Some new results on old constructions of harmonic maps

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Harmonic maps

Let $\varphi : (M, g) \to (N, h)$ be a smooth map between compact smooth Riemannian manifolds. The **energy** or **Dirichlet integral** of φ is

$$E(\varphi) = \int_M e(\varphi)\,\omega_g = \int_M \tfrac{1}{2}|\mathrm{d}\varphi|^2\,\omega_g$$

where ω_g = volume measure and, for any $x \in M$,

$$|\mathrm{d}\varphi_x|^2 = \text{Hilbert–Schmidt norm}^2 \text{ of } \mathrm{d}\varphi_x = g^{ij} h_{\alpha\beta}\,\varphi_i^\alpha \varphi_j^\beta\,.$$

The map φ is called **harmonic** if the first variation of E for variations φ_t of the map φ vanishes at φ, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}E(\varphi_t)\big|_{t=0} = 0$. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\varphi_t)\big|_{t=0} = -\int_M \langle \tau(\varphi), v\rangle\,\omega_g$$

where $v = \partial\varphi_t/\partial t|_{t=0}$ is the **variation vector field** of (φ_t), and $\tau(\varphi) = \operatorname{div}\mathrm{d}\varphi$ is the **tension field of** φ given as follows:

Tension field of a map φ

$$
\begin{aligned}
\tau(\varphi) &= \operatorname{div} \mathrm{d}\varphi = \operatorname{Trace} \nabla \mathrm{d}\varphi = \sum_{i=1}^{m} \nabla \mathrm{d}\varphi(e_i, e_i) \\
&= \sum_{i=1}^{m} \left\{ \nabla^{\varphi}_{e_i} \left(\mathrm{d}\varphi(e_i) \right) - \mathrm{d}\varphi(\nabla^M_{e_i} e_i) \right\};
\end{aligned}
$$

$$
\begin{aligned}
\tau(\varphi)^{\gamma} &= g^{ij} \left(\frac{\partial^2 \varphi^{\gamma}}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial \varphi^{\gamma}}{\partial x^k} + L^{\gamma}_{\alpha\beta} \frac{\partial \varphi^{\alpha}}{\partial x^i} \frac{\partial \varphi^{\beta}}{\partial x^j} \right) \\
&= \Delta^M \varphi^{\gamma} + g(\operatorname{grad} \varphi^{\alpha}, \operatorname{grad} \varphi^{\beta})\, L^{\gamma}_{\alpha\beta}\,.
\end{aligned}
$$

Δ^M = **Laplacian on functions** $f : M \to \mathbb{R}$:

$$
\Delta^M f = \operatorname{div} \operatorname{grad} f = \operatorname{div} \mathrm{d}f = -\mathrm{d}^* \mathrm{d}f = \operatorname{Trace} \nabla \mathrm{d}f
$$

$$
= \sum_{i=1}^{m} \left\{ e_i \left(e_i(f) \right) - \left(\nabla^M_{e_i} e_i \right) f \right\} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|}\, g^{ij} \frac{\partial f}{\partial x^j} \right).
$$

Examples of harmonic maps

$\varphi : M \to N$ is harmonic $\iff$

$$\tau(\varphi) \equiv \mathrm{Trace}\, \nabla \mathrm{d}\varphi = 0 \quad \textit{(Harmonic equation)}$$

1. $\varphi : \mathbb{R}^m \supseteq U \to \mathbb{R}^n$ is harmonic iff $\Delta\varphi = 0$
   ($\Delta =$ usual Laplacian on $\mathbb{R}^m$).

2. $\varphi : (M, g) \to \mathbb{R}^n$ is harmonic iff $\Delta^M \varphi = 0$
   ($\Delta^M =$ Laplace–Beltrami operator on (M, g)).

3. $\varphi : \mathbb{R} \supseteq U \to N$ or $S^1 \to N$ is harmonic iff it defines a **geodesic** parametrized linearly.

4. **Holomorphic maps** between Kähler manifolds are harmonic; in fact they give absolute minima of the energy functional.

5. **Harmonic morphisms**, i.e., maps which preserve Laplace's equation, are harmonic maps.

Weakly conformal maps

A smooth map $\varphi : (M, g) \to (N, h)$ is called **weakly conformal** (with conformality factor λ) if $\varphi^* h = \lambda^2 g$; explicitly, for all $p \in M$,

$$h\big(\mathrm{d}\varphi_p(X), \mathrm{d}\varphi_p(Y)\big) = \lambda(p)^2 \, g(X, Y) \qquad (X, Y \in T_p M) \, ;$$

equivalently,

$$\mathrm{d}\varphi_p^* \circ \mathrm{d}\varphi_p = \lambda(p)^2 \ \mathrm{Id}_{T_p M} \ .$$

In local coordinates, this reads

$$h_{\alpha\beta}\varphi_i^\alpha \varphi_j^\beta = \lambda^2 g_{ij} \, .$$

A. Sanini: *A non-constant map* φ *is extremal with respect to variations* **of the metric** *iff* $\dim M = 2$ *and* φ *is weakly conformal.*
Proof. The Euler–Lagrange operator for such variations is the **stress-energy tensor** $S(\varphi) = e(\varphi) g - \varphi^* h$. If this is zero, taking the trace shows that $\dim M = 2$, then comparing with the above, shows that φ is weakly conformal (with $\lambda^2 = e(\varphi)$).

Conformal invariance in 2 dimensions

Harmonic maps from *surfaces* are **conformally invariant** in the sense that the composition $\varphi \circ \psi$ of a harmonic map $\varphi : M^2 \to N$ from a surface with a weakly conformal map $\psi : M'^2 \to M^2$ of surfaces is harmonic.
In fact, if (x, y) are isothermal coordinates on M^2, the harmonic equation reads

$$\nabla^{\varphi}_{\partial/\partial x}\Big(\frac{\partial \varphi}{\partial x}\Big) + \nabla^{\varphi}_{\partial/\partial y}\Big(\frac{\partial \varphi}{\partial y}\Big) = 0\,.$$

If we write $z = x + \mathrm{i}y$, this reads

$$\nabla^{\varphi}_{\partial/\partial \bar{z}}\Big(\frac{\partial \varphi}{\partial z}\Big) = 0\,, \quad \text{equivalently,} \quad \nabla^{\varphi}_{\partial/\partial z}\Big(\frac{\partial \varphi}{\partial \bar{z}}\Big) = 0\,.$$

Hence the concept of *harmonic map from a Riemann surface* is well defined. Further, $\varphi : M^2 \to N$ is harmonic if and only if $\partial\varphi/\partial z$ is holomorphic with respect to the Koszul–Malgrange holomorphic structure on $\varphi^{-1}TN \to M$.

Harmonic maps and minimal surfaces

Let $\varphi : M^2 \to N$ be a weakly conformal map with conformality factor λ. Then, away from branch points, the mean curvature is $2\,\lambda^2$ times the tension field, hence:

A weakly conformal map is harmonic if and only if it is minimal away from branch points.

Such a map is called a *minimal branched immersion*.

[**W**. and others] *Any harmonic map from the* 2*-sphere is weakly conformal and so is a minimal branched immersion*.
Proof. The $(2,0)$-part of the stress energy tensor

$$S(\varphi)_{2,0} \equiv \left\langle \frac{\partial\varphi}{\partial z}\,,\,\frac{\partial\varphi}{\partial z}\right\rangle$$

is a holomorphic section of $T^*_{1,0}S^2$. Such a section must vanish since this bundle has negative degree.

Harmonic maps from surfaces to $\mathbb{C}P^n$

Holomorphic maps $M^2 \to \mathbb{C}P^n$ are harmonic, as are **antiholomorphic maps**.

When $n = 1$ and $M^2 = S^2$, this accounts for all harmonic maps.

When $n > 1$, we can find harmonic maps which are not $\pm$-holomorphic, as follows:

Given a harmonic non-$\pm$-holomorphic map $\varphi = [\Phi] : M^2 \to \mathbb{C}P^n$ it has two **Gauss transforms**, a ∂'- and a ∂''-transform:

$$G'(\varphi) = \left[\pi^{\perp}_{\varphi} \frac{\partial \Phi}{\partial z}\right], \quad G''(\varphi) = \left[\pi^{\perp}_{\varphi} \frac{\partial \Phi}{\partial \bar{z}}\right].$$

These can be extended over points where the derivatives are zero, and are both harmonic. If φ is holomorphic (resp. antiholomorphic), only $G'(\varphi)$ (resp. $G''(\varphi)$) is defined.

[J. **Eells–W**., 1983] *All harmonic maps from $S^2 \to \mathbb{C}P^n$ are obtained from holomorphic maps by applying the ∂'-Gauss transform up to n times.*

An example

Let $f(z) = [F(z)] = [1, z^3, z^2]$. This defines a (smooth) holomorphic map from S^2 to $\mathbb{C}P^2$ (note that $f(\infty) = [0,1,0]$).

We calculate its Gauss transform $G'(f)$ at each z as the orthogonal complement of $f(z)$ in the **first osculating space** $f_{(1)}(z)$ spanned by $F(z)$ and $F'(z)$.

Now $F'(z) = (0, 3z^2, 2z)$ so that $(F \wedge F')(z) = (-z^4, -2z, 3z^2)$; note that this is zero at $z = 0$ because f has a branch point there. However, it equals $z\,\psi(z)$ where $\psi(z) = (1, z^3, z^2) \wedge (0, 3z, 2)$ is non-zero for all z. So $f_{(1)}(z)$ is the plane spanned by $(1, z^3, z^2)$ and $(0, 3z, 2)$ and $G'(f)(z)$ is the orthogonal complement of the line $f(z)$ in that plane. **Hence $G'(f)$ is smooth at the branch point** $z = 0$. Calculating at $z = \infty$ by replacing z by $w = 1/z$, we see that $G'(f)$ is smooth there too.

We thus obtain a harmonic map $G'(f) : S^2 \to \mathbb{C}P^2$ which is not holomorphic or antiholomorphic.

Skip to Explicit talk

Smoothness of the Gauss transform

For any harmonic map $\varphi : M^2 \to \mathbb{C}P^n$, its Gauss transform $G^{(1)}(\varphi)$ is a smooth map, but **is the map** $G^{(1)} : \varphi \mapsto G^{(1)}(\varphi)$ **smooth, or even continuous?** In general, no:
Example Let $f_t : S^2 \to \mathbb{C}P^2$ be $f_t(z) = [F_t(z)]$, where

$$F_t(z) = (1, tz + z^3, z^2) \quad (z \in \mathbb{C}, t \in \mathbf{R})$$

(so that $f_t(\infty) = [0, 1, 0]$). Note that f_t is a smooth family of full holomorphic maps. Then $G'(f_t)(0) = [0, 1, 0]\ \forall t \neq 0$ but $G'(f_0)(0) = [0, 0, 1]$.

Details We compute $F_t'(z) = (0, t + 3z^2, 2z)$ and so

$$F_t \wedge F_t'(z) = (tz^2 - z^4, -2z, t + 3z^2)\,.$$

If $t \neq 0$, at $z = 0$ this equals $(0, 0, t) = t(1, 0, 0) \wedge (0, 1, 0)$.

Details of the last example, continued

It follows that the first associated curve has the value

$$f_{t(1)}(0) = \text{span}\{(1,0,0),(0,1,0)\}\text{and} \quad G'(f_t)(0) = [0,1,0]\,.$$

However, if $t = 0$, then

$$F_t \wedge F_t'(z) = (-z^4, -2z, 3z^2) = z\psi(z)$$

where $\psi(z) = (-z^3, -2, 3z)$. In particular
$\psi(0) = (0,-2,0) = 2(1,0,0) \wedge (0,0,1)$ so that

$$f_{0(1)}(0) = \text{span}\{(1,0,0),(0,0,1)\}\text{and } G'(f_0)(0) = [0,0,1]\,.$$

This shows that $f_{t(1)}$ *and* $G'(f_t)$ *do not vary continuously with* t. The reason for this is that f_t is unramified when $t \neq 0$ but ramified with ramification index 1 at $z = 0$ when $t = 0$, and that in the presence of ramification, $f_{t(1)}$ involves division of polynomials by their common factor, a discontinuous process when the degree of the factor changes.

Recovering smoothness

• $\mathsf{Hol}_k(S^2, \mathbb{C}P^2)$ = (connected) component of the space of holomorphic maps of degree k;
• $\mathsf{Harm}_{d,E}(S^2, \mathbb{C}P^2)$ = the component of the space of harmonic maps of degree d and energy $4\pi E$.

[**Lemaire–W**., 1996] *The Gauss transform* $\mathsf{Hol}_k(S^2, \mathbb{C}P^2) \to \mathsf{Harm}(S^2, \mathbb{C}P^2)$ *restricts to a diffeomorphism*:

$$\mathsf{Hol}^{\mathrm{full}}_{k,r}(S^2, \mathbb{C}P^2) \to \mathsf{Harm}^{\mathrm{full}}_{d,E}(S^2, \mathbb{C}P^2)\,.$$

where r = ramification index, $d = k - r - 2$, $E = 3k - r - 2$.

Example Let $f_t : S^2 \to \mathbb{C}P^2$ be defined by

$$F_t(z) = \big(z^4 + 1, (1 - 3t^2)z^3 + (-3t + t^3)z, 2tz^2 + (1 - t^2)\big)$$

$(z \in \mathbb{C}, t \in \mathbb{R})$ (so that $f_t(\infty) = [1, 0, 0]$). When $t \neq 0$, f_t is ramified at $z = \pm\sqrt{t}$ with index 1; when $t = 0$, these ramification points coalesce into a ramification point at $z = 0$ of index 2.

Then $G'(f_t)$ **is a smooth family of harmonic maps**.

Details of the smooth example

Identifying S^2 with $\mathbb{C} \cup \{0\}$ by stereographic projection, let $f_t : S^2 \to \mathbb{C}P^2$ be defined by

$$F_t(z) = \big(z^4 + 1, (1 - 3t^2)z^3 + (-3t + t^3)z, 2tz^2 + (1 - t^2)\big)$$

$(z \in \mathbb{C}, t \in \mathbb{R})$ (so that $f_t(\infty) = [1, 0, 0]$).
Identifying $\Lambda^2\mathbb{C}^3$ with $\mathbb{C}^3$ we have $(F_t \wedge F_t')(z) = (z^2 - t)\psi(z)$ where

$$\psi(z) = \big((-2t + 6t^3)z^2 + (-3 + t^2)(1 - t^2), \\ 4z(tz^2 + 1), (-1 + 3t^2)z^4 + 8tz^2 + 3 - t^2\ \big)$$

which shows that, if $t \neq 0$, f_t is ramified at $z = \pm\sqrt{t}$ with index 1, but if $t = 0$, these ramification points coalesce into a ramification point at $z = 0$ of index 2. Further $f_{t(1)}(z) = [\psi(z)]$. We see from this that $f_t \in \mathrm{Hol}^{\mathrm{full}}_{4,2}(\mathbb{C}P^2)$ for all t and that $f_{t(1)}$ **and so** $G'(f_t)$ **vary smoothly with** t**, even though each root does not**.

Second Variation

Let $\varphi_{t,s}$ be a 2-parameter variation of φ;

write $v = \frac{\partial \varphi_{t,s}}{\partial t}\Big|_{(0,0)}$ and $w = \frac{\partial \varphi_{t,s}}{\partial s}\Big|_{(0,0)}$. The **Hessian** of φ is:

$$H_\varphi(v,w) = \frac{\partial^2 E}{\partial t \partial s}(\varphi_{t,s})\Big|_{(0,0)} = \int_M \langle J_\varphi v, w\rangle \, \omega_g$$

where

$$J_\varphi v = \Delta^\varphi v - \mathrm{Trace}\, R^N(\mathrm{d}\varphi, v)\mathrm{d}\varphi$$

is called the *Jacobi operator* along φ, and
$\Delta^\varphi v = -\,\mathrm{Trace}\, \nabla^2$ is the Laplacian on $\varphi^{-1}TN$.

Observation If φ_t is a one-parameter family of maps with $\varphi_0 = \varphi$ and $\frac{\partial \varphi_t}{\partial t}\big|_{t=0} = v$, then

$$J_\varphi(v) = -\frac{\partial}{\partial t}\tau(\varphi_t)\Big|_{t=0}.$$

Corollary If φ_t is a one-parameter family of *harmonic* maps with $\varphi_0 = \varphi$, then $v = \frac{\partial \varphi_t}{\partial t}\big|_{t=0}$ is a *Jacobi field along* φ, i.e., $J_\varphi(v) = 0$.

Some examples of Jacobi fields

1. **For constant maps or maps to a Euclidean space** v is Jacobi if and only if it is harmonic: $\Delta^M v = 0$, and so constant.

2. **Killing fields** give Jacobi fields.

3. **Parallel vector fields**. A vector field U on (M, g) is called *parallel* if $\nabla^M U = 0$. Parallel vector fields are Killing.

4. **Geodesics**. Coincides with usual notion of Jacobi field.

5. **Holomorphic vector fields** A vector field along a holomorphic map between Kähler manifolds is Jacobi if and only if it is holomorphic.

6. **Conformal vector fields** along conformal diffeomorphisms of surfaces are Jacobi fields.
In higher dimensions those that are not Killing fields give $H_{\mathrm{Id}}(v, v) < 0$.

Skip to Integrability of Jacobi fields

Some examples of Jacobi fields

7. **For the identity map** Jacobi operator is

$$J_{\mathrm{Id}} = \Delta^H - 2\ \mathrm{Ric}^M = 0. \tag{1}$$

[K. Yano and T. Nagano] who called Jacobi fields *geodesic vector fields*.

8. **When** (M, g) **is Einstein** with $\mathrm{Ric}^M = c\ \mathrm{Id}$ (c constant), formula (1) reads

$$J_{\mathrm{Id}} = \Delta^H - 2c\ \mathrm{Id}\ .$$

[R.T. Smith] *The nullity of identity map = the multiplicity of* $2c$ *as an eigenvalue of* Δ^M *plus the dimension of the space of Killing fields*.

9. **Jacobi fields on spheres**. $\mathrm{nullity}(\mathrm{Id}_{S^m}) = m(m+1)/2$ if $m \neq 2$, and 6 if $m = 2$.

10. **Harmonic variations [G. Toth]**. $\varphi_t = \exp(tv)$ is harmonic for all $t \in \mathbb{R} \implies v$ is Jacobi.

Some general results on Jacobi fields

An interpretation [M.-J. Ferreira]

A vector field v along a smooth map $\varphi : M \to N$ is harmonic as a map $M \to (TN, \text{complete lift metric})$ if and only if φ is harmonic and v is a Jacobi field along it.

The negative curvature case

Suppose that $Riem(N) \leq 0$. Then any Jacobi field v along a harmonic map $M \to N$ is parallel along the map.

For the identity map $N \to N$, v is Jacobi iff it is Killing.

If, further, $\mathrm{Ric}(N) < 0$ at some point, any Jacobi field along the identity map is zero.

Integrability of Jacobi fields

Definition A Jacobi field v along a harmonic map φ is said to be integrable if there is a one-parameter family (φ_t) of harmonic maps with $\varphi_0 = \varphi$ and $\left.\dfrac{\partial \varphi_t}{\partial t}\right|_{t=0} = v$.

Big Question: When are all Jacobi fields integrable?

Why is this important?

1. [**D. Adams and L. Simon**] Let (M, g) and (N, h) be real-analytic Riemannian manifolds. If all Jacobi fields along harmonic maps from M to N are integrable, then $\mathrm{Harm}(M, N)$ is a real-analytic manifold **with tangent space the space of Jacobi fields**.

2. [**Adams–Simon, R. Gulliver and B. White**] If all Jacobi fields are integrable, we get information on the *singular set* of a harmonic map and rate of *blowing up* near the singular set.

Cases where all Jacobi fields are integrable

1. Jacobi fields along **constant maps** and **maps into Euclidean spaces**;

2. [**W. Ziller, 1977**] $M = S^1$ and $N =$ a globally symmetric space;

3. [**Besse, 1978**] $M = S^1$ and $N =$ a manifold all of whose geodesics are closed (i.e., periodic) and of the same length;

4. [**T. Sunada, 1979**] M arbitrary, $N =$ a locally symmetric space with $\mathrm{Riem}\,(N) \leq 0$;

5. [**Gulliver and White, 1989**] $M = N = S^2 = \mathbb{C}P^1$;

6. [**Lemaire and W., 2002**] $M = S^2$ and $N = \mathbb{C}P^2$;

Cases where *not* all Jacobi fields are integrable

1. [**Algrem and others, 1970s**] $M = S^1$ and $N =$ a surface of revolution with $R = 0$ along a geodesic of revolution and $R < 0$ elsewhere;

2. [**Mukai, 1997**] $M = S^1 \times S^1$ and $N = S^3$;

3. [**Lemaire and W., 2008**] $M = S^2$, $N = S^n$ $(n = 3, 4)$;

4. [**In progress**] $M = S^2$, $N = \mathbb{C}P^n$ $(n \geq 3)$, $N = S^n$ $(n \geq 5)$.

Skip to Jacobi fields for $S^2 \to S^4$

The space of harmonic maps for negative curvature

1. **Riem** $N < 0$ [**Hartman, 1967**]

$\mathrm{Harm}_f(M,N) = \{\text{point}\}$, unless f is constant or has image a circle. *In this case, all Jacobi fields are zero.*

2. **Riem** $N \leq 0$ [**Schoen–Yau, 1979**]

If M, N are C^ω, then $\mathrm{Harm}_f(M,N)$ is parametrized by a compact manifold N_0. Precisely, there is a map
$F : M \times N_0 \to N$ such that

$$\mathrm{Harm}_f(M,N) = \{F|_{M\times\{t\}} : t \in N_0\},$$

and, for any $p \in M$, $F|_{\{p\}\times N_0}$ is an isometric immersion onto a totally geodesic submanifold.
However, in this case, it is not necessarily true that all Jacobi fields are integrable.

Harmonic maps from S^2 to S^4

1. [**Calabi, 1967**] Each harmonic map $\varphi : S^2 \to S^4$ is $+$ or $-$ the projection of a horizontal holomorphic maps $f : S^2 \to \mathbb{C}P^3$ into the twistor space $\mathbb{C}P^3$ of S^4; the $\pm$-sign is called the *spin* of φ.

If f has degree d, then φ has energy $4\pi d$. The components of $\mathrm{Harm}(S^2, S^4)$ are thus parametrized by $d \in \{1, 2, \dots\}$.

2. (**Full and non-full maps**) *Either* φ is a $\pm$-holomorphic map $S^2 \to S^2$ followed by a totally geodesic inclusion of S^2 in S^4 *or* φ is (linearly) full.

Harmonic maps from S^2 to S^4, contd

3. [**Verdier, 1985**] The harmonic maps of energy $4\pi d$ form a connected complex algebraic variety $\mathrm{Harm}_d(S^2, S^4)$ of pure dimension $2d + 4$. This has three irreducible components if $d \geq 3$.

a. One component is the subset $\mathrm{Harm}_d^{\mathrm{NF}}(S^2, S^4)$ of **non-full** harmonic maps of energy $4\pi d$; this is a (non-empty) closed complex manifold of dimension $2d + 4$.

b. The subset of **full** harmonic maps $\mathrm{Harm}_d^{\mathrm{LF}}(S^2, S^4)$ is empty for $d = 1, 2$, and is a non-empty with two connected components $\mathrm{Harm}_d^{+}(S^2, S^4)$ and $\mathrm{Harm}_d^{-}(S^2, S^4)$ for $d \geq 3$; these components are smooth manifolds when $d = 3, 4, 5$ [**Bolton and Woodward, 2004**].

For any $d \geq 3$, their closures are the other two irreducible components of $\mathrm{Harm}_d(S^2, S^4)$.

Jacobi fields along harmonic maps from S^2 to S^4

Theorem [Lemaire and W., 2006]. (i) There is a one-to-one correspondence between the Jacobi fields along a full harmonic map $\varphi : S^2 \to S^4$ (of positive spin) and the infinitesimal horizontal holomorphic deformations of its twistor lift $f : S^2 \to \mathbb{C}P^3$.

(ii) Let $C_d =$ set of limit points of $\mathrm{Harm}_d^{\mathrm{LF}}(S^2, S^4)$ in $\mathrm{Harm}_d^{\mathrm{NF}}(S^2, S^4)$ ('collapse points'). C_d is non-empty for $d \geq 3$ [**Ejiri–Kotani**, **Guest–Ohnita**, **Montiel–Ros**].

(iii) For $\varphi \in \mathrm{Harm}_d^{\mathrm{NF}}(S^2, S^4)$, all Jacobi fields along φ are integrable if and only if $\varphi \notin C_d$.

(iv) Let $\varphi : S^2 \to S^2 \subset S^4$ be in C_d. Consider φ as in $\mathrm{Harm}_d(S^2, S^3)$. **Then there are non-integrable Jacobi fields along it, although $\mathrm{Harm}_d(S^2, S^3)$ is a manifold.**

Example of a collapse

For $t \in \mathbb{R}$, define $f_t : S^2 \to \mathbb{C}P^3$ by

$$f_t = [z, -3tz^2, z^3 + 1, t] \,.$$

For $t \neq 0$ this is a full horizontal holomorphic map of degree 3.

As $t \to 0$, it approaches the non-full horizontal holomorphic map $f_0 = [z, 0, z^3 + 1, 0]$.

Setting $\varphi_t = \pi \circ f_t$ gives a family of full harmonic maps $\varphi_t : S^2 \to S^4$ which 'collapse' to a non-full harmonic map $\varphi_0 : S^2 \to S^2$, $z \mapsto (z^3 + 1)/z$.

Thus $\varphi_0 \in C_3$, showing it is non-empty.

Skip to Maps to the unitary group

$\mathrm{Harm}(S^2, S^n)$

[**Calabi 1967**] $\mathrm{Harm}_d^{\mathrm{LF}}(S^2, S^{2m})$ is non-empty if and only if $d \geq \frac{1}{2}m(m+1)$.

[**Barbosa 1975**] For $d = \frac{1}{2}m(m+1)$, $\mathrm{Harm}_d^{\mathrm{LF}}(S^2, S^{2m}) = \mathrm{O}(2m+1, \mathbb{C})$.

[**Kotani 1992**] $\mathrm{Harm}_d(S^2, S^n)$ is path connected.

[**Furuta–Guest–Kotani–Ohnita 1992**] calculate $\pi_1 \mathrm{Harm}_d(S^2, S^n)$.

[**Bolton and Woodward 2001**] For $d = \frac{1}{2}m(m+1) + 1$, $\mathrm{Harm}_d^{\mathrm{LF}}(S^2, S^{2m}) = \mathbb{C}P^2 \setminus Q \times \mathrm{O}(2m+1, \mathbb{C})$.

[**Bolton and Woodward, conjecture, proved by L. Fernandez, 2003, 2005**] $\dim_{\mathbb{C}} \mathrm{Harm}_d(S^2, S^{2m}) = 2d + m^2$.

$\mathrm{Harm}(S^2, \mathbb{C}P^n)$

[**Guest–Ohnita 1993 + Crawford 1997**]
The connected components of $\mathrm{Harm}(S^2, \mathbb{C}P^n)$ are $\mathrm{Harm}_{k,E}(S^2, \mathbb{C}P^n)$.

[**Segal, Crawford 1993**] The inclusions

$$\mathrm{Hol}_k^{\mathrm{LF}}(S^2, \mathbb{C}P^n) \hookrightarrow \mathrm{Hol}_k(S^2, \mathbb{C}P^n) \hookrightarrow C^0(S^2, \mathbb{C}P^n)$$

are homotopy equivalences, the first to dimension $2(k-n)$, the second to dimension $k(2n-1)$.

[**Crawford 1993**] determines the homology of $\mathrm{Hol}_k^{\mathrm{LF}}(S^2, \mathbb{C}P^2)$.

Harmonic maps from unitons

The Gauss transform is an example of Uhlenbeck's operation of **adding a uniton** (also called **flag transform**) which transforms harmonic maps $M^2 \to \mathrm{U}(n)$ into others, as follows:
Any smooth map φ defines a connection $A^\varphi = \frac{1}{2}\varphi^{-1}\mathrm{d}\varphi$, and thus a covariant derivative $D^\varphi = \mathrm{d} + A^\varphi$, on the trivial bundle $\underline{\mathbb{C}}^n = M^2 \times \mathbb{C}^n$; writing $A^\varphi = A^\varphi_z \mathrm{d}z + A^\varphi_{\bar{z}} \mathrm{d}\bar{z}$, since A^φ_z represents $\partial\varphi/\partial z$, Harmonicity of φ is equivalent to holomorphicity of A^φ_z.

A **uniton** for φ is a subbundle $\underline{\alpha}$ of the trivial bundle which is **holomorphic**, i.e, closed under $D^\varphi_{\bar{z}}$, and **closed under** A^φ_z.
Then [K. **Uhlenbeck**, 1989]:

(i) If $\varphi : M^2 \to \mathrm{U}(n)$ is harmonic and $\underline{\alpha}$ is a uniton for φ, then $\widetilde{\varphi} = \varphi(\pi_\alpha - \pi_\alpha^\perp)$ is harmonic.

(ii) Any harmonic map $\varphi : S^2 \to \mathrm{U}(n)$ can written as a finite product of unitons:

$$\varphi = \text{const.}\ (\pi_{\alpha_1} - \pi_{\alpha_1}^\perp) \cdots (\pi_{\alpha_r} - \pi_{\alpha_r}^\perp)\,. \qquad (2)$$

◂ Return to Unique Factorization Theorem ▸▸ Skip to General Construction

Building all harmonic maps

To build all harmonic maps we add unitons $\underline{\alpha}_1, \underline{\alpha}_2, \ldots$ starting with the constant map, giving a sequence of harmonic maps
$\varphi_0 = \text{const.}, \ \varphi_1 = \varphi_0(\pi_{\alpha_1} - \pi_{\alpha_1}^{\perp})$,
$\varphi_2 = \varphi_1(\pi_{\alpha_2} - \pi_{\alpha_2}^{\perp}) = \varphi_0(\pi_{\alpha_1} - \pi_{\alpha_1}^{\perp})(\pi_{\alpha_2} - \pi_{\alpha_2}^{\perp}), \ldots$.
To do this, we must know the possible unitons at each stage.
For this, we must find a meromorphic basis for the trivial bundle $\underline{\mathbb{C}}^n$ with respect to $D_{\bar{z}}^{\varphi_i}$ for each i; in general, this involves solving $\overline{\partial}$-*problems*.

[M.J. **Ferreira**, B. **Simões**, **W**., preprint, 2008]
Every harmonic map can be built by adding unitons which are given in terms of projections along previous unitons of meromorphic functions, without solving $\overline{\partial}$-problems.

This gives a completely explicit algebraic parametrization of all harmonic maps $S^2 \to \mathrm{U}(n)$ by meromorphic functions.

The general construction

Theorem

All non-constant harmonic maps from $S^2 \to \mathrm{U}(n)$ are given as a product (2) *in the following way for some $r \in \{0, 1, \ldots, n\}$:*

Choose an $r \times n$ array of meromorphic functions $H_{i,j} : S^2 \to \mathbb{C}^n$, ($i = 0, \ldots, r-1$, $j = 1, \ldots, n$); these could be zero.

For $i = 0, 1, \ldots, r-1$, let $\underline{\alpha}_{i+1}$ be the sum of subbundles $\underline{\alpha}_{i+1}^{(k)}$ ($k = 0, \ldots, i$) with $\underline{\alpha}_{i+1}^{(k)}$ spanned by

$$\sum_{s=k}^{i} C_s^i H_{s-k,j}^{(k)} \quad (j = 1, \ldots, n)\,.$$

Here $C_s^i = \sum_{1 \leq i_1 < \cdots < i_s \leq i} \pi_{i_s}^{\perp} \cdots \pi_{i_1}^{\perp}$, and the superscript $^{(k)}$ denotes the k'th derivative.

The first few formulae

(i) $\underline{\alpha}_1$ is the subbundle $\underline{\alpha}_1 = \underline{\alpha}_1^{(0)} = \text{span}\{H_{0,j}\}$.

(ii) $\underline{\alpha}_2$ is the sum of subbundles $\underline{\alpha}_2 = \underline{\alpha}_2^{(0)} + \underline{\alpha}_2^{(1)}$ where

$$\underline{\alpha}_2^{(0)} = \text{span}\{H_{0,j} + \pi_{\alpha_1}^{\perp} H_{1,j}\},$$

$$\underline{\alpha}_2^{(1)} = \text{span}\{\pi_{\alpha_1}^{\perp} H_{0,j}^{(1)}\}.$$

(iii) $\underline{\alpha}_3$ is the sum $\underline{\alpha}_3 = \underline{\alpha}_3^{(0)} + \underline{\alpha}_3^{(1)} + \underline{\alpha}_3^{(2)}$ where

$$\underline{\alpha}_3^{(0)} = \text{span}\{H_{0,j} + (\pi_{\alpha_1}^{\perp} + \pi_{\alpha_2}^{\perp}) H_{1,j} + \pi_{\alpha_2}^{\perp}\pi_{\alpha_1}^{\perp} H_{2,j}\},$$

$$\underline{\alpha}_3^{(1)} = \text{span}\{(\pi_{\alpha_1}^{\perp} + \pi_{\alpha_2}^{\perp}) H_{0,j}^{(1)} + \pi_{\alpha_2}^{\perp}\pi_{\alpha_1}^{\perp} H_{1,j}^{(1)}\},$$

$$\underline{\alpha}_3^{(2)} = \text{span}\{\pi_{\alpha_2}^{\perp}\pi_{\alpha_1}^{\perp} H_{0,j}^{(2)}\}.$$

◂ Return to Unitons to meromorphic data ▸▸ Skip to Grassmannian model

Examples

A smooth (resp. holomorphic) map $\alpha : M^2 \to G_k(\mathbb{C}^n)$ is the same thing as a smooth (resp. holomorphic) subbundle $\underline{\alpha}$ of $\underline{\mathbb{C}}^n$. The Grassmannians $G_k(\mathbb{C}^n)$ are included in $\mathrm{U}(n)$ via the *Cartan embbedding* $B \mapsto \pi_B - \pi_B^\perp$. This is a totally geodesic isometry which identifies the disjoint union $G_*(\mathbb{C}^n)$ of these Grassmannians with the subset $\{B \in \mathrm{U}(n) : B = B^*\} = \{B \in \mathrm{U}(n) : B^2 = I\}$.

A harmonic map into $G_*(\mathbb{C}^n)$ is thus the same as a harmonic map into $\mathrm{U}(n)$ with image in this subset.

1. If the data $(H_{i,j})$ has only one non-zero entry $H_{0,1}$, then $\underline{\alpha}_1 = \mathrm{span}\{H_{0,1}\}$ and $\underline{\alpha}_2 = \mathrm{span}\{H_{0,1}\} + \mathrm{span}\{\pi_{\alpha_1}^\perp(H_{0,1}^{(1)})\}$. The formula gives $\varphi_2 =$ the ∂'-*Gauss transform* $G'(\underline{\alpha}_1)$ of $\underline{\alpha}_1$.

2. More generally, if each column of $(H_{i,j})$ has only one non-zero entry, then the unitons are *nested*: $\underline{\alpha}_i \subset \underline{\alpha}_{i+1}$, and φ is an **isotropic map** into a Grassmannian.

Unique factorization–1

The data gives a sequence of unitons satisfying the following **covering condition** due to G. Segal: $\pi_i(\underline{\alpha}_{i+1}) = \underline{\alpha}_i$ for all i.

Theorem

Given a harmonic map $\varphi : M^2 \to U(n)$ there is a unique factorization (2) *into proper unitons satisfying the covering condition with $\underline{\alpha}_1$ full.*

Proof. We recall that any harmonic map φ gives rise to a loop of smooth maps Φ_λ called an **extended solution for** φ as follows. With $A^\varphi = \frac{1}{2}\varphi^{-1}\mathrm{d}\varphi$, write $A^\varphi = A^\varphi_z \mathrm{d}z + A^\varphi_{\bar{z}} \mathrm{d}\bar{z}$ and set $A^\varphi_\lambda = \frac{1}{2}(1-\lambda^{-1})A^\varphi_z \mathrm{d}z + \frac{1}{2}(1-\lambda)A^\varphi_{\bar{z}}\mathrm{d}\bar{z}$. Then φ harmonic implies $\mathrm{d}A^\varphi_\lambda + [A^\varphi_\lambda, A^\varphi_\lambda] = 0$, the integrability condition for solving

$$\tfrac{1}{2}\Phi_\lambda^{-1}\mathrm{d}\Phi_\lambda = A^\varphi_\lambda\,.$$

We let Φ_λ be any smooth solution. Note that Φ_1 is constant, we take it to be I, and Φ_{-1} equals φ up to left equivalence, i.e., up to left multiplication by an element of $\mathrm{U}(n)$.

Unique factorization–2

A harmonic map $\varphi : M^2 \to \mathrm{U}(n)$ is said to be of **finite uniton number** if if can be written as the product of unitons:

$$\varphi = \varphi_0(\pi_{\alpha_1} - \pi_{\alpha_1}^{\perp}) \cdots (\pi_{\alpha_r} - \pi_{\alpha_r}^{\perp}).$$

The minimal number of unitons required is called the **uniton number** of φ. An extended solution for φ is:

$$\Phi = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^{\perp}) \cdots (\pi_{\alpha_r} + \lambda\pi_{\alpha_r}^{\perp}).$$

A harmonic map φ is of finite uniton number if and only if it has a *polynomial extended solution*:

$$\Phi = T_0^{\Phi} + \lambda T_1^{\Phi} + \cdots + \lambda^r T_r^{\Phi}.$$

This is unique if we insist that the $\underline{\mathrm{Im}}\, T_0^{\Phi}$ be a **full** subbundle. The degree of Φ is then the uniton number of φ.

Unique factorization–3

Given a harmonic map φ of uniton number r, we find a uniton factorization into proper covering unitons inductively as follows.

(i) Set $\Phi_r = \Phi$, the unique extended solution of φ with $\underline{\underline{\mathrm{Im}}}\, T_0^{\Phi}$ full.

(ii) Suppose we have defined an extended solution Φ_i of degree i (for some harmonic map φ_i) with non-zero constant term, set $\underline{\alpha}_i^{\perp} = \underline{\underline{\mathrm{Im}}}\,(T_i^{\Phi_i})^*$, and $\Phi_{i-1} = \Phi_i(\pi_{\alpha_i}^{\perp} + \lambda\pi_{\alpha_i})$. This is a polynomial of degree $i-1$ with non-zero constant term.

(iii) The process terminates after exactly r steps with $\Phi_0 = I$ giving a factorization into proper unitons satisfying the covering condition. The fullness of $\underline{\underline{\mathrm{Im}}}\, T_0^{\Phi}$ translates into fullness of $\underline{\alpha}_1$.

(iv) The Φ_i are extended solutions for the product of the first i unitons: $\Phi = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^{\perp})\cdots(\pi_{\alpha_i} + \lambda\pi_{\alpha_i}^{\perp})$.

(v) Clearly $\underline{\underline{\mathrm{Im}}}\,(T_i^{\Phi_i})^* = \underline{\underline{\mathrm{Im}}}(\pi_{\alpha_i}^{\perp} \circ \cdots \circ \pi_{\alpha_1}^{\perp})$. The covering condition implies that this is equal to $\underline{\alpha}_i^{\perp}$, establishing uniqueness.

Unitons to meromorphic data–1

Theorem

Let $\underline{\alpha}_1, \underline{\alpha}_2, \ldots$ be a sequence of unitons satisfying the covering condition. Then they are given by the formulae above *for some meromorphic data $H_{i,j}$.*

Proof.

1. We start with a constant map φ_0, so that the first uniton $\underline{\alpha}_1$ is a holomorphic subbundle of $\underline{\mathbb{C}}^n$, and so is given by the span of some meromorphic functions $K_0 = H_0$.
This gives us a harmonic map $\varphi_1 = \varphi_0(\pi_{\alpha_1} - \pi_{\alpha_1}^{\perp})$.

2. The meromorphic sections K_0 of $\underline{\alpha}_1$ remain meromorphic for $(\underline{\mathbb{C}}^n, D_{\bar{z}}^{\varphi_1})$ — a special property of this first step.

Meromorphic sections of $\underline{\alpha}_1^{\perp}$ are all of the form $\pi_{\alpha_1}^{\perp} H_1$ where H_1 is a meromorphic vector. We conclude that the most general meromorphic section of $(\underline{\mathbb{C}}^n, D_{\bar{z}}^{\varphi_1})$ is of the form $K_1 = H_0 + \pi_{\alpha_1}^{\perp} H_1$ so that $\underline{\alpha}_2$ must be spanned by sections of this form.

Unitons to meromorphic data–2

3. We set $\varphi_2 = \varphi_1(\pi_{\alpha_2} - \pi_{\alpha_2}^{\perp})$.

The sections of the form K_1 give a basis for $\underline{\alpha}_2$, and are meromorphic with respect to $D_{\bar{z}}^{\varphi_1}$ but they are not, in general, meromorphic with respect to $D_{\bar{z}}^{\varphi_2}$. However sections of the form

$$H_0 + (\pi_{\alpha_1}^{\perp} + \pi_{\alpha_2}^{\perp})H_1$$

are meromorphic with respect to $D_{\bar{z}}^{\varphi_i}$ and project under π_{α_2} to K_1.

To complete a basis we prove that for any meromorphic H_2, $\pi_{\alpha_2}^{\perp}\pi_{\alpha_1}^{\perp}(H_2)$ is a meromorphic section of $\underline{\alpha}_2^{\perp}$; by the covering condition, this gives **all** such, so all meromorphic sections of $(\underline{\mathbb{C}}^n, D_{\bar{z}}^{\varphi_1})$ are of the form

$$H_0 + (\pi_{\alpha_1}^{\perp} + \pi_{\alpha_2}^{\perp})H_1 + \pi_{\alpha_2}^{\perp}\pi_{\alpha_1}^{\perp}(H_2).$$

Hence $\underline{\alpha}_3$ must be spanned by sections of this type.

4. We set $\varphi_3 = \varphi_2(\pi_{\alpha_3} - \pi_{\alpha_3}^{\perp})$, ...

□

Grassmannian model interpretation–1

Let $\Omega^{\mathrm{alg}}\mathrm{U}(n)$ denote the **algebraic loop group** consisting of all maps $\gamma : S^1 \to \mathrm{U}(n)$ with $\gamma(1) = I$ of the form

$$\gamma(\lambda) = \sum_{i=s}^{t} T_i \lambda^i$$

for some integers s, t and some $T_i \in \mathbf{gl}(n, \underline{\mathbb{C}})$.

Let $\mathcal{H}$ denote the Hilbert space $L^2(S^1, \mathbb{C}^n)$ and let $\mathcal{H}_+$ denote the closed subspace of elements of the form $\sum_{k \geq 0} \lambda^k a_k$ where $a_k \in \mathbb{C}^n$.

The loop group $\Omega^{\mathrm{alg}}\mathrm{U}(n)$ acts on $\mathcal{H}$ and the map $\gamma \mapsto \gamma(H_+)$ identifies $\Omega^{\mathrm{alg}}\mathrm{U}(n)$ with the *algebraic Grassmannian*, consisting of all subspaces W of $\mathcal{H}$ such that $\lambda W \subset W$ and $\lambda^s \mathcal{H}_+ \subset W \subset \lambda^t \mathcal{H}_+$ for some s, t.

Grassmannian model interpretation–2

A polynomial extended solution can be thought of as a smooth map $\Phi : M^2 \to \Omega^{\mathrm{alg}}\mathrm{U}(n)$.
Setting $W = \Phi(\mathcal{H}_+)$ defines a map W from M^2 to the algebraic Grassmannian.
If Φ has degree r, the subspace W satisfies $\lambda^r\mathcal{H}_+ \subset W \subset \mathcal{H}_+$, and so can be considered to be in the quotient space $\mathcal{H}_+/\lambda^r\mathcal{H}_+$.
We thus obtain a map $W : M^2 \to G_*(\mathcal{H}_+/\lambda^r\mathcal{H}_+)$ into the Grassmannian of the finite-dimensional vector space $\mathcal{H}_+/\lambda^r\mathcal{H}_+$.
Harmonic maps $\varphi : M^2 \to \mathrm{U}(n)$ of uniton number at most r correspond to holomorphic maps $W : M^2 \to \mathcal{H}_+/\lambda^r\mathcal{H}_+ \cong \mathbb{C}^{rn}$, or equivalently, holomorphic subbundles $\underline{W}$ of the trivial bundle $M^2 \times \mathcal{H}_+/\lambda^r\mathcal{H}_+ \cong \underline{\mathbb{C}}^{rn}$ satisfying

$$\lambda \underline{W}_{(1)} \subset \underline{W}\,. \tag{3}$$

Here, $\underline{W}_{(1)}$ denotes the subbundle spanned by (local) sections of $\underline{W}$ and their first derivatives (with respect to any complex coordinate on M^2).

Grassmannian model interpretation–3

Holomorphic subbundles $\underline{W}$ of $M^2 \times \mathcal{H}_+/\lambda^r\mathcal{H}_+$ satisfying (3) are given [M. Guest] by taking an arbitrary holomorphic map $X : M^2 \to G_*(\mathcal{H}_+/\lambda^r\mathcal{H}_+) \cong G_*(\mathbb{C}^{rn})$, equivalently, holomorphic subbundle $\underline{X}$ of $\underline{\mathbb{C}}^{rn}$, and setting $\underline{W}$ equal to the coset

$$\underline{W} = \underline{X} + \lambda \underline{X}_{(1)} + \lambda^2 \underline{X}_{(2)} + \cdots + \lambda^{r-1}\underline{X}_{(r-1)} + \lambda^r \mathcal{H}_+ \,. \tag{4}$$

Theorem

Let φ be given by an array $(H_{i,j})$. Set

$$L_{i,j} = \textstyle\sum_{k=0}^{i} \binom{i}{k} H_{k,j} \tag{5}$$

and let $\underline{X}$ be the subbundle spanned by $\sum_{i=0}^{r-1} \lambda^i L_{i,j}$. Then $\underline{W}$ is the Grassmannian model of φ.

Harmonic maps into a Grassmannian–1

Uhlenbeck proved that Φ_λ *is a type one extended solution of a harmonic map* $\varphi : M^2 \to G_*(\mathbb{C}^n)$ *if and only if*

$$\Phi_\lambda = Q\Phi_{-\lambda}\Phi_{-1}^{-1}Q \tag{6}$$

for some $Q \in \mathrm{U}(n)$ *with* $Q^2 = I$*, equivalently,* $Q = \pi_C - \pi_C^\perp$ *for some* $C \in G_*(\mathbb{C}^n)$*. Further, in this case,* $Q\Phi_{-1}$ *is a harmonic map into a Grassmannian.*

Hence

Lemma

Let Φ *be the type one extended solution of a harmonic map of finite uniton number, say* r*, and let* $\underline{W} = \Phi\mathcal{H}_+ \in \mathcal{H}_+/\lambda^r\mathcal{H}_+$ *be the corresponding Grassmannian model. Then* Φ *is the extended solution of a harmonic map into a Grassmannian if and only if there exists* $Q \in \mathrm{U}(n)$ *with* $Q^2 = I$ *such that*

$$\underline{W}_\lambda = Q\underline{W}_{-\lambda} \quad (\lambda \in \mathbb{C})\,. \tag{7}$$

Harmonic maps into a Grassmannian–2

Definition

Say that a polynomial $L \in \mathcal{H}_+/\lambda^r\mathcal{H}_+$ is *Q-adapted* if its coefficients have image alternately in C and $C^\perp$, i.e., $L(\lambda) = \sum_{i=0}^{r-1} L_i\lambda^i$ and *either*
(+) L_i has image in C for i even, and in $C^\perp$ for i odd, *or*
(−) L_i has image in $C^\perp$ for i even, and in C for i odd;
equivalently, L lies in *either* the (+1)- *or* the (−1)-eigenspace of the involution $L(\lambda) \mapsto QL(-\lambda)$ on $\mathcal{H}_+/\lambda^r\mathcal{H}_+$.

Remark. When $Q = I$, a polynomial is Q-adapted if and only if it is even or odd, i.e, has coefficients of all odd or all even powers of λ equal to zero.

Corollary

Φ is the extended solution of a harmonic map into a Grassmannian if and only if $\underline{W}$ has a spanning set consisting of Q-adapted polynomials, or, equivalently, $\underline{W}$ is given by (4) for some $\underline{X}$ which has a spanning set consisting of Q-adapted polynomials.

Ha maps into a Grassmannian–Example 1

Let $r = 3$, $Q = I$.

Let $L_{0,1}$ and $L_{2,1}$ be arbitrary meromorphic vectors, and let $\underline{X}$ be spanned by the even polynomial $L_{0,1} + \lambda^2 L_{2,1}$. Then $\underline{W}$ is spanned by this and by the odd polynomial $\lambda L_{0,1}^{(1)}$; both polynomials are Q-invariant with $Q = I$.

Via (5), $\underline{X}$ corresponds to a single column $(H_{0,1}, 0, H_{2,1})^T$ of data where $H_{0,1} = L_{0,1}$ and $H_{2,1} = L_{2,1} + L_{0,1}$. Taking $H_{0,1} = L_{0,1}$ and $H_{2,1} = L_{2,1}$ gives the same unitons and so the same harmonic map. $\underline{\alpha}_1 = \mathrm{span}\{H_{0,1}\}$, $\underline{\alpha}_2 = (\underline{\alpha}_1)_{(1)} = \mathrm{span}\{H_{0,1}, H_{0,1}^{(1)}\}$ and $\underline{\alpha}_3 = \mathrm{span}\{H_{0,1} + \pi_2^{\perp} H_{2,1}, \pi_1^{\perp} H_{0,1}^{(1)}, \pi_2^{\perp} H_{0,1}^{(2)}\}$.

From (28), $\varphi : M^2 \to G_2(\mathbb{C}^4)$ is given by $\underline{\varphi} = \mathrm{span}\{H_{0,1} + \pi_2^{\perp} H_{2,1}, \pi_2^{\perp} H_{0,1}^{(2)}\}$.

Assuming that $\underline{\alpha}_1$ is full, φ has uniton number three.

Ha maps into a Grassmannian–Example 2

$r = 2$, $Q \neq I$. Let $Q = \pi_C - \pi_C^\perp$ where C is a proper subspace of $\mathbb{C}^n$. Let $L_{i,j}$ $(i = 0, 1,\ j = 1, 2)$ be meromorphic vectors where $L_{0,1}$ and $L_{1,2}$ have values in C but $L_{0,2}$ and $L_{1,1}$ have values in $C^\perp$.

Let $\underline{X}$ be spanned by $L_{0,j} + \lambda L_{1,j}$ $(j = 1, 2)$. Then $\underline{W}$ is spanned by these and by $\lambda L_{0,j}^{(1)}$ $(j = 1, 2)$, and all four polynomials are Q-adapted. Note that (5) gives $H_{0,j} = L_{0,j}$ and $H_{1,j} = L_{1,j} - L_{0,j}$, but we obtain the same unitons, and so the same harmonic map, from the data $H_{i,j} = L_{i,j}$.

Then $\underline{\alpha}_1 = \mathrm{span}\{L_{0,1} = H_{0,1}, L_{0,2} = H_{0,2}\}$; we can choose this data such that $\underline{\alpha}_1$ is full — it suffices to have no linear relation between their components.

$\underline{\alpha}_2 = \mathrm{span}\{\pi_1 L_{0,j} + \pi_1^\perp L_{1,j} = H_{0,j} + \pi_1^\perp H_{1,j},\ \pi_1^\perp L_{0,j}^{(1)} = \pi_1^\perp H_{0,j}^{(1)}\ (j = 1, 2)\}$.

So $\varphi = Q(\pi_1 - \pi_1^\perp)(\pi_2 - \pi_2^\perp) : M^2 \to G_*(\mathbb{C}^n)$ is a harmonic map of uniton number two into a Grassmannian.

Isotropic maps

Suppose that $\underline{X}$ is spanned by monomials of the form $\lambda^k L_k$; equivalently, the data $(H_{i,j})$ is in the 'diagonal' form

$$\begin{bmatrix} H_{0,1} & \cdots & H_{0,d_1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & \cdots & 0 & H_{1,d_1+1} & \cdots & H_{1,d_2} & 0 & \cdots & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & H_{2,d_2+1} & \cdots & H_{2,d_3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

then the unitons $\underline{\alpha}_i$ are *nested*, i.e., $\underline{\alpha}_i \subset \underline{\alpha}_{i+1}$. The projections in (28) are unnecessary and $\underline{\alpha}_{i+1} = \mathrm{span}\{H^{(\ell)}_{i-\ell,k} : 1 \le k \le d^{(0)}_{i+1}, 0 \le \ell \le i\}$. The harmonic map $\varphi = (\pi_1 - \pi_1^\perp)\cdots(\pi_r - \pi_r^\perp)$ has image in a Grassmannian and is given by the formula

$$\underline{\varphi}(^\perp) = \sum_{k=0}^{[(r-1)/2]} \underline{\alpha}^\perp_{r-1-2k} \cap \underline{\alpha}_{r-2k} = \underline{\alpha}^\perp_{r-1} \cap \underline{\alpha}_r \oplus \underline{\alpha}^\perp_{r-3} \cap \underline{\alpha}_{r-2} \oplus \cdots ,$$

where we set $\underline{\alpha}_0$ equal to the zero subbundle. We thus obtain a harmonic map into a Grassmannian which is isotropic in the sense that it is invariant under the natural S^1 action of C.-L. Terng.

Explicit formulae for any factorization

Recall that polynomial extended solution $\Phi : M^2 \to \Omega^{\mathrm{alg}}\mathrm{U}(n)$ of degree r defines a holomorphic subbundle $\underline{W} = \Phi(\mathcal{H}_+)$ of the trivial bundle $\mathcal{H}_+/\lambda^r\mathcal{H}_+ \cong M^2 \times \mathbb{C}^n$, which satisfies $\underline{W}_{(1)} \subset \underline{W}$.

G. Segal showed that the sequence of harmonic maps $\Phi_0 = I, \Phi_1, \ldots, \Phi_r = \Phi$ coming from a factorization into unitons

$$\Phi_i = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^{\perp}) \cdots (\pi_{\alpha_i} + \lambda\pi_{\alpha_i}^{\perp}) \quad (i = 0, 1, \cdots, r)$$

corresponds, via $\underline{W}_i = \underline{\Phi}_i\mathcal{H}_+$, to a filtration

$$\underline{W} = \underline{W}_r \subset \underline{W}_{r-1} \subset \cdots \subset \underline{W}_1 \subset \underline{W}_0 = H_+,$$

with $\lambda\underline{W}_i \subset \underline{W}_{i+1}$.

Theorem (M. Svensson and JCW, 2009)

For any factorization/filtration, we have the following explicit formula for the unitons:

$$\underline{\alpha}_i = \sum_{s=0}^{i-1} S_s^{i-1} P_s \underline{W}_i. \tag{8}$$

Further, the projection converts a holomorphic basis of $\underline{W}_i$ into a basis holomorphic with respect to the harmonic map φ_{i-1}.

In fact, the formula (8) is equivalent to

$$\underline{\alpha}_i = P_0 \Phi_{i-1} \underline{W}_i$$

and we can show that $P_0\Phi_{i-1}$ converts the operators $\partial_{\bar{z}}$ and $\lambda\partial_z$ on W_{i-1} into $D_{\bar{z}}$ and $-A_z^{\varphi_{i-1}}$ on $\underline{\mathbb{C}}^n$, respectively.

Example

Set $\underline{W}_i = \underline{W} + \lambda^i \mathcal{H}_+$. Then we obtain the factorization satisfying Segal's covering condition

$$\pi_i(\underline{\alpha}_{i+1}) = \underline{\alpha}_i \quad \text{for all } i\,.$$

Note that the rank of the unitons increases with i. The $\underline{W}_i$ in (8) can be replaced by $\underline{W}$ and we obtain the 'Lisbon' formulae for the unitons.

Example

Set $\underline{W}_i = (\lambda^{i-r}\underline{W}) \cap \mathcal{H}_+ + \lambda^i \mathcal{H}_+$. Then we obtain Uhlenbeck's factorization which satisfies the 'dual' covering condition

$$\pi_{i+1}(\underline{\alpha}_i) = \underline{\alpha}_{i+1} \quad \text{for all } i\,.$$

Note that the rank of the unitons decreases with i. This time, we cannot replace the $\underline{W}_i$ in (8) by $\underline{W}$. We obtain the formulae of B. Dai and C.-L. Terng.

Harmonic maps into $0(n)$ and $\mathrm{Sp}(n)$

Given any λ-closed subspace W_r with $\lambda^r\mathcal{H}_+ \subset W_r \subset \mathcal{H}_+$ and $r \geq 2$, consider the subspace

$$W_{r-2} = \lambda^{-1}W_r \cap \mathcal{H}_+ + \lambda^{r-2}\mathcal{H}_+$$

This can be obtained from W_r by first doing a 'Segal'-step:

$$W_{r-1} = W_r + \lambda^{r-1}\mathcal{H}_+$$

and then an 'Uhlenbeck'-step:

$$W_{r-2} = \lambda^{-1}W_{r-1} \cap \mathcal{H}_+ + \lambda^{r-2}\mathcal{H}_+$$

or vice versa.

Say W_r is *real* if $r = 2s$ and $\overline{W}^{\perp} = \lambda^{1-r}W_r$; equivalently $\lambda^{-s}W = \Phi\mathcal{H}_+$ for some $\Phi \in \Omega^{\mathrm{alg}}\mathrm{O}(n)$.

Lemma

If W_r is real so is W_{r-2}.

Apply this to the (type one) extended solution Φ of a harmonic map of finite uniton number into $\mathrm{O}(n)$, starting with $W_r = \Phi\mathcal{H}_+$.

This gives a factorization of harmonic maps of finite uniton number from a surface to $\mathrm{O}(n)$, with explicit formulae provided by (8) given a meromorphic basis of W.

Similarly, for $\mathrm{Sp}(n)$, replacing 'real' by 'symplectic' and complex conjugation by J, obtaining an explicit version of R. Pacheco's factorization (2006).

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