\Delta =$ usual Laplacian on $\mathbb{R}^m$).

2. $\varphi : (M, g) \to \mathbb{R}^n$ is harmonic iff $\Delta^M \varphi = 0$

   ($\Delta^M =$ Laplace–Beltrami operator on (M, g)).

3. $\varphi : \mathbb{R} \supseteq U \to N$ or $S^1 \to N$ is harmonic iff it defines a **geodesic** parametrized linearly.

4. A map $\varphi : M \to N$ between Kähler manifolds is called **holomorphic** (resp. **antiholomorphic** if its differential intertwines the almost complex structures, i.e., $\mathrm{d}\varphi \circ J^M = J^N \circ \mathrm{d}\varphi$ (resp. $\mathrm{d}\varphi \circ J^M = -J^N \circ \mathrm{d}\varphi$). *Such maps are harmonic*; in fact they give absolute minima of the energy functional.

Weakly conformal maps

A smooth map $\varphi : (M, g) \to (N, h)$ is called **weakly conformal** (with conformality factor λ) if $\varphi^* h = \lambda^2 g$; explicitly, for all $p \in M$,

$$h\big(\mathrm{d}\varphi_p(X), \mathrm{d}\varphi_p(Y)\big) = \lambda(p)^2\, g(X, Y) \qquad (X, Y \in T_p M)\,;$$

In local coordinates, this reads

$$h_{\alpha\beta}\varphi_i^\alpha \varphi_j^\beta = \lambda^2 g_{ij}. \tag{1}$$

Remarks

• A *weakly conformal harmonic map* from a surface is the same thing as a **minimal branched immersion**

• *Any harmonic map from the* 2*-sphere* S^2 *is weakly conformal and so is a minimal branched immersion*.

• [**J. Sacks and K. Uhlenbeck** (1977)] Bijection between harmonic maps of finite energy $\mathbb{C} \to N$ and harmonic maps $S^2 \to N$.

Conformal invariance in 2 dimensions

Harmonic maps *from surfaces* are **conformally invariant (in the domain)** in the sense that the composition $\varphi \circ \psi$ of a harmonic map $\varphi : M^2 \to N$ from a surface with a conformal or weakly conformal map $\psi : M'^2 \to M^2$ of surfaces is harmonic.
In fact, if (x, y) are isothermal coordinates on M^2, the harmonic equation reads

$$\nabla^{\varphi}_{\partial/\partial x}\Big(\frac{\partial \varphi}{\partial x}\Big) + \nabla^{\varphi}_{\partial/\partial y}\Big(\frac{\partial \varphi}{\partial y}\Big) = 0 \,.$$

If we write $z = x + \mathrm{i}y$, this reads

$$\nabla^{\varphi}_{\partial/\partial \bar{z}}\Big(\frac{\partial \varphi}{\partial z}\Big) = 0 \,, \quad \text{equivalently,} \quad \nabla^{\varphi}_{\partial/\partial z}\Big(\frac{\partial \varphi}{\partial \bar{z}}\Big) = 0 \,. \tag{2}$$

Hence the concept of *harmonic map from a Riemann surface* is well defined. Further, $\varphi : M^2 \to N$ is harmonic iff $\partial\varphi/\partial z$ is holomorphic with respect to the Koszul–Malgrange holomorphic structure on $\varphi^{-1}TN \to M^2$.

EXCURSION: Harmonic morphisms

$\phi : M \to N$ is called a **harmonic morphism** if, for every harmonic function $f : V \to \mathbb{R}$ defined on an open subset V of N with $\phi^{-1}(V)$ non-empty, the composition $f \circ \phi$ is harmonic on $\phi^{-1}(V)$.

Harmonic morphisms TO *surfaces* are **conformally invariant**: the composition $\psi \circ \phi$ of a harmonic morphism $\phi : M \to N^2$ to a surface with a weakly conformal map $\psi : N'^2 \to N^2$ is harmonic.

DEFINITION. A map is called **H***orizontally* **W***eakly* **C***onformal* if

$$g^{ij} \phi_i^\alpha \phi_j^\beta = \lambda^2(p)\, h^{\alpha\beta} \qquad \text{(cf. weak conformality, eqn. (1))}$$

[**B. Fuglede** (1978), **T. Ishihara** (1979)] *A smooth map $\phi : M \to N$ is a harmonic morphism iff it is harmonic and HWC.*

[**P. Baird and J. Eells** (1981)] *A HWC map is a harmonic morphism, iff its* **fibres are minimal** *at regular points.* In particular a conformal submersion with minimal fibres is a harmonic morphism. See the book by **P. Baird and JCW** (2003) for more information.

The genesis of twistor lifts 1: Gauss maps

Let M^2 be a Riemann surface and $\varphi : M^2 \to \mathbb{R}^n$ an immersion.
The *Gauss map* $\gamma\colon M^2 \to \widetilde{G}_2(\mathbb{R}^n)$ is defined by $\gamma(p) = \mathrm{Im}(\mathrm{d}\varphi_p)$.

Theorem [S.S. Chern (1965)] Let $\varphi : M^2 \to \mathbb{R}^n$ be conformal. Then its Gauss map is antiholomorphic iff φ is harmonic, i.e., minimal.

Proof. $\widetilde{G}_2(\mathbb{R}^n) \cong Q_{n-2} = \big\{[\zeta_0, \dots, \zeta_{n-1}] \in \mathbb{C}P^{n-1} : \sum {\zeta_i}^2 = 0\big\}$.
In a local complex coordinate $z = x + \mathrm{i}y$ on M^2,

$$\gamma(z) = \mathrm{span}\Big\{\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\Big\} = \Big[\frac{\partial\varphi}{\partial x} + \mathrm{i}\frac{\partial\varphi}{\partial y}\Big] = \Big[\frac{\partial\varphi}{\partial \bar{z}}\Big] \in Q_{n-2}.$$

Since the harmonicity equation (2) reads $(\partial/\partial z)(\partial\varphi/\partial\bar{z}) = 0$, φ harmonic implies γ antiholomorphic.

Conversely, γ antiholomorphic implies that $\dfrac{\partial}{\partial z}\Big(\dfrac{\partial\varphi}{\partial\bar{z}}\Big) = \lambda\dfrac{\partial\varphi}{\partial\bar{z}}$. But since LHS is a multiple of the mean curvature, it is normal, so $\lambda = 0$, i.e., φ is harmonic. This extends to weakly conformal maps.

The genesis of twistor lifts 2: Gauss lifts

Let Z be the trivial bundle $\pi : Z = \mathbb{R}^n \times \widetilde{G}_2(\mathbb{R}^n) \to \mathbb{R}^n$, then we can think of the Gauss map as a map $\psi : M^2 \to Z$ called the **Gauss lift** of φ which covers φ, i.e., $\varphi = \pi \circ \psi$. Then Chern's result reads: *a weakly conformal map φ is harmonic iff its Gauss lift ψ is* ***vertically antiholomorphic****, i.e., the vertical part of its differential intertwines the complex structure J^M on M^2 with $-J^{\mathcal{V}}$ where $J^{\mathcal{V}}$ is the complex structure on $G_2(\mathbb{R}^n)$.*

Now suppose that $n = 4$. Then each point $w \in Z$ defines an almost complex structure J_w on $T_{\pi(w)}N$, namely, rotation by $+\pi/2$ on w and $w^{\perp}$. This lifts to an almost complex structure $J^{\mathcal{H}}$ on $\mathcal{H}_w$. Then φ *is conformal iff it is* **horizontally holomorphic***, i.e., the horizontal part of its differential intertwines the complex structure J^M with $J^{\mathcal{H}}$.* To combine the statements for vertical and horizontal parts, set $J_1 = (J^{\mathcal{V}}, J^{\mathcal{H}})$ and $J_2 = (-J^{\mathcal{V}}, J^{\mathcal{H}})$. Then

Proposition *Let $\varphi : M^2 \to N$ be an immersion. Then φ is harmonic iff its Gauss lift $\psi : M^2 \to (Z, J_2)$ is holomorphic.*

Twistor spaces - 1

Definition (**F.E. Burstall and J.H. Rawnsley** (1990))

A *twistor fibration of N (for harmonic maps)* is
• an almost complex manifold (Z, J) (called the *twistor space*), and
• a fibration $\pi : Z \to N$ such that

for every holomorphic map $\psi : M^2 \to (Z, J)$ from a Riemann surface, the composition $\varphi = \pi \circ \psi : M^2 \to N$ is harmonic.

φ is called the *twistor projection of* ψ,
ψ is called *a twistor lift of* φ.

Problems

• Find twistor fibrations for given Riemannian manifolds N, e.g., symmetric spaces.
• Find twistor lifts of harmonic maps $M^2 \to N$ from surfaces into those symmetric spaces.

Twistor spaces - 2

First problem solved by Burstall and Rawnsley (1990):
Twistor fibrations of compact inner symmetric spaces N are given by canonical fibrations of generalized flag manifolds.

- For $N =$ a complex Grassmannian $G_k(\mathbb{C}^n)$, the twistor spaces Z are geometric flag manifolds:

$$\begin{aligned} Z &= F_{d_0,\dots,d_t} = \mathrm{U}(n)/\mathrm{U}(d_0) \times \cdots \times \mathrm{U}(d_t) \\ &= \text{orthogonal decompositions } \psi = (\psi_0, \dots, \psi_t) \text{ of } \mathbb{C}^n. \end{aligned}$$

- The twistor projection $\pi : F_{d_0,\dots,d_t} \to G_k(\mathbb{C}^n)$ is given by combining the even-numbered legs of ψ:

$$\pi_e : F_{d_0,\dots,d_t} \to G_k(\mathbb{C}^n), \qquad \psi = (\psi_0, \dots, \psi_t) \mapsto \sum_{j=0}^{[t/2]} \psi_{2j}.$$

Z has a canonical Kähler structure J_1 from embedding in product of complex Grassmannians. However, we give Z the *almost* complex structure J_2 obtained from J_1 by reversing orientation on the vertical spaces.

Twistor spaces - 3

Second problem: contributions of [Svensson–JCW, preprint (2012)]

Given a harmonic map $\varphi : M^2 \to G_k(\mathbb{C}^n)$ from a surface to complex projective space, **Burstall** (1986) showed that φ (or its orthogonal complement) has a twistor lift if and only if it is *nilconformal* (see below). We extend this result to

(i) all other classical simply connected compact inner Riemannian symmetric spaces:

- the oriented real Grassmannians $\widetilde{G}_k(\mathbb{R}^n) = \mathrm{SO}(n)/\big(\mathrm{SO}(n-k) \times \mathrm{SO}(k)\big)$ with $k(n-k)$ even,
- the quaternionic Grassmannians $G_k(\mathbb{H}^m) = \mathrm{Sp}(m)/\big(\mathrm{Sp}(k) \times \mathrm{Sp}(m-k)\big)$,
- the space $\mathrm{SO}(2m)/\mathrm{U}(m)$ of +ve orthogonal complex strs on $\mathbb{R}^{2m}$ and the space $\mathrm{Sp}(m)/\mathrm{U}(m)$ of Lagrangian subspaces of $\mathbb{C}^{2m}$;

(ii) the exceptional symmetric space $G_2/\mathrm{SO}(4)$.

Twistor spaces - 4

Second problem: further contributions of [Svensson–JCW]

• We extend Burstall's result as follows: given a nilconformal harmonic map φ from any of the above symmetric spaces and a uniton α which is adapted to φ, we construct a twistor lift $\psi : M^2 \to F_{d_0,\dots,d_t}$ of $\pm\varphi$ associated to the uniton.

• For harmonic maps from a surface to the above symmetric spaces of finite uniton number, we give explicit formulae for the map and its twistor lift in terms of holomorphic data.

Harmonic maps into Grassmannians-1

We identify smooth maps $\varphi : M^2 \to G_k(\mathbb{C}^n)$ with smooth subbundles of the trivial bundle $\underline{\mathbb{C}}^n = M^2 \times \mathbb{C}^n$.

Any subbundle φ of $\underline{\mathbb{C}}^n$ inherits a metric by restriction, and a connection ∇_φ by orthogonal projection:

$$(\nabla_\varphi)_W(v) = \pi_\varphi(\partial_W v) \qquad (W \in TM),\ v \in \Gamma(\varphi)\,);$$

here $\Gamma(\cdot)$ denotes the space of (smooth) sections of a bundle.

By the ∂' *and* ∂''*-second fundamental forms of* φ we mean the vector bundle morphisms $A'_\varphi, A''_\varphi : \varphi \to \varphi^\perp$ defined on each coordinate domain (U, z) by

$$A'_\varphi(v) = \pi_\varphi^\perp(\partial_z v) \quad \text{and} \quad A''_\varphi(v) = \pi_\varphi^\perp(\partial_{\bar z} v) \qquad (v \in \Gamma(\varphi)\,)$$

where $\partial_z = \partial/\partial z$ and $\partial_{\bar z} = \partial/\partial \bar z$. These represent the derivatives of φ, so φ *is harmonic iff* A'_φ *is holomorphic iff* A''_φ *antiholomorphic*.

Harmonic maps into Grassmannians-2

Since the second fundamental forms are holomorphic, we can fill out zeros to get the *Gauss bundles* $G'(\varphi) = \operatorname{Im} A'_\varphi$, $G''(\varphi) = \operatorname{Im} A''_\varphi$, *subbundles of* $\varphi^\perp$;
further, *if* φ *is harmonic, then so are its Gauss bundles*.

We can repeat the process to get *higher Gauss bundles*:
$G^{(1)}(\varphi) = G'(\varphi)$, $G^{(i)}(\varphi) = G'(G^{(i-1)}(\varphi))$ and similarly, $G^{(-1)}(\varphi) = G''(\varphi)$, $G^{(-i)}(\varphi) = G'(G^{(-i+1)}(\varphi))$. On setting $G^{(0)}(\varphi) = \varphi$, this gives a sequence $G^{(i)}(\varphi) : i \in \mathbb{Z}$ of harmonic maps from φ called the *harmonic sequence* of φ.

In particular, *if we start with a full holomorphic map* $f : M^2 \to \mathbb{C}P^n$*from a surface to complex projective space, we obtain a sequence of harmonic maps* $G^{(i)}(f)$ $i = 0, 1, \ldots, n$ *with the first one holomorphic and the last one antiholomorphic*.

First idea for obtaining twistor lifts

- Use the Gauss bundle $G'(\varphi)$ or $G'(\varphi^\perp)$,

i.e., use the *image* of A'_φ or $A'_{\varphi^\perp}$;

or, dually,

- use the the *kernel* of A'_φ or $A'_{\varphi^\perp}$.

Example: Strongly Conformal Maps

A harmonic map $\varphi : M^2 \to G_*(\mathbb{C}^n)$ is called *strongly conformal* if $A'_{\varphi^\perp} \circ A'_\varphi = 0$, equivalently, $G'(\varphi)$ and $G''(\varphi)$ are orthogonal.

Let $\varphi : M^2 \to \mathbb{C}P^n$ be a strongly conformal harmonic map which is neither holomorphic nor antiholomorphic, then is has a twistor lifts:

The case $n = 2$. *$\varphi^\perp$ has the J_2-holomorphic twistor lift $\psi : M^2 \to F_{1,1,1}$ given by*

$$\psi = \big(G''(\varphi), \varphi, G'(\varphi)\big).$$

The case $n > 2$. This involves some choice: Let W be a holomorphic subbundle of $\varphi^\perp$ satisfying

$$\operatorname{Im} A'_\varphi \subset W \subset \ker A'_{\varphi^\perp}.$$

Set $V = W^\perp \cap \varphi^\perp$. Then (V, φ, W) is a J_2-holomorphic lift of $\varphi^\perp$, and every J_2-holomorphic twistor lift of $\varphi^\perp$ into a flag manifold with three legs is given this way.

Example: superconformal tori

Consider the isometric minimal immersion of the torus $\mathbb{C}/\langle 2\pi/\sqrt{3},\ 2\pi\mathrm{i}\rangle$ into $\mathbb{C}P^2$ given by the harmonic map

$$\varphi(z) = \left[\mathrm{e}^{z-\overline{z}},\, \mathrm{e}^{\zeta z-\overline{\zeta z}},\, \mathrm{e}^{\zeta^2 z-\overline{\zeta^2 z}}\,\right], \tag{3}$$

where $\zeta = \mathrm{e}^{2\pi\mathrm{i}/3}$. Set $\varphi_i = G^{(i)}(\varphi)\ \ (i \in \mathbb{Z})$. Then

$$\varphi_i(z) = \left[\mathrm{e}^{z-\overline{z}},\, \zeta^i\mathrm{e}^{\zeta z-\overline{\zeta z}},\, \zeta^{2i}\mathrm{e}^{\zeta^2 z-\overline{\zeta^2 z}}\,\right];$$

The φ_i are cyclic of order 3 and mutually orthogonal. In particular $G''(\varphi) = \varphi_2$ so that φ is strongly conformal. Also $G^{(3)}(\varphi) = \varphi$; a map φ with $G^{(r)}(\varphi) = \varphi$ for some $r \in \mathbb{N}$ is called *superconformal* [**Bolton and Woodward** (1994), **Bolton, Pedit and Woodward** (1995)]. $\varphi^\perp$ has the J_2-holomorphic twistor lift:

$$\psi = (\varphi_2, \varphi, \varphi_1) = \big(G''(\varphi), \varphi, G'(\varphi)\big).$$

Horizontal lifts of complex-isotropic maps

A harmonic map $\varphi : M^2 \to \mathbb{C}P^n$ is called *complex isotropic* or *superminimal* if its (higher) Gauss bundles $G^{(i)}(\varphi)\ \ (i \in \mathbb{Z})$ are mutually orthogonal. *All harmonic maps $S^2 \to \mathbb{C}P^n$ are complex isotropic*. We can construct all such harmonic maps from holomorphic maps as follows:

Theorem (**Eells and JCW** (1983))

There is a bijection between (i) pairs (f, r) where $f : M^2 \to \mathbb{C}P^n$ is a full holomorphic map and $r \in \{0, 1, \dots, n\}$, and (ii) full complex-isotropic harmonic maps $\varphi : M^2 \to \mathbb{C}P^n$, given by $\varphi = G^{(r)}(f)$, with inverse $f = G^{(-r)}(\varphi)$ where r is the largest integer such that $G^{(-r)}(\varphi)$ is non-zero.

We can see this construction in a twistorial way: given a full holomorphic map $f : M^2 \to \mathbb{C}P^n$ and $r \in \{0, 1, \dots, n\}$, set $\psi = (\psi_0,\ \psi_1,\ \psi_2) = \left(\sum_{i=0}^{r-1} G^{(i)}(f),\ G^{(r)}(f),\ \sum_{i=r+1}^{n} G^{(i)}(f)\right)$, then $\varphi^\perp = \pi_e(\psi)$, i.e., $\varphi = \psi_1$.

Horizontal lifts of complex-isotropic maps, continued

Conversely, given a complex-isotropic harmonic map $\varphi : M^2 \to \mathbb{C}P^n$, set $(\psi_0,\ \psi_1,\ \psi_2) = \left(\sum_{i\in\mathbb{N}} G^{(-i)}(\varphi),\ \varphi,\ \sum_{i\in\mathbb{N}} G^{(i)}(\varphi)\right)$. Then ψ is a twistor lift of $\varphi^\perp$, i.e., $\varphi^\perp = \pi_e(\psi)$.

This twistor lift has image in the horizontal spaces of π_e, so it is not only J_2-holomorphic but it is J_1-holomorphic as well.

HOW DO WE FIND LIFTS OF MORE GENERAL HARMONIC MAPS?

To proceed, set

$$A_z^\varphi = -A'_\varphi - A'_{\varphi^\perp}.$$

This is an endomorphism of the trivial bundle $\underline{\mathbb{C}}^n = M^2 \times \mathbb{C}^n$ which restricts to $-A'_\varphi$ on φ and to $-A'_{\varphi^\perp}$ on $\varphi^\perp$.

Interpretation: Harmonic maps into Lie groups [Uhlenbeck (1989)]

Any smooth map $\varphi : M^2 \to G$ from a surface to a Lie group defines a connection form $A^\varphi = \frac{1}{2}\varphi^{-1}\mathrm{d}\varphi =$ half the pull-back of the Maurer–Cartan form, and thus a covariant derivative $D^\varphi = \mathrm{d} + A^\varphi$, on the trivial bundle $\underline{\mathbb{C}}^n = M^2 \times \mathbb{C}^n$.
Write

$$D^\varphi = D^\varphi_z \mathrm{d}z + D^\varphi_{\bar z}\mathrm{d}\bar z$$

and

$$A^\varphi = A^\varphi_z \mathrm{d}z + A^\varphi_{\bar z}\mathrm{d}\bar z.$$

Since A^φ_z represents $\partial\varphi/\partial z$, φ *harmonic if and only if* A^φ_z *is holomorphic in the sense that* $D^\varphi_{\bar z} A^\varphi_z = 0$.
We can include the complex Grassmannians $G_k(\mathbb{C}^n)$ in $\mathrm{U}(n)$ via the totally geodesic Cartan embedding

$$G_k(\mathbb{C}^n) \hookrightarrow \mathrm{U}(n), \quad V \mapsto \pi_V - \pi_{V^\perp};$$

then we obtain the formula of last slide: $A^\varphi_z = -A'_\varphi - A'_{\varphi^\perp}$.

Nilconformality

Similarly, any of the symmetric spaces we consider can be embedded in $\mathrm{U}(n)$ as a totally geodesic submanifold and we have the general fact: *a map into a totally geodesic submanifold is harmonic iff it is harmonic as a map into the ambient manifold*.

We say that a smooth map $\varphi : M^2 \to \mathrm{U}(n)$ (or to a totally geodesic submanifold N of $\mathrm{U}(n)$) is *nilconformal* if A^φ_z is nilpotent, i.e., $(A^\varphi_z)^r = 0$ for some non-negative integer r. Note that

• strongly conformal maps $\varphi : M^2 \to N$ are nilconformal, in fact, they are characterized by $\left(A^\varphi_z|_\varphi\right)^2 = 0$ which implies $(A^\varphi_z)^3 = 0$;

• *nilconformal maps* $\varphi : M^2 \to N$ *are weakly conformal*, so that nilconformal harmonic maps are minimal branched immersions.

SECOND IDEA FOR OBTAINING TWISTOR LIFTS

Use images or kernels of *powers* of A^φ_z. To get real and quaternionic cases, we need to be a bit more general as follows:

A_z^φ-filtrations

Definition

[**Svensson–JCW**, preprint (2012)] Let $\varphi : M^2 \to \mathrm{U}(n)$ be a smooth map. Let (Z_i) be a finite sequence of subbundles of $\underline{\mathbb{C}}^n$ which forms a *filtration* of $\underline{\mathbb{C}}^n$:

$$\underline{\mathbb{C}}^n = Z_0 \supset Z_1 \supset \cdots \supset Z_t \supset Z_{t+1} = \underline{0}\,.$$

We call (Z_i) an *A_z^φ-filtration (of length t)* if, for each $i = 0, 1, \ldots, t$,

(i) Z_i is a holomorphic subbundle, i.e., $\Gamma(Z_i)$ is closed under $D_{\bar{z}}^\varphi$;

(ii) A_z^φ maps Z_i into the smaller subbundle Z_{i+1} .

We say that φ is *nilconformal* if A_z^φ is nilpotent, i.e., $(A_z^\varphi)^r = 0$ for some non-negative integer r. Then, *A_z^φ-filtrations exist iff φ is nilconformal*.

Examples of A_z^φ-filtrations

Given a filtration (Z_i) of $\underline{\mathbb{C}}^n$ of length t, we define its *legs* ψ_i by

$$\psi_i = Z_i \ominus Z_{i+1}\,, \quad \text{equivalently,} \;\; Z_i = \sum_{j \geq i} \psi_j \quad (i = 0, 1, \ldots, t+1)\,. \tag{4}$$

Let $\varphi : M^2 \to \mathrm{U}(n)$ be a nilconformal harmonic map and let t be the least $t \in \mathbb{N}$ with $(A_z^\varphi)^{t+1} = 0$. We obtain two A_z^φ-filtrations of length t:

(i) $Z_i = \mathrm{Im}(A_z^\varphi)^i$, the *filtration by A_z^φ-images*;

(ii) $\widehat{Z}_i = \ker(A_z^\varphi)^{t+1-i}$, the *dual* of (i). Note that $Z_i \subseteq \widehat{Z}_i$ $\forall i$.

The legs of suitable A_z^φ-filtrations give twistor lifts for harmonic maps into any of the classical simply connected compact inner Riemannian symmetric spaces and for $G_2/\mathrm{SO}(4)$.

Twistor spaces for real Grassmannians

Let $d_0, d_{-1}, \ldots, d_{-s}$ be positive integers and set $d_i = d_{-i}$ for $i = 0, \ldots, s$. Define a submanifold of the complex flag manifold $F_{d_{-s},\ldots,d_s} = \mathrm{U}(n)/\mathrm{U}(d_{-s}) \times \cdots \times \mathrm{U}(d_s)$ by

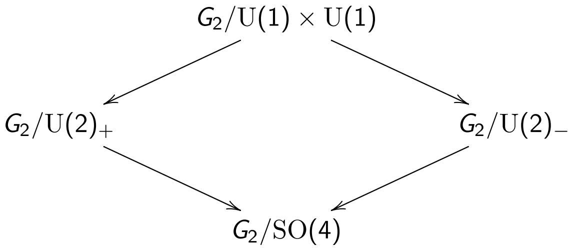
$$F^{\mathbb{R}}_{d_{-s},\ldots,d_0} = \{\psi = (\psi_{-s}, \ldots, \psi_0, \ldots, \psi_s) \in F_{d_{-s},\ldots,d_s} : \psi_i = \overline{\psi}_{-i} \,\forall i\}.$$

Note that $F^{\mathbb{R}}_{d_0,\ldots,d_s}$ is a complex submanifold of $F_{d_{-s},\ldots,d_s}$ with respect to the complex structures J_1 and J_2. Hence the twistor fibration over the complex Grassmanian restricts to a twistor fibration: $\pi_0^{\mathbb{R}} : F^{\mathbb{R}}_{d_{-s},\ldots,d_0} \to \widetilde{G}_k(\mathbb{R}^n) = \mathrm{SO}(n)/\mathrm{SO}(k) \times \mathrm{SO}(n-k)$ given by $\pi_0^{\mathbb{R}}(\psi) = \sum \psi_{2j}$.
Here, $k = \sum_{j \in \mathbb{Z}} d_{2j}$ and $n - k = 2\sum_{j \in \mathbb{N}} d_{2j+1}$; note that $n-k$ is even.

If we took an even number of legs instead, we would get twistor spaces for $\mathrm{SO}(2m)/\mathrm{U}(m)$.

Harmonic maps into $G_2/\mathrm{SO}(4)$

Think of $\mathbb{R}^7$ as the space of imaginary octonions. Recall that $\mathbb{R}^7$ has a vector product $\times$ given by $v \times w =$ the imaginary part of the octonion product $v \cdot w$ $\ (v, w \in \mathbb{R}^7)$.
$G_2/\mathrm{SO}(4) =$ 3-dimensional subspaces V of $\mathbb{R}^7$ which are *associative*, i.e., closed under the vector product. Note that any such V has a natural orientation coming from a basis $\{v, w, v \times w\}$ of V. Thus $G_2/\mathrm{SO}(4)$ is a submanifold of $\widetilde{G}_3(\mathbb{R}^7)$.
$G_2/\mathrm{SO}(4)$ has three twistor spaces, cf. [**Salamon** (1989)]:



with every fibre isomorphic to $\mathbb{C}P^1$.

The twistor spaces of $G_2/\mathrm{SO}(4)$

$G_2/\mathrm{U}(2)_+$ is the quaternionic twistor space with fibre at $V \in G_2/\mathrm{SO}(4)$ consisting of all *positive* almost Hermitian structures on $V^\perp$, equivalently isotropic 2-dimensional subspaces of $V^\perp \otimes \mathbb{C}$ which are *coassociative*, i.e., enjoy the property $v \times w = 0$ for all $v, w \in V^\perp \otimes \mathbb{C}$.

On the other hand, $G_2/\mathrm{U}(2)_-$ is the twistor space with fibre at $V \in G_2/\mathrm{SO}(4)$ consisting of all *negative* almost Hermitian structures on $V^\perp$, equivalently isotropic 2-dimensional subspaces of $V^\perp \otimes \mathbb{C}$ which are *not* coassociative.

For $G_2/\mathrm{U}(1) \times \mathrm{U}(1)$, see the next slide.

Unified description of the twistor spaces of $G_2/\mathrm{SO}(4)$

Recall:
$F = F^{\mathbb{R}}_{d_{-s},\ldots,d_0} = \{\psi = (\psi_{-s},\ldots,\psi_s) \in F_{d_{-s},\ldots,d_s} : \psi_{-i} = \overline{\psi_i}\ \forall i\}$.
The twistor projections $\pi : F \to \widetilde{G}_k(\mathbb{R}^n)$ to real Grassmannians are given by the sum of the even legs: $\pi(\psi) = \sum_j \psi_{2j}$.
We can give a unified description of all the twistor spaces as submanifolds of the flag manifolds F: We say that (ψ_i) is a *G_2-flag* if (G2): $\psi_i \times \psi_j \subseteq \psi_{i+j}\ \forall i,j$. Then

$$G_2/\mathrm{U}(2)_+ = \{(\psi_{-1},\psi_0,\psi_1) \in F^{\mathbb{R}}_{2,3} : \text{G2 holds}\}$$

$$G_2/\mathrm{U}(2)_- = \{(\psi_{-2},\psi_{-1},\psi_0,\psi_1,\psi_2) \in F^{\mathbb{R}}_{1,2,1} : \text{G2 holds}\}$$

$$G_2/\mathrm{U}(1)\times\mathrm{U}(1) = \{(\psi_{-3},\psi_{-2},\psi_{-1},\psi_0,\psi_1,\psi_2,\psi_3) \in F^{\mathbb{R}}_{1,1,1,1} : \text{G2 holds}\}$$

Since (G2) implies that $\pi(\psi) = \sum_j \psi_{2j}$ is associative, the projection π restricts to projections from these twistor spaces to $G_2/\mathrm{SO}(4)$, cf. **Kobak** (1993).

Lifts to the twistor spaces of $G_2/\mathrm{SO}(4)$

Theorem

Let $\varphi : M^2 \to G_2/\mathrm{SO}(4)$ be harmonic. Then φ has a J_2-holomorpic lift to one of the above three twistor spaces iff it is nilconformal.

Idea of proof. A harmonic map $\varphi : M^2 \to \widetilde{G}_3(\mathbb{R}^7)$ has a lift to one of the above three flag manifolds F if and only if it is nilconformal. If now, φ is associative, we show that the legs satisfy (*). For example, φ has a lift to $F^{\mathbb{R}}_{2,3}$ if and only if it is strongly conformal. We know that $v \times w \in \varphi$ for all $v, w \in \varphi$. Differentiating this twice gives $A'_\varphi v \times A'_\varphi w \in \varphi$, showing that the Gauss bundle $G'(\varphi)$ is closed under the vector product. By strong conformality, it is isotropic. Either it is of rank 2 and so coassociative as required, or it's of rank 1 and can be extended to a unique coassociative subbundle of $\varphi^\perp$. Either way, we get our twistor lift to $G_2/\mathrm{U}(2)_+$.

Almost complex maps into S^6

The vector product on $\mathbb{R}^7$ defines an almost complex structure J on S^6 by $J_u(v) = u \times v$. A map $M^2 \to S^6$ is called *almost complex* if it intertwines J^M and J. Such maps are harmonic.
The next result is a slight reformulation of [**Bolton,Vranken and Woodward** (1994), **Bolton, Pedit and Woodward** (1995)]

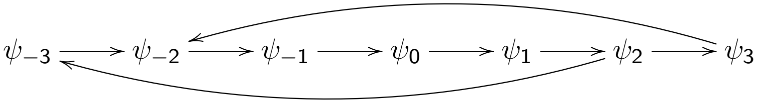
Theorem

Let $F : M^2 \to S^6$ be a non-constant almost complex map. Then either (i) *F is a weakly conformal map into a totally geodesic $S^2 = \Pi^3 \cap S^6$ where Π^3 is associative 3-dimensional subspace of $\mathbb{R}^7$ or*
(ii) *there is a G_2-flag*

$$\psi = (\psi_{-3}, \psi_{-2}, \psi_{-1}, \psi_0, \psi_1, \psi_2, \psi_3) : M^2 \to G_2/\mathrm{U}(1) \times \mathrm{U}(1)$$

with $\psi_0 = f = \operatorname{span} F$ and we have a diagram showing the only possible second fundamental forms A'_{ψ_i, ψ_j} :

Maps into $G_2/\mathrm{SO}(4)$ from a.c. maps – 1



Almost complex maps into S^6 can be extended to harmonic maps into $G_2/\mathrm{SO}(4)$ as follows:

Proposition

Let $F : M^2 \to S^6$ be an almost complex map and let α be a holomorphic line subbundle of $F^{-1}T^{1,0}S^6$. Set $\varphi = \overline{\alpha} \oplus f \oplus \alpha$ where $f = \operatorname{span} F$. Then φ is a harmonic map from M^2 to $G_2/\mathrm{SO}(4)$.

This essentially follows from the reduction theorem of [**Burstall and JCW**, 1986] and the fact that $\alpha \times \overline{\alpha} \subset f$ so that φ is associative.

Maps into $G_2/\mathrm{SO}(4)$ from a.c. maps – 2

Choosing $\alpha = G'(f)$ we obtain

Theorem (**Svensson and JCW**, preprint (2013))

There is a one-to-one correspondence between

1. *almost complex maps* $F : M^2 \to S^6$ *with image not contained in a totally geodesic* S^2*, and*
2. *strongly conformal harmonic maps* $\varphi : M \to G_2/\mathrm{SO}(4)$ *with* (a) $\varphi^\perp$ *strongly conformal,* (b) $\operatorname{rank} G'(\varphi) = 1$ *and* (c) $G'(\varphi^\perp) \times G''(\varphi^\perp)$ *a non-constant subbundle,*

given by

$$F \mapsto \varphi = \overline{G'(f)} \oplus f \oplus G'(f) \quad \textit{with inverse} \quad \varphi \mapsto F = \mathrm{i}\,\overline{L} \times L/|L|^2,$$

where $f = \operatorname{span} F$ *and* L *is any non-zero (local) section of* $G'(\varphi^\perp)$*.*

Maps into $G_2/\mathrm{SO}(4)$ from a.c. maps – 3

The following additional constructions use the twistor method.

Proposition

Let $F : M^2 \to S^6$ be an almost complex map with image not contained in a totally geodesic S^2. Set $\psi_3 = G^{(1)}(f) \times G^{(2)}(f)$. Then $\varphi = \overline{\psi}_3 \oplus f \oplus \psi_3$ is a strongly conformal harmonic map $M^2 \to G_2/\mathrm{SO}(4)$ with $\operatorname{rank} G'(\varphi) = 2$.

Theorem

Let $F : M^2 \to S^6$ be an isotropic almost complex map. Then $\varphi = G^{(-i)}(f) \oplus f \oplus G^{(i)}(f)$ is a harmonic map into $G_2/\mathrm{SO}(4)$ for $i = 1, 2, 3$.

All isotropic almost complex maps are given by a twistor construction of [**Bryant** (1982)], see [**Bolton–Woodward** (1997)] in terms of superhorizontal holomorphic maps. Such holomorphic maps are constructed by [**Fernández**, preprint (2011)], see also [**Martins**, preprint (2012)]. Skip to references

Harmonic maps into Grassmannians-3

Recall the totally geodesic **Cartan embedding**:

$$G_*(\mathbb{C}^n) = \cup_k G_k(\mathbb{C}^n) \hookrightarrow \mathrm{U}(n), \quad V \mapsto \pi_V - \pi_{V^\perp}.$$

This identifies $G_*(\mathbb{C}^n)$ with $\sqrt{I} := \{g \in \mathrm{U}(n) \mid g^2 = I\}$. Note that it maps $V^\perp \in G_*(\mathbb{C}^n)$ to $-V$. So
there is an identification between

$$\textit{Harmonic maps } \varphi : M^2 \to G_*(\mathbb{C}^n)$$

and

$$\textit{Harmonic maps } \varphi : M^2 \to \mathrm{U}(n) \textit{ which have image in } \sqrt{I}.$$

Further, $A_z^\varphi = -A'_\varphi - A'_{\varphi^\perp}$, so the conditions
A_z^φ *holomorphic* and A'_φ *holomorphic*
are both equivalent to harmonicity of $\varphi : M^2 \to G_*(\mathbb{C}^n)$.

Holomorphicity of twistor lifts

Recall the twistor projection which takes the sum of the even legs
The $(1,0)$ horizontal and vertical spaces w.r.t. J_1 are given by

$$\mathcal{H}^{J_1}_{(1,0)} = \sum_{\substack{i,j=0,\dots,t \\ i<j,\ j-i \text{ odd}}} \mathrm{Hom}(\psi_i, \psi_j)\,, \quad \mathcal{V}^{J_1}_{(1,0)} = \sum_{\substack{i,j=0,\dots,t \\ i<j,\ j-i \text{ even}}} \mathrm{Hom}(\psi_i, \psi_j).$$

For J_2, $\mathcal{H}^{J_2}_{(1,0)} = \mathcal{H}^{J_1}_{(1,0)}$; but, for $\mathcal{V}^{J_2}_{(1,0)}$ the condition $i < j$ is changed to $j < i$. Hence

Proposition

Let $\psi = (\psi_0, \psi_1, \dots, \psi_t) : M^2 \to F$ be a smooth map. Then

(i) *ψ is J_1-holomorphic iff $A'_{\psi_i,\psi_j} = 0$ when $i - j$ is $+ve$;*

(ii) *ψ is J_2-holomorphic iff*

$$A'_{\psi_i,\psi_j} = 0 \textit{ when } i - j \textit{ is } +ve \textit{ and odd, or } j - i \textit{ is } +ve \textit{ and even.} \quad \square$$

Filtrations for maps into Grassmannians-1

Let $\varphi : M^2 \to G_*(\mathbb{C}^n)$ be a smooth map to a Grassmannian. Say that a filtration (Z_i) is *alternating for* φ if its legs $\psi_i = Z_i \ominus Z_{i+1}$ satisfy

$$\psi_i \subset (-1)^i \varphi \qquad \text{for } i = 0, 1, \dots, t. \tag{5}$$

This is equivalent to $\varphi = \sum_j \psi_{2j}$, i.e., $\pi_e \circ \psi = \varphi$.

Proposition

Let $\varphi : M^2 \to G_*(\mathbb{C}^n)$ be a smooth map. Then formulae (4) *define a one-to-one correspondence between*
(i) *J_2-holomorphic lifts $(\psi_i) : M^2 \to F$ of φ to a complex flag manifold (with all ψ_i non-zero) and*
(ii) *strict A_z^φ-filtrations (Z_i) which are alternating for φ.*
Further, $F = F_{d_0,\dots,d_{t+1}}$ where (Z_i) has length t, and $d_i = \operatorname{rank} \psi_i$ $(i = 0, 1, \dots, t+1)$.

Filtrations for maps into Grassmannians-2

The only problem is that the two A_z^φ-filtrations we know — the filtration by A_z^φ-images and its dual — are not alternating. However, they *split* in the sense that

$$Z_i = U_i \oplus V_i \quad \text{where} \quad U_i \subset \varphi \quad \text{and} \quad V_i \subset \varphi^\perp. \tag{6}$$

There is a procedure for turning split A_z^φ-filtrations into strict alternating ones.
Applied to the A_z^φ-filtration by images, the procedure gives $\widetilde{Z}_i = \widetilde{U}_i + \widetilde{V}_i$ $(i = 1, 2, \dots)$ with

$$\widetilde{U}_i = \operatorname{Im}(A_z^\varphi)^{2i-1}|_{\varphi^\perp} = \operatorname{Im}\big((A'_{\varphi^\perp} \circ A'_\varphi)^{i-1} \circ A'_{\varphi^\perp}\big)$$

$$\widetilde{V}_i = \operatorname{Im}(A_z^\varphi)^{2i}|_{\varphi^\perp} = \operatorname{Im}(A'_\varphi \circ A'_{\varphi^\perp})^i$$

The legs of $\widetilde{Z}_i$ give the twistor lift of **Burstall** (1986).

Extended solutions [**Uhlenbeck** (1989)]

Any harmonic map $\varphi : M^2 \to \mathrm{U}(n)$ gives rise to a loop Φ_λ of smooth maps called an **extended solution associated to** φ as follows: Set $A^\varphi = \frac{1}{2}\varphi^{-1}\mathrm{d}\varphi$, write $A^\varphi = A^\varphi_z \mathrm{d}z + A^\varphi_{\bar z}\mathrm{d}\bar z$ and set $A^\varphi_\lambda = \frac{1}{2}(1-\lambda^{-1})A^\varphi_z \mathrm{d}z + \frac{1}{2}(1-\lambda)A^\varphi_{\bar z}\mathrm{d}\bar z$.
Then φ harmonic implies $\mathrm{d}A^\varphi_\lambda + [A^\varphi_\lambda, A^\varphi_\lambda] = 0$, the integrability condition for solving

$$\tfrac{1}{2}\Phi_\lambda^{-1}\mathrm{d}\Phi_\lambda = A^\varphi_\lambda\,.$$

We let Φ_λ be any smooth solution — these exist locally, at least. Note that Φ_1 is constant, we take it to be I, so that Φ is a mapping from M^2 to the *loop group* $\Omega\mathrm{U}(n) = \{\text{smooth maps } (S^1, 1) \to (\mathrm{U}(n), I)\}$. Further Φ_{-1} equals φ up to left multiplication by an element of $\mathrm{U}(n)$.
We say that φ has *finite uniton number* if it has an associated extended solution which is polynomial in λ, in which case its *(minimal) uniton number* is the minimum of the degrees of such extended solutions.

The Grassmannian model [Segal (1989)]

Set $\mathcal{H} = L^2(S^1, \mathbb{C}^n)$ and $\mathcal{H}_+ = \{\sum_{k\geq 0} a_k \lambda^k \in \mathcal{H} \mid a_k \in \mathbb{C}^n\}$.
We have a map $\Omega \mathrm{U}(n) \ni \Phi \mapsto W = \Phi \mathcal{H}_+ \subset \mathcal{H}$. W is called the *Grassmannian model of* Φ. Φ is an extended solution iff (i) W is holomorphic, i.e. $\partial_{\bar{z}}(\Gamma(W)) \subset \Gamma(W)$, and (ii) $\Gamma(W)$ is closed under the operator $F = \lambda \partial_z$, i.e., $F\big(\Gamma(W)\big) \subset \Gamma(W)$, in which case we also call W an *extended solution*.
The map restricts to a map from
$\Omega^{\mathrm{alg}} \mathrm{U}(n) = \{\gamma \in \Omega \mathrm{U}(n) = \sum_{i=s}^{r} \lambda^k T_k$ for some integers $r \geq s$ and $n \times n$ complex matrices $T_k\}$ to

$$\{W \subset \mathcal{H} |\ \lambda^r \mathcal{H}_+ \subset W \subset \lambda^s \mathcal{H}_+ \text{ for some } r, s \in \mathbb{N},\ \lambda W \subset W\}.$$

It restricts further to a map from $\Omega_r \mathrm{U}(n) =$
$\{$polynomials $\sum_{k=0}^{r} \lambda^k T_k \subset \Omega_{\mathrm{alg}} \mathrm{U}(n)$ of degree at most $r\}$ to

$$\{W \subset \mathcal{H} |\ \lambda^r \mathcal{H}_+ \subset W \subset \mathcal{H}_+,\ \lambda W \subset W\}.$$

The simplest example

Example

Let $\alpha : M^2 \to G_*(\mathbb{C}^n)$ be holomorphic. Then an associated extended solution is $\Phi = \pi_\alpha + \lambda \pi_\alpha^\perp$. In fact, $\alpha = \Phi_{-1}$.

• Note this is also an extended solution associated to the antiholomorphic map $\alpha^\perp$, but $\alpha^\perp = -\Phi_{-1}$.

Hence, both α and $\alpha^\perp$ have uniton number 1.

• An associated extended solution with $\Psi_{-1} = \alpha^\perp$ is $\Psi = \lambda\Phi$.

FOR LATER USE

• Note that Φ is ν-invariant, i.e.,satisfies (7).

• $\Phi = \pi_\alpha + \lambda \pi_\alpha^\perp$ is a uniton factorization with one uniton.

• The Grassmannian model associated to α is $W = \alpha + \lambda \mathcal{H}_+$.

W is closed under $\lambda \mapsto -\lambda$.

Extended solutions for maps into Grassmannians

Let $\nu : \mathcal{H} \to \mathcal{H}$ be the involution $\lambda \mapsto -\lambda$. Then $W = \Phi\mathcal{H}_+$ *is closed under* ν *iff* Φ *is* ν*-invariant in the sense that*

$$\Phi_\lambda \Phi_{-1} = \Phi_{-\lambda} \qquad (\lambda \in S^1)\,; \tag{7}$$

this condition implies that $\varphi = \Phi_{-1}$ satisfies $\varphi^2 = I$, which means that it has image in a complex Grassmannian $G_*(\mathbb{C}^n)$. Conversely,

Lemma

(i) *Let* $\varphi : M^2 \to G_*(\mathbb{C}^n)$ *be a harmonic map from a Riemann surface which has an associated extended solution. Then it has a* ν*-invariant extended solution* Ψ *with* $\Psi_{-1} = \varphi$*.*

(ii) *Suppose that* $\varphi : M^2 \to G_*(\mathbb{C}^n)$ *is a harmonic map of uniton number* r*. Then it has a* ν*-invariant polynomial extended solution* Ψ *of degree* r *or* $r+1$ *with* $\Psi_{-1} = \varphi$*.*

Harmonic maps from unitons

A **uniton** for φ is a subbundle α of the trivial bundle $\underline{\mathbb{C}}^n$ which is **holomorphic**, i.e, closed under $D^{\varphi}_{\bar{z}}$, and **closed under** A^{φ}_z.

Example

(i) Any holomorphic subbundle of $\underline{\mathbb{C}}^n$ which is contained in the kernel of A^{φ}_z is a uniton, called a *basic* uniton.
(ii) Any holomorphic subbundle of $\underline{\mathbb{C}}^n$ which contains the image of A^{φ}_z is a uniton, called an *antibasic* uniton.

Then [**Uhlenbeck** (1989)]:

(i) *If* $\varphi : M^2 \to \mathrm{U}(n)$ *is harmonic and* $\underline{\alpha}$ *is a uniton for* φ*, then* $\widetilde{\varphi} = \varphi(\pi_\alpha - \pi_\alpha^\perp)$ *is harmonic.*

(ii) *Any harmonic map* $\varphi : S^2 \to \mathrm{U}(n)$ *can written as a finite product of unitons*:

$$\varphi = \mathrm{const.}\ (\pi_{\alpha_1} - \pi_{\alpha_1}^\perp) \cdots (\pi_{\alpha_r} - \pi_{\alpha_r}^\perp)\,. \tag{8}$$

called a **uniton factorization** of the harmonic map φ.

Unitons and extended solutions

(i) and (ii) of previous page are proved by showing the stronger results for *extended solutions*:

(i) *Let* $\Phi : M^2 \to \mathrm{U}(n)$ *be an extended solution. Then* α *is a uniton for* φ *iff* $\widetilde{\Phi} = \Phi(\pi_\alpha + \lambda\pi_\alpha^\perp)$ *is an extended solution.*

(ii) *Any extended solution* $\Phi : S^2 \to \mathrm{U}(n)$ *can written as a finite product of unitons*:

$$\Phi = \ (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp)\cdots(\pi_{\alpha_r} + \lambda\pi_{\alpha_r}^\perp)\,; \tag{9}$$

here α_i is a uniton for $\Phi_{i-1} = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp)\cdots(\pi_{\alpha_{i-1}} + \lambda\pi_{\alpha_{i-1}}^\perp)\,.$

Unitons factorizations and the Grassmannian model

[**Segal**] Setting $W_i = \Phi_i \mathcal{H}_+$ defines an equivalence between uniton factorizations (9) and filtrations

$$W = W_r \subset W_{r-1} \subset \cdots \subset W_0 = \mathcal{H}_+ \tag{10}$$

by extended solutions satisfying

$$\lambda W_{i-1} \subset W_i \subset W_{i-1} \qquad (i = 1, 2, \dots, r). \tag{11}$$

Example

With $\Phi = \pi_\alpha + \lambda \pi_\alpha^\perp$, we have

$$W_1 = W = \alpha + \lambda \mathcal{H}_+ \quad \text{and} \quad W_0 = \mathcal{H}_+.$$

These satisfy (11).

Two factorizations

Let Φ be a polynomial extended solution of degree r and set $W = \Phi\mathcal{H}_+$.

Example (Segal (1989))

Set $W_i = W + \lambda^i\mathcal{H}_+$. Then (W_i) defines a filtration (10) satisfying (11), and so a uniton factorization (9); we call this the *Segal factorization of* Φ *(or* W*)*, as it appears in Segal (1989). Each α_i appearing in the corresponding uniton factorization (9) is an antibasic uniton for Φ_{i-1}.

Example (Uhlenbeck (1989))

Set $W_i = \lambda^{r-i}W \cap \mathcal{H}_+$. Again, (W_i) defines a filtration (10) satisfying (11), and so a uniton factorization (9) which we call the *Uhlenbeck factorization of* Φ *(or* W*)*. Each α_i appearing in (9) is a basic uniton for Φ_{i-1}.

Twistor lifts from unitons

Let $\Phi : M^2 \to \Omega\mathrm{U}(n)$ be a polynomial extended solution of degree r and set $W = \Phi\mathcal{H}_+$.

Proposition

Given a uniton factorization (9) *such that each uniton α_i is a basic uniton for Φ_{i-1}, let $W_i = \Phi_i\mathcal{H}_+$ be the associated filtration and set*

$$\begin{aligned} Z_i &= P_0\Phi^{-1}Y_i \quad \textit{where} & (12)\\ Y_i &= \lambda^i W_{r-i} + \lambda W \quad (i = 0, 1, \ldots, r), \quad Y_{r+1} = \lambda W. & (13) \end{aligned}$$

Then (i) $Z_i = P_0\Phi^{-1}Y_i$ is an A_z^{φ}-filtration given in terms of the unitons α_i in (9) *by*

$$Z_i = \mathrm{Im}(\pi_{\alpha_r}^{\perp} \circ \cdots \circ \pi_{\alpha_{r-i+1}}^{\perp}). \qquad (14)$$

Examples

Let Φ be a polynomial extended solution of degree r; $W = \Phi\mathcal{H}_+$.

Example (Twistor lift from Uhlenbeck unitons)

The Uhlenbeck factorization has basic unitons. For a ν-invariant extended solution, the corresponding A_z^{φ}-filtration is alternating. Its legs thus give a twistor lift of $\varphi = \Phi_{-1}$, which we call the *canonical twistor lift*.

Example (Factorization giving Burstall lift)

For any $i \in \mathbb{N}$, let $W_{(i)}$ denote the ith osculating space spanned by derivatives of local holomorphic sections of W up to order i. Suppose there is a $t \in \mathbb{N}$ such that $(P_0 W)_{(t)} = \underline{\mathbb{C}}^n$. Then we have a *uniton factorization*:

$$W = W_{(0)} \subset W_{(1)} \subset \cdots \subset W_{(r)} \subset \cdots \subset W_{(t)} = \mathcal{H}_+;$$

this is the *factorization by A_z-images* of the author. It has basic unitons and the corresponding A_z^{φ}-filtration is that leading to the Burstall twistor lift.

The canonical filtration for a harmonic map into a Grassmannian

Proposition

Let Φ be a ν-invariant polynomial extended solution of degree r. Set $W = \Phi\mathcal{H}_+$ and $\varphi = \Phi_{-1} : M^2 \to G_*(\mathbb{C}^n)$. Then the canonical A_z^{φ}-filtration is alternating, i.e, setting $\psi_i = Z_i \ominus Z_{i+1}$ $(i = 0, 1, \ldots, r)$ we have $\psi_i \subset (-1)^i\varphi$ so that $\varphi = \sum_j \psi_{2j}$.

Applying the lemma above which says that a harmonic map $\varphi : M^2 \to G_*(\mathbb{C}^n)$ of uniton number r has a ν-invariant polynomial extended solution Ψ of degree r or $r+1$ with $\Psi_{-1} = \varphi$, we obtain

Corollary

Let $\varphi : M^2 \to G_k(\mathbb{C}^n)$ be a harmonic map of finite uniton number r. Then there is a J_2-holomorphic twistor lift $\psi : M^2 \to F$ of φ or $\varphi^\perp$ into some flag manifold $F = F_{d_0,d_1,\ldots,d_t}$ with $t \leq min(r+1, n-1)$.

S^1-invariant solutions – 1

Consider the action of S^1 on $\Omega\mathrm{U}(n)$ given by

$$(\gamma \cdot \eta)(\lambda) = \eta(\lambda\gamma)\eta(\gamma)^{-1} \qquad (\gamma \in S^1, \eta \in \Omega\mathrm{U}(n)).$$

An extended solution Φ is S^1-invariant if $\Phi_\lambda \Phi_\gamma = \Phi_{\lambda\gamma}$ $\forall \lambda, \gamma \in S^1$.

Proposition

Let $\Phi : M^2 \to \Omega\mathrm{U}(n)$ be an S^1-invariant polynomial extended solution of degree r. Let $\beta_1, \ldots, \beta_r$ and $\gamma_1, \ldots, \gamma_r$ be the unitons in the Segal and Uhlenbeck factorizations of Φ, respectively. Then

- $\beta_1 \subset \beta_2 \subset \cdots \subset \beta_r$;
- β_i *are holomorphic subbundles with respect to the standard holomorphic structure on* $\underline{\mathbb{C}}^n$;
- $\partial\beta_i \subset \beta_{i+1}$;
- $\gamma_i = \beta_{r+1-i}$.

S^1-invariant solutions – 2

The corresponding extended solution and harmonic map $\varphi = \Phi_{-1}$ are given by

$$\Phi = (\pi_{\beta_1} + \lambda\pi_{\beta_1^\perp}) \cdots (\pi_{\beta_r} + \lambda\pi_{\beta_r^\perp}) \tag{15}$$

$$\varphi = \beta_1 + \beta_2^\perp \cap \beta_3 + \cdots + \beta_r^\perp : M^2 \to G_*(\mathbb{C}^n).$$

The canonical twistor lift $\psi = (\psi_0, \psi_1, \ldots, \psi_r) : M^2 \to F_{d_0,\ldots,d_r}$ of φ defined by Φ is given by $\psi_i = \beta_{i+1} \ominus \beta_i$.

This is *superhorizontal* i.e., the derivative of ψ lies in the *superhorizontal distribution*, by which we mean the subbundle of the $(1,0)$-horizontal bundle given by $\sum_{i=0}^{r-1} \mathrm{Hom}(\psi_i, \psi_{i+1})$.

In particular, ψ *is horizontal and both J_1- and J_2-holomorphic*.

Superhorizontal sequences of length 2

Example

With $r = 2$, the extended solution (15) and corresponding $W = \Phi\mathcal{H}_+$ are given by

$$\Phi = \pi_{\beta_1} + \lambda\pi_\varphi + \lambda^2\pi_{\beta_2}^\perp \quad \text{and} \quad W = \beta_1 + \lambda\beta_2 + \lambda^2\mathcal{H}_+ \,. \qquad (16)$$

The resulting harmonic map $\varphi = \Phi_{-1}$ is given by $\varphi = \beta_1 \oplus \beta_2^\perp$, equivalently $\varphi^\perp = \beta_2 \ominus \beta_1$; this is *strongly isotropic* [**Erdem–Wood** (1983)] in the sense that $G^{(i)}(\varphi^\perp)$ and $G^{(j)}(\varphi^\perp)$ are orthogonal for all integers $i \neq j$.

All strongly isotropic harmonic maps $M^2 \to G_*(\mathbb{C}^n)$ are obtained from a ∂'-pair $\beta_1 \subset \beta_2$ in this way [Erdem–Wood], in particular, all harmonic maps $S^2 \to \mathbb{C}P^{n-1}$ [Eells–Wood].

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