Countable homogeneous multipartite graphs

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Abstract

We give a classification of all the countable homogeneous multipartite graphs. This generalizes the similar result for bipartite graphs given in [5].

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1 Introduction

In this paper we describe the classification of all the countable homogeneous multipartite graphs. This generalizes the classification of the countable homogeneous bipartite graphs given for instance in [5], and it may also be viewed as the solution of a (very) special case of a problem posed in [3], concerning the classification of all countable homogeneous 'n-graphs'. By 'homogeneous' we mean that any isomorphism between finite substructures extends to an automorphism. (We often refer to an isomorphism between finite substructures as a *finite partial automorphism*.) Note that this notion is sometimes referred to as 'ultrahomogeneity', for instance in [9], in order to distinguish it from a related, but distinct notion of homogeneity in model theory. We begin by reviewing earlier work on classifications of classes of homogeneous graphs and related structures.

Gardiner [4] extended the work of Sheehan [13] to classify all the finite homogeneous graphs. They were independently classified by Golfand and Klin [6]. Lachlan and Woodrow [9] extended this to the countably infinite case. Lachlan in [11] classified the finite homogeneous directed graphs, and in [10] classified the countable homogeneous tournaments. Cherlin [3] then classified the countable homogeneous directed graphs in which for any two vertices x and y there is an edge between x and y in at most one direction, and the countable homogeneous n-tournaments. What is particularly interesting about Cherlin's work on the homogeneous directed graphs is that Henson in [7] had shown that there are $2^{\aleph_0}$ of these, which led many to believe that a full classification would not be possible. Cherlin, however, showed that the method used to construct the $2^{\aleph_0}$ examples, could in fact be used to construct all but countably many of the homogeneous directed graphs, and these remaining ones could then be divided into finitely many parametrized families.

Of course graphs are not the only homogeneous structures that have been classified. Homogeneous linear, and partial orders have been extensively studied. Several of these are discussed in [16], and here we just mention the countable homogeneous partial orders, which were classified by Schmerl [12] and the coloured versions of these, see [15]. We remark that we carry over various features of the organization of [15]. In particular, we adopt a notion of 'reduced coding graph', which plays a similar role to that of 'reduced skeleton' in [15], where perfect matchings and their complements are disallowed, and this considerably simplifies the overall structure of the proof.

One respect in which the present paper is more involved than [15] is that a key point in that case was that the whole structure was controlled by the relations between 3-component substructures. To explain this fully would require the definition of 'component'. Since we do not wish to go into this in

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detail, we just mention that in a countable homogeneous coloured partial order there is a notion of ('interdense') component such that the possibilities for components are given by a straightforward adaptation of Schmerl's classification, and given this, the main focus of the effort is in working out how the components 'fit together'. Now the notion of component was heavily reliant on the transitivity of the relation. In the general multipartite case this is absent, and so the only sensible analogue of component is one of the parts that the graph is partitioned into under the multi-partition. But now there is no particular reason to believe that the whole structure will be controlled by the restrictions to just three components.

We shall not formulate our results for the most part in model-theoretic terms, since they have a much more combinatorial flavour. However, officially, and for the record, we work in a fixed countable relational language L, and seek to study and classify certain L-structures. This will enable us to employ Fraïssé's Theorem freely throughout, which links homogeneous structures in a relational language with amalgamation classes of finite structures. In practice, for the whole of this paper, the relations will all be unary or binary, and all the binary relations will be symmetric.

To give some model-theoretic justification we do however briefly state the connections between homogeneity and quantifier elimination, insofar as they apply in the current context. A countable structure over a finite relational language is homogeneous if and only if it is $\aleph_0$-categorical and admits elimination of quantifiers. Furthermore, if we make no cardinality restriction on the number of relations, but assume $\aleph_0$-categoricity, then quantifier elimination is equivalent to homogeneity. It follows that Fraïssé's construction allows one easily to generate many $\aleph_0$ -categorical structures out of amalgamation classes, and in particular that applies here.

For us, an *amalgamation class* is a family $\mathcal{C}$ of finite structures in a finite or countable relational language which is closed under isomorphism and the formation of substructures, and which has the amalgamation property: if $A_0, A_1, A_2 \in \mathcal{C}$ and $f_i : A_0 \hookrightarrow A_i$ $(i = 1, 2)$ are embeddings, then A_1, A_2 can be amalgamated over A_0 in $\mathcal{C}$; meaning that there are $A \in \mathcal{C}$ and embeddings $g_i : A_i \hookrightarrow A$ $(i = 1, 2)$ such that $g_1 f_1 = g_2 f_2$. We say that A is an *amalgam* of A_1 and A_2 over A_0.

We remark that two other properties are also often listed, namely the 'joint embedding property' which says that any two structures in $\mathcal{C}$ can be embedded in some member of $\mathcal{C}$, and the requirement that there are only countably many isomorphism types of members of $\mathcal{C}$. We do not need to list these separately here, since we are allowing the 'empty structure' as a member of $\mathcal{C}$ (since $\mathcal{C}$ is closed under forming substructures, and the language is relational). And again, as the language is countable, there are automatically only countably many finite structures up to isomorphism.

For any countable structure $\mathcal{A}$, we let its *age* be the family of finite structures which are isomorphic to substructures of $\mathcal{A}$. Fraïssé's theorem then says the following:

> The age of any countable homogeneous structure is an amalgamation class.
> Conversely, if $\mathcal{C}$ is an amalgamation class of finite structures in a countable relational language, then there is a countable homogeneous structure $\mathcal{A}$, unique up to isomorphism, having $\mathcal{C}$ as its age.

This theorem allows us to pass freely between homogeneous structures and amalgamation classes. In verifying the amalgamation property we note that it suffices to show that any 'two-point' amalgamations can be carried out. This is because we may then repeat finitely many times to amalgamate more points than this. By a 'two-point amalgamation' we understand an amalgamation in which A_1 and A_2 each have only one more point than A_0. In practice we usually also assume that A_0 is a substructure of A_1 and A_2, and furthermore that $A_0 = A_1 \cap A_2$. This can always be achieved by taking isomorphic copies.

The general problem posed in [3] is to classify all the countable homogeneous n-graphs for n a positive integer. Here an n-graph is a structure whose domain is partitioned into n pieces, each of which carries a graph structure, and such that between different parts there are finitely many possible edge types (which for ease we may think of as 'colours'; in the 'monochromatic' case, the two edge types are just 'joined' and 'not joined'). This is a more general type of structure than the ones we are considering here, which correspond to the special case in which each of the graphs on the parts is empty, and such that for each pair of distinct parts there are just two colours (joined or not joined). The general problem seems quite hard, but the one in which the graphs on the parts are empty, but more than two colours are allowed between different parts is more tractable, and indeed we give the solution in the bipartite case. The signature for an n-graph consists of n unary predicates to distinguish the parts, and for each pair of distinct parts, finitely many binary relations for the edge types. For multipartite graphs as studied here, the signature consists of finitely or countably many unary predicates and a symmetric binary relation. The unary predicates are interpreted as pairwise disjoint sets, called *parts*, whose union is the whole structure, and the binary relation is interpreted as a graph relation such that there are no edges between members of the same part. It is also possible to view the graph relation as a family of binary relations, one for each pair of parts, and this is the viewpoint we shall adopt since, for example, we may wish to consider the complement of the relation between two parts without necessarily complementing the whole graph relation. We shall write V_i as i varies for the parts, and R_{ij} for the relation between V_i and V_j.

To classify all the countable homogeneous multipartite graphs turns out to be considerably more complicated and interesting than originally expected. Now, for bipartite graphs, there are five cases, which are called 'empty', 'complete', 'perfect matching' or its complement, and 'generic'. We might hope that there would be analogues, or combinations of these which would apply to more than two parts, and indeed that is what happens, but the combinations in which they can arise, and the possible restrictions which need to be considered, turn out to be quite involved.

We begin by giving four basic lemmas, which will be used throughout, and which will aid the exposition. By a *restriction* of a multipartite graph G we mean an induced subgraph of G whose vertex set is a union of some set of parts of G.

Lemma 1.1. *Any restriction of a homogeneous multipartite graph to a subset of its set of parts is also homogeneous.*

Proof: Let a finite partial automorphism of a restriction be given. Then it is also a finite partial automorphism of the whole structure, which by homogeneity extends to an automorphism. Since any automorphism preserves the parts, this automorphism restricts to an automorphism of the restriction, which extends the originally given partial automorphism. □

Lemma 1.2. *If G_1 and G_2 are multipartite graphs on the same vertex set such that each relation of G_2 is either equal to the corresponding relation of G_1, or to its complement, then G_1 is homogeneous if and only if G_2 is homogeneous.*

Proof: It suffices to show that if G_1 is homogeneous then so is G_2. The point is that for a permutation θ of the domain of G_1, θ is an automorphism of G_1 if and only if it is an automorphism of G_2 (with a similar statement for partial maps). For suppose that θ is an automorphism of G_1, and let $x \in V_i$ and $y \in V_j$ where $i \neq j$, and we shall show that $x\, R_2\, y \Leftrightarrow \theta x\, R_2\, \theta y$, where R_1 and R_2 are the relations between V_i and V_j in G_1 and G_2 respectively. If $R_1 = R_2$, then $x\, R_2\, y \Leftrightarrow x\, R_1\, y \Leftrightarrow \theta x\, R_1\, \theta y$ (since θ is an automorphism of G_1) $\Leftrightarrow \theta x\, R_2\, \theta y$. If R_2 is the complement of R_1, then $x\, R_2\, y \Leftrightarrow \neg x\, R_1\, y \Leftrightarrow \neg \theta x\, R_1\, \theta y \Leftrightarrow \theta x\, R_2\, \theta y$. □

The third of these frequently used lemmas concerns perfect matchings, which when viewed as functions are just bijections.

Lemma 1.3. *Suppose that G is a multipartite graph and that parts V_i and V_j are related by a perfect matching or its complement. Then G is homogeneous if and only if $G - V_j$ is homogeneous, and the map from $G - V_j$ to $G - V_i$ induced by the perfect matching is an isomorphism, or is obtained from an isomorphism by reversing some (but not necessarily all) the relations between V_i and the other components.*

Proof: First suppose that G is homogeneous, and let θ be the map from $G - V_j$ to $G - V_i$ which is equal to the given perfect matching on V_i, and fixes all other elements. Then it is immediate that θ preserves all the relations on $\bigcup_{k \neq i,j} V_k$. We just have to show that θ takes R_{ik} to R_{jk} or its complement for $k \neq i, j$. So this says that either $(\forall x \in V_i)(\forall y \in V_k)(x\, R_{ik}\, y \Leftrightarrow \theta x\, R_{jk}\, y)$ or $(\forall x \in V_i)(\forall y \in V_k)(x\, R_{ik}\, y \Leftrightarrow \neg \theta x\, R_{jk}\, y)$. Suppose for a contradiction that there are $x, x' \in V_i$ and $y, y' \in V_k$ such that $x\, R_{ik}\, y \wedge x'\, R_{ik}\, y' \wedge \theta x\, R_{jk}\, y \wedge \neg \theta x'\, R_{jk}\, y'$. By homogeneity there is an automorphism g taking x to x' and y to y'. Since $\theta x\, R_{jk}\, y$ and g preserves θ, we have $\theta x'\, R_{jk}\, y'$, which is a contradiction. Other cases are handled similarly.

Conversely, suppose that there is a perfect matching or its complement between V_i and V_j, and that $G-V_j$ is homogeneous. Since by Lemma 1.2 homogeneity is unaltered by taking the complement of any one relation, we suppose that the relation between V_i and V_j is a perfect matching, given by a bijection θ from V_i to V_j. We are also assuming that the extension of θ to $G-V_j$ is an isomorphism (which may however reverse some of the relations), and we have to show that G itself is homogeneous. Let p be a finite partial automorphism of G. We extend p to another finite partial automorphism which is closed under the action of θ and θ^{-1}, by letting $q(x) = y$ if $p(x) = y$, or $p(\theta x) = \theta y$, or $p(\theta^{-1}x) = \theta^{-1}y$. We need to check that this is indeed a partial automorphism. For this, it suffices to show that q preserves the relations. Since there is no change of these on $\bigcup_{k \neq i,j} V_k$ from p to q, we consider the cases $x \in V_i$ and $z \in V_k$ where $k \neq i$ as sufficiently typical, and x does not lie in dom p. If $k = j$ then θx must lie in dom p and $qx = \theta^{-1}p\theta x$. Hence $x\, R\, z \Leftrightarrow z = \theta(x) \Leftrightarrow \theta q(x) = pz \Leftrightarrow q(x)\, R\, p(z) = q(z)$. If $k \neq j$, then treating the case where θ gives an isomorphism of R_i to R_j (the case where it is an anti-isomorphism is similar), we have $x\, R_i\, z \Leftrightarrow \theta x\, R_j\, z \Leftrightarrow p\theta x\, R_j\, pz \Leftrightarrow qx = \theta^{-1}p\theta x\, R_i\, p(z) = q(z)$. Other cases are handled similarly.

Now we use the fact that $G - V_j$ is homogeneous to extend q to an automorphism φ of $G - V_j$, and φ is extended to an automorphism of G by letting $\varphi x = \theta\varphi\theta^{-1}x$ for $x \in V_j$. By definition of q, this also extends p, as required. □

The most significant parts of our classification are carried out by considering the class $O(G)$ of finite structures 'minimally omitted' by a multipartite graph G. The definitions are as follows. A finite m-partite graph A is *realized* in an m-partite graph G if A can be embedded in G. Otherwise it is *omitted.* It is *minimally omitted* if it is omitted, but any proper subgraph is realized.

Lemma 1.4. *Suppose that G_1 and G_2 are countable homogeneous multipartite graphs. Then G_1 and G_2 are isomorphic if and only if they minimally omit the same class of finite multipartite graphs.*

Proof: By Fraïssé's theorem, G_1 and G_2 are isomorphic if and only if they have the same age, and this is equivalent to saying that they omit the same class of finite graphs, since a finite graph is omitted if and only if it is *not* in the age, and this is clearly equivalent to saying that they minimally omit the same class of finite graphs. □

We remark here that we can quite easily deduce what the situation is for arbitrary countable homogeneous multipartite graphs, that is where infinitely many parts are allowed, from the classification of those for finitely many parts, by a compactness-type argument as follows. Namely, a countable multipartite graph is homogeneous if and only if each of its restrictions to finitely many parts is homogeneous. In one direction this follows from Lemma 1.1. Conversely, suppose that all the restrictions of G to finitely many parts are homogeneous, and we see that G is also homogeneous. For this, it suffices to show that its age $\mathcal{C}$ has the amalgamation property. But if $A_0, A_1, A_2 \in \mathcal{C}$ and $f_i : A_0 \hookrightarrow A_i$ $(i = 1, 2)$ are embeddings of elements of $\mathcal{C}$, then all the structures lie in some restriction of G to a finite set of parts, so by the amalgamation property there, can be amalgamated in the restriction, and hence in $\mathcal{C}$. So from now on we always just consider multipartite graphs on finitely many parts.

As was seen in [15], the presence of perfect matchings and their complements makes for some technical difficulties when considering amalgamation classes, and it is better if possible to eliminate them during most of the discussion, and just restore them at the end, when summarizing the complete classification. This is where Lemma 1.3 comes in, and it is employed to reduce the classification problem to that in which there are no perfect matchings or their complements between the parts, which we call 'reduced' graphs.

This means that by far the most involved and interesting case of the classification is that in which all the bipartite restrictions of the homogeneous multipartite graph under consideration are generic. Let us call such an m-partite graph an *m-generic* graph. Most of our effort is devoted to describing what these can be. For the tripartite case, this is straightforward. There are just 9 possibilities, corresponding to the completely generic case (that is, the Fraïssé limit of the class of *all* finite tripartite graphs), and 8 in which a 'triangle' is omitted. Here by *triangle* we mean a tripartite graph having exactly one vertex in each part. Since each of the three pairs of vertices can be joined by an edge or a non-edge, we get $2^3 = 8$ possibilities (all essentially the 'same' graph in the sense of

Lemma 1.2). Sometimes 'triangle' means the (usual) special case, with three edges, and in view of Lemma 1.2, there is usually no loss of generality in restricting to this case.

To give a general treatment, we slightly relax the notion of 'm-generic' to 'reduced', where all bipartite restrictions are generic, complete, or empty. The point here is that we may construe an empty or complete relation between two parts as saying that an edge or non-edge respectively is omitted, and this observation enables us to treat the reduced case in the same way as the m-generic one, with little extra complication. There are then 26 reduced countable homogeneous tripartite graphs that are not 3-generic, 8 of them where all the bipartite restrictions are complete or empty, 12 with one generic bipartite restriction, and the other two empty or complete, and 6 with two generic bipartite restrictions, and just one complete or empty.

Guided by this case, we naturally expect that for more than 3 parts, reduced graphs may also be described by which particular graphs are omitted. We say that a graph having no more than one vertex in each part is *monic*. A conjecture made in [8], based on the tripartite, quadripartite, and quintipartite cases, was that any minimally omitted graph is monic. We are able to verify this, which is done in section 6, and is a major step towards reaching an overall description.

In view of these remarks, we aim to describe the m-partite reduced homogeneous graphs in terms of the (monic) structures that they minimally omit. In the typical case, if two (monic) graphs are omitted, and are both defined on V_iV_j, they will agree there, meaning that either both have an edge between V_i and V_j, or both have a non-edge. If this does *not* happen, then we say that they *differ*, and an omitted graph is said to *differ* if it differs from some other omitted graph. It turns out that the quadripartite case is more complicated than the tripartite, in that here it is possible to minimally omit a family of graphs, any two of which differ on some edge type in this sense. These families comprise four tripartite graphs, and are referred to as 'omission quartets'. This represents an added complication at this level. The fear is that as we increase the number of parts, other, possibly more complicated families of (differing) graphs can be minimally omitted, and any overall classification would get more and more involved. The 'non-complication theorem' (also conjectured in [8]) says that this does not happen; that is, the worst type of differing omission that can arise is given by omission quartets. This is also proved in section 6.

Now we explain how we plan to 'keep track' of the structures we are classifying. We need to have a family of 'classifiers', structures which are simpler than the objects being classified, which will nevertheless encode their structure in a reasonably direct way. The structures we use are called 'coding graphs'. These correspond to what were called 'coding trees' in [1, 2], and 'skeletons' in [15]. The precise definition will be given below, but we give the idea here. A coding graph Γ for the m-partite reduced case (called a 'reduced coding graph') will be a complete monic m-partite graph, but with extra structure giving us information about which multipartite graph G we are talking about, consisting of a cardinality label between 1 and $\aleph_0$ on each vertex, telling us the cardinality of the corresponding part in G, and a family $\mathcal{G}$ of monic graphs on the same m parts, representing the minimally omitted structures, which as we saw in Lemma 1.4 is sufficient to determine the whole structure, given homogeneity. Although the complete graph structure on Γ plays no apparent role so far, it was the method followed in [8] and [14], where labels E, C, Ge on its edges told us the corresponding type of bipartite restriction, empty, complete, or generic, and this information may be recovered from $\mathcal{G}$, so it may be convenient to regard Γ as having edge labels, even if they were not part of the official definition. In the general definition of 'coding graph', we should have to incorporate labels to deal with perfect matchings and their complements which may arise, so now we allow also PM and CPM as edge labels. Here again, however, an alternative treatment is offered, which also deals neatly with the free use of complementation as in Lemma 1.2, by allowing a ± 1 label on each edge to tell us whether it was complemented or not, and this is sufficient to allow us to recover the PM and CPM labels too, so these do not need to be explicitly included.

The next step is to define which multipartite graph is represented by which coding graph. The formal definition is given in section 3, but in terms of the vertex and edge labels, a coding graph Γ on m vertices *represents* an m-partite graph G if there is a 1–1 correspondence between the vertices of Γ and the parts of G which respects the vertex labels, and such that each edge label correctly represents the corresponding bipartite restriction (so for instance if uv is labelled PM, then the relation between the parts of G corresponding to u and v is a perfect matching). There are extra conditions explained later, which involve saying that $\mathcal{G}$ accurately describes the minimally omitted graphs on the 'reduced part' of G (called its 'core'), and also that the ± 1 labels are correctly realized.

In terms of coding graphs, the main results given are that any coding graph represents a unique countable homogeneous multipartite graph, and that any countable homogeneous multipartite graph is represented by a unique coding graph. This is what we view as constituting a 'classification', and is the content of the main theorem, Theorem 7.2.

The paper is organized as follows. In the next section, we give for completeness a proof of the classification of the countable homogeneous bipartite graphs, since this is such an important ingredient in what follows. In addition, in this instance, we are able to deal with the case of several cross types, which we now think of as 'colours'.

In section 4 we move on to the tripartite case, and we give a more streamlined treatment than in [8] or [14], making free use of Lemmas 1.1, 1.2, and 1.3. The idea here is to illustrate the methods which will be applied more generally later on in the paper in a relatively concrete instance. In particular, we subdivide into the non-reduced case where, by taking a complement if necessary, we may assume that there is a perfect matching, and the reduced case. In the latter instance we shall see that for each homogeneous graph there is in fact at most one minimally omitted structure, and in view of Lemma 1.4, this suffices to provide a classification.

In order to simplify the presentation in section 4, we first give some of the general framework in section 3, including the precise definition of 'reduced' multipartite graph, the definitions of coding graph and reduced coding graph, and the reduction of the general classification problem to the reduced case. The main result which is proved is that any countable multipartite graph has a reduced restriction, obtained by suitably removing parts which are related by a perfect matching or it complement to other parts, and the original structure is homogeneous if and only its reduction is. The reduction is 'essentially' unique. This result justifies restricting attention in the principal analysis to the reduced case.

In section 5 we look at reduced homogeneous quadripartite graphs, which illustrate some important and typical features of the general situation. We first give the definition of 'omission quartet', and prove that the class of finite quadripartite graphs omitting a given omission quartet is an amalgamation class, and hence defines a homogeneous structure. Apart from this, it is also possible to omit a wider class of (monic) graphs, which apart from the case of omission quartets agree on shared edge types. Given these ideas, we can complete the classification of the countable homogeneous quadripartite graphs.

The groundwork has now been laid for tackling the general case. The fact that things 'get no worse' beyond 4 parts is encapsulated in the two key results conjectured in [8] and proved there for up to five parts, and in general in [14], the 'Non-monic realization theorem', Theorem 6.2, which says that any finite graph minimally omitted in a reduced homogeneous multipartite graph is monic, and the 'Non-complication theorem', Theorem 6.9, which says that in a reduced homogeneous graph, any two differing minimally omitted graphs lie in an omission quartet. Proofs of these two results are presented in section 6.

Finally in section 7 we are able to put together all these ingredients and complete the overall classification. We thought at one stage that this would have to be inductive in nature; in other words, the classification for m-partite graphs would involve reference to the classifications already assumed to be obtained for smaller values of m, but in the end it was possible to be quite explicit. It is not an easy or short classification, but all the same it is transparent in the sense that for the reduced case, we just need to know which graphs are minimally omitted, and the possibilities for how this can happen are explicitly described (and for each m there are only finitely many), and then the general case is built up from this by methods requiring some 'book-keeping' to keep track, but no more than this. In conclusion we describe some more complicated configurations of omission quartets which can arise on more than four parts, called 'omission families', which make the final classification more transparent.

2 Classification of countable homogeneous bipartite graphs, including ones with coloured edges

We begin by giving the classification of the countable homogeneous bipartite graphs, as described in [5], as it will be the basis for all that follows. In fact, in the bipartite case we are able to classify a wider class of structures, which has the countable homogeneous bipartite graphs as a special case. Namely, we allow the complete bipartite graph to have its edges 'coloured' by colours from a finite set C where $|C| \geq 1$, and we may call such a graph 'C-edge-coloured', and the original case 'monochromatic' to distinguish it. The monochromatic case is then essentially the same as the 2-coloured case (where the two colours are 'joined' and 'not joined'). We write F for the colouring function. Thus, if V_0 and V_1 are the two parts, F maps $V_0 \times V_1$ onto C (since if it was not onto, we could make do with a smaller colour set). First we recall the meanings of the terms we shall use.

A bipartite graph is *empty* if it has no edges at all, and it is *complete* if all vertices from V_0 and V_1 are joined by an edge. It is a *perfect matching* if it is a 1–1 correspondence between V_0 and V_1. The *complement* of a bipartite graph G is the bipartite graph having the same vertex sets V_0 and V_1, but having precisely those edges which are not in G, and this gives us the notion of a *complement of a perfect matching.* We say that G is *generic* if $|V_0| = |V_1| = \aleph_0$ and for any finite disjoint subsets X and Y of V_i where $i = 0$ or 1, there is $x \in V_{1-i}$ joined to all members of X and to no members of Y.

The edge-coloured case can also be related to the monochromatic case by the notion of 'colour restriction'. The *colour restriction* of an edge-coloured bipartite graph G to a colour c just treats edges coloured c as 'edges' and all other edges as 'non-edges'. This means that any automorphism of G is also an automorphism of any colour restriction, but not necessarily conversely. We say that the C-coloured bipartite graph G is *generic* (or *C-generic*) if $|V_0| = |V_1| = \aleph_0$ and for any map α from a finite subset of V_i into C for $i = 0$ or 1 there is $x \in V_{1-i}$ such that for all $y \in \mathrm{dom}\,\alpha$, $F\{x, y\} = \alpha(y)$.

We remark that for bipartite graphs the first four types occur in complementary pairs—empty and complete, and perfect matching and its complement. The complement of a generic bipartite graph is also generic however. Note that if $|C| \geq 2$, any colour restriction of a C-generic bipartite graph is generic.

Lemma 2.1. *Any C-generic graph G is the Fraïssé limit of the class of all finite bipartite C-edge-coloured graphs.*

Proof: If $|C| = 1$ this is immediate, so now suppose that $|C| > 1$. The main point is to see that the given condition implies homogeneity. For let p_0 be a finite partial automorphism of G, and we shall show that there is an automorphism θ of G extending p_0. For this it suffices, using back-and-forth, to show how to extend any given finite partial automorphism p of G extending p_0 to another such which includes a given point x in its domain or range. As a typical case we suppose that we have to add x to the domain, and that $x \in V_0 - \mathrm{dom}\,p$. Let $N = |V_0 \cap \mathrm{range}\,p|$, and as $|V_1| = \aleph_0$, pick distinct points $y_0, y_1, \ldots, y_N \in V_1 - \mathrm{range}\,p$ and distinct $c_0, c_1 \in C$. For $0 \leq n \leq N$ define α_n by

$$\alpha_n(y) = \begin{cases} F\{x, p^{-1}y\} & \text{if } y \in V_1 \cap \mathrm{range}\,p \\ c_0 & \text{if } y = y_n \\ c_1 & \text{if } y = y_i \text{ where } i \neq n \end{cases}$$

By the definition of 'generic' there are $z_n \in V_0$ such that for all $y \in \mathrm{dom}\,\alpha_n$, $F\{z_n, y\} = \alpha_n(y)$. Since $\alpha_n \neq \alpha_{n'}$ if $n \neq n'$, all the z_n are distinct, so at least one must lie outside range p. Let z_n be such. Then $q = p \cup \{(x, z_n)\}$ is the desired extension of p, since for each $y \in V_1 \cap \mathrm{dom}\,p$, $F\{x, y\} = \alpha_n(py) = F\{z_n, py\}$.

Now that we know that G is homogeneous, to show that it is the Fraïssé limit of the class of all finite bipartite C-edge-coloured graphs we just have to see that it embeds every such finite graph A. Let A_0 and A_1 be the two parts of A, and write $A_1 = \{y_n : n < N\}$. Let $A' = A_0' \cup A_1$ where $A_0' = A_0 \cup \{x_n : n < N\}$ and the x_n are distinct new points where $F\{x_n, y_i\}$ is equal to c_0 if $i = n$ and is c_1 otherwise. The hypothesis of C-genericity ensures that each $A_0' \cup \{y_n\}$ embeds in G, and by homogeneity we may suppose that the embeddings all agree on A_0'. The fact that the y_n are differently joined to $\{x_i : i < N\}$ guarantees that under the union of the embeddings, no distinct y_ns are identified, so this gives an embedding of the whole of A' into G, and hence also of A. $\square$

Theorem 2.2. *If G is a countable homogeneous C-edge-coloured bipartite graph where $1 \leq |C| < \aleph_0$, then one of the following holds:*

(i) $|C| = 1$ and all edges have the same colour,

(ii) $|C| = 2$ and the edges of one colour are a perfect matching, and those of the other colour are its complement,

(iii) $|C| \geq 2$ and G is C-generic.

Hence, if G is a countable homogeneous bipartite graph, it is empty or complete, a perfect matching or its complement, or generic.

Proof: We first show that the restriction G_c of G to any colour $c \in C$ is empty, complete, a perfect matching or its complement, or generic.

Let V_0 and V_1 be the two parts of G. We first note that any 1–1 map from a finite subset of V_i to V_i for $i = 0$ or 1 is a partial automorphism, so extends to an automorphism. Hence $\mathrm{Aut}(G) \leq \mathrm{Aut}(G_c)$ acts highly transitively on each V_i. Let $G(x)$ be the set of neighbours of a vertex x in G_c. It follows that if $G(x)$ is empty or equal to the whole of V_1 for some $x \in V_0$, then the same holds for every $x \in V_0$. The former would imply that G_c is empty, contrary to F onto, and the latter gives all edges coloured by c, which is clause (i).

From now on we therefore assume that for any $x \in V_0$, $G(x)$ is a proper subset of V_1, and similarly for $y \in V_1$, $G(y)$ is a proper subset of V_0. From this it follows that if $x \neq y$ in V_0, then $G(x) \neq G(y)$. For as $G(x)$ is a proper non-empty subset of V_1 we can find $u \in G(x)$ and $v \in V_1 - G(x)$. By transitivity there is an automorphism taking u to v, and the image z of x under this automorphism must be different from x, and $F\{x, u\} = c \neq F\{x, v\}$. By double transitivity we can fix x and map z to y by an automorphism. Since $G(x) \neq G(z)$ it follows that $G(x) \neq G(y)$ too. The similar statement holds for $x \in V_1$.

If $|G(x)| = 1$ for some $x \in V_0$, then again using transitivity, this holds for all $x \in V_0$, and under these circumstances we show that C has only two members, and the c-coloured edges of G form a perfect matching. Let $f(x)$ be the unique member of $G(x)$ and we show that f is a bijection from V_0 to V_1. It is onto since we already know that $G(y) \neq \emptyset$ for all $y \in V_1$. It is 1–1 since $x \neq y$ in V_0 implies that $G(x) \neq G(y)$ (as we have seen) which implies that $f(x) \neq f(y)$. This case therefore gives a perfect matching.

A similar argument applies if for some x, $|V_1 - G(x)| = 1$, giving rise to the complement of a perfect matching.

From now on we suppose that for some (any) $x \in V_0$, $G(x)$ and $V_1 - G(x)$ both have size at least 2, and that the same applies for $x \in V_1$. By homogeneity, $|G(x)|$ has the same value, n say, for every $x \in V_0$. We show that n cannot be finite. If so, then by high transitivity, for any subsets A and B of V_1 of size n, there is an automorphism of G_c taking A to B, and it follows that since there is *some* such A which is the neighbour set of a member of V_0, that they *all* are. Now choose elements $a_1, a_2, \ldots, a_{n+2}$ of V_1, which exist since $V_1 - G(x)$ has size at least 2. Let $A_1 = \{a_1, \ldots, a_n\}$, $A_2 = \{a_1, \ldots, a_{n-1}, a_{n+1}\}$, and $A_3 = \{a_1, \ldots, a_{n-2}, a_{n+1}, a_{n+2}\}$, and let $x, y, z \in V_0$ be such that A_1, A_2, and A_3 are their respective neighbour sets. By double transitivity there is an automorphism fixing x and taking y to z. This fixes A_1 (setwise) but takes A_2 to A_3, which is impossible since $A_1 \cap A_2$ and $A_1 \cap A_3$ have different sizes.

The conclusion is that $G(x)$ is infinite for each $x \in V_0$, and similarly so are $V_1 - G(x)$, and $G(y)$ and $V_0 - G(y)$ for each $y \in V_1$. From this we can deduce that G_c is generic. To verify the property characterizing what this means, let X and Y be finite disjoint subsets of V_i for $i = 0$ or 1, $i = 0$ say. Pick any $y' \in V_1$. By what we have shown, $G(y')$ and $V_0 - G(y')$ are both infinite, and so we can choose $X' \subseteq G(y')$ and $Y' \subseteq V_0 - G(y')$ having the same sizes as X and Y respectively. By high transitivity there is an automorphism taking X' to X and Y' to Y. Let y be the image of y' under this automorphism. Then y is joined to all members of X and to none of Y, so is as required.

Now we go back to the main proof. In view of clause (i) we may suppose that $|C| \geq 2$ (that is, at least two colours arise as values of F), and hence that no restriction G_c is empty or complete.

If some G_c is the complement of a perfect matching then looking at pairs of vertices, we see that we cannot add another complement of perfect matching or generic. Thus the only possibility is to add a perfect matching, specifically the one corresponding to the complement of a perfect matching which we have, and this gives us clause (ii).

So let us now assume that we have no complements of perfect matchings. Assume further that we have a perfect matching in some colour restriction, and we show that G can only have one perfect matching as a colour restriction. Since we have one perfect matching, $|V_0| = |V_1| > 2$ (in the case where $|V_0| = |V_1| = 2$, the complement of the perfect matching is itself a perfect matching, but in any case it is still covered by clause (ii)). Suppose then that $d \in V_1$ is matched with a and b in V_0 by two distinct perfect matchings, and let $c \in V_0 - \{a, b\}$. By double transitivity of $\mathrm{Aut}(G)$, there is an automorphism fixing a and taking b to c. This must also fix d (since it preserves the first perfect matching), but then it does not preserve the second perfect matching after all.

We next show that it is not possible to have both a perfect matching and a generic as colour restrictions, using essentially the same argument as in the previous paragraph. Since the complement of a perfect matching is not one of the edge relations, there must be at least two others and so (by transitivity of $\mathrm{Aut}(G)$ on V_1) there are $a, b, c \in V_0$ and $d \in V_1$ such that a and d are related by the perfect matching and b, c are related to d by two other distinct relations. As before there is an automorphism fixing a and taking b to c, but this must fix d and so does not preserve the second relation after all.

It remains to show that if all colour restrictions are generic, then G is itself C-generic. Let α map a finite subset of V_0 into C (the argument being the same for V_1). Since each colour restriction G_c is generic, for each $c \in C$ and $y \in V_1$, $|\{x \in V_0 : F\{x, y\} = c\}| = \aleph_0$. Hence there is a subset X of V_0 such that for each c, if $X_c = \{x \in X : F\{x, y\} = c\}$, then $|X_c| = |\alpha^{-1}\{c\}|$. By high transitivity of $\mathrm{Aut}(G)$ on V_0, there is an automorphism θ of G such that $\theta(X_c) = \alpha^{-1}\{c\}$ for each c, and then if $x \in \mathrm{dom}\,\alpha$, $F\{x, \theta(y)\} = c \Leftrightarrow F\{\theta^{-1}x, y\} = c \Leftrightarrow \theta^{-1}x \in X_c \Leftrightarrow x \in \theta X_c \Leftrightarrow x \in \alpha^{-1}\{c\} \Leftrightarrow \alpha(x) = c$ so that $F\{x, \theta(y)\} = \alpha(x)$, and $\theta(y)$ is the required witness. □

3 Reduced structures and coding graphs

In this section we describe the passage between a general countable homogeneous multipartite graph and a reduced one. Recall that a multipartite graph is said to be *reduced* if all relations between its parts are generic, empty, or complete. The key result which will enable us to restrict to this case is Lemma 1.3.

The main classification of all the countable homogeneous multipartite graphs will be carried out using 'coding graphs'. In this section, we first give the definition of *coding graph*, and *graph encoded* by a coding graph. As in [15] it is easiest to restrict attention to 'reduced' coding graphs (encoding countable homogeneous reduced multipartite graphs). The reason for this is that handling the amalgamation class corresponding to a perfect matching gives rise to annoying and not particularly illuminating technicalities; for instance, one cannot just take the family of all finite perfect matchings contained in the given perfect matching (even in the finite case), as this is not closed under substructures. We could also eliminate 'complete' (being the complement of 'empty') but for the general theory it is smoother to retain it.

In order to describe the classification of the reduced m-partite graphs, we need to focus attention on which finite m-partite graphs are omitted. Now any m-partite graph having a subgraph which is omitted is also omitted, so the only relevant consideration is which finite m-partite graphs are minimally omitted, as defined in section 1. For an m-partite graph G, let $O(G)$ be the family of finite m-partite graphs which are minimally omitted by G. The intention is that all the countable reduced homogeneous m-partite graphs may be classified by the possible values of $O(G)$. The justification for this is provided by Lemma 1.4.

In order to make the classification at all satisfactory however, we need to provide further information on the possible values for $O(G)$ for reduced G. In particular we wish the list of countable homogeneous reduced m-partite graphs (and hence also the list of all countable homogeneous m -partite graphs) to be given 'effectively', which will also show that there are only countably many of them up to isomorphism, The key steps for this purpose are as follows.

The first is to show that any minimally omitted m-partite graph is monic . This was called the 'non-monic realization conjecture' in [8], and was proved in [14]. This at once implies that there are only finitely many possibilities for $O(G)$, which simplifies things enormously. (It immediately follows that the class we seek over all m is countable, but does not yet give effectivity.) We present the proof in Theorem 6.2.

The next is to describe how the different members of $O(G)$ may interact. First suppose that for any two parts i and j, all members of $O(G)$ (which we now know to be monic) that are defined on V_iV_j *agree*, meaning that any two members of $O(G)$ defined on V_iV_j either both have an edge, or both have a non-edge. It is then easy to check that the family of all finite m-partite graphs omitting $O(G)$ is an amalgamation class, and hence gives rise to a Fraïssé-generic structure. It is however possible for two members of $O(G)$ not to agree on some edge type (and we then say that they *differ*). It was shown in [8] that there is a countable homogeneous quadripartite graph whose minimally omitted structures are all defined on just three parts, and that any two of them differ (on a single edge type). The configuration of these four omitted graphs is called an *omission quartet* (defined in detail in section 5), and it seems at first sight that the presence of such a configuration in just the quadripartite case makes the prospects for classification at higher levels remote. It was however shown by Jenkinson that in up to 6 parts, this is the most complicated configuration which can occur, and he conjectured that this held in general. Again this conjecture was proved by Seidel in [14], and it means that we are able to give a complete, and fairly explicit, classification in terms of the possible values of $O(G)$.

With the above in mind, we define a *reduced coding graph* Γ to be a finite complete graph on vertices $\{0, 1, \ldots, m-1\}$ with labels on vertices (labelling function φ) in $(\mathbb{N} - \{0\}) \cup \{\aleph_0\}$ (for the cardinality of the corresponding component), and a family $\mathcal{G}$ of monic graphs on subsets of $\{0, 1, \ldots, m-1\}$ of size at least 2, fulfilling the following conditions:

(i) if two members of $\mathcal{G}$ differ on an edge type then they are members of an omission quartet (of members of $\mathcal{G}$),

(ii) if $\varphi(x) \neq \aleph_0$ then for every $y \neq x$, there is a graph on $\{x, y\}$ in $\mathcal{G}$ (i. e. a single edge or a single non-edge).

A *coding graph* is a finite graph Γ on vertices $\{0, 1, \ldots, m-1\}$ with labels on edges (labelling function ψ) in $\{\pm 1\}$ and a map θ onto a reduced coding graph $\Gamma' = (\Gamma', \varphi, \mathcal{G})$, such that Γ' is a subgraph of Γ fixed pointwise by θ.

The meaning of these definitions is not immediately apparent, so we may provide the following expanded version (which explains the connection with the treatment in [8, 14]). A reduced coding graph Γ is intended to encode a countable homogeneous reduced m-partite graph G, and $\mathcal{G}$ tells us which graphs are minimally omitted by G. In particular, a bipartite restriction of G is empty or complete (labelled E, C respectively) if there is a corresponding member of $\mathcal{G}$ with an edge, or non-edge, respectively, and otherwise is generic (labelled Ge). This means that we do not need to include the labelling of the edges of Γ explicitly in the reduced coding graph, since this information can be discovered anyway from the members of $\mathcal{G}$. In a coding graph we may also label distinct members of the same $\theta^{-1}(x)$ for $x \in \Gamma'$ by PM or CPM if the edge label is $+1$ or -1, and the edge labels between members of $\theta^{-1}(x)$ and $\theta^{-1}(y)$ for distinct x and y control whether the map arising as in Lemma 1.3 is an isomorphism or anti-isomorphism (the latter meaning that it is an isomorphism between one graph and the complement of the other).

After 'coding graph' is defined abstractly in this fashion, we then have to explain which multipartite graph is represented by which coding graph. The above description makes it fairly clear what is intended. Thus a reduced coding graph Γ on m vertices *represents* (or *encodes*) a reduced m-partite graph G if there is a 1–1 correspondence between the vertices of Γ and the parts of G such that the label at a vertex equals the cardinality of the corresponding part of G, and $\mathcal{G}$ is the family of graphs which are minimally omitted by G. Similarly, a (general) coding graph Γ on m vertices *represents* an m-partite graph G if there is a 1–1 correspondence between the vertices v_i of Γ and the parts V_i of G such that the bipartite restriction on V_iV_j in G is a perfect matching or its complement if $\theta(v_i) = \theta(v_j)$ and $\psi(\{v_i, v_j\}) = +1$ or -1 respectively, $\theta\Gamma$ represents the restriction of G to $\bigcup\{V_i : i \text{ is a vertex of } \theta\Gamma\}$, and if $i \neq j$ and $\theta(i) = \theta(j)$, then for any $k \neq i, j$, the natural map between $V_i \cup V_k$ and $V_j \cup V_k$ provided by the matching between V_i and V_j is an isomorphism if $\psi(\{v_i, v_k\}) = \psi(\{v_j, v_k\})$, and it is an anti-isomorphism otherwise.

In these terms, the main result, Theorem 7.2, says that coding graphs provide a classification of all the countable homogeneous multipartite graphs (on finitely many parts). The precise meaning of this is as follows: any coding graph on m vertices encodes a countable homogeneous m-partite graph, which is uniquely determined up to isomorphism, and conversely, any countable homogeneous m-partite graph G is encoded by a coding graph of size m. This coding graph need not be uniquely determined, but any two coding graphs for G will only differ on the values of ψ.

For the remainder of this section we explain the passage between a general countable homogeneous m-partite graph and its reduced 'core', which precisely corresponds, for the coding graphs, to the map θ from a coding graph to a reduced coding graph. The definition is as follows. If G is an m-partite graph, then a *core* for G (written $core(G)$)), is the restriction of G to some set of its parts such that $core(G)$ is reduced, and every part of G not in $core(G)$ is related to a part of $core(G)$ by a perfect matching or its complement. The corresponding notion of a core of a coding graph is just that if Γ is a coding graph and θ is the map as in the definition of what this means, then $\Gamma' = \theta\Gamma$ is its *core*.

Theorem 3.1. *For any countable homogeneous m-partite graph G there is a core, $core(G)$, which is reduced and is also homogeneous. For any two cores G_1 and G_2 of G, there is a bijection of the vertices of G_1 to those of G_2, which carries parts of G_1 to parts of G_2, and which is an isomorphism of each bipartite restriction of G_1 to the corresponding bipartite restriction of G_2 or its complement. Furthermore, the reduced coding graph of $core(G)$ is the core of any coding graph of G. Conversely, for any countable reduced homogeneous k-partite graph encoded by the reduced coding graph Γ', and for any m-partite coding graph Γ having Γ' as core, there is a countable homogeneous m-partite graph G encoded by Γ and having G' as core.*

Proof: As a simple example that the core need not be unique, if G is bipartite and is a perfect matching, then we at once get two possible cores, by omitting either part, though of course in this case, the fact that the two cores are isomorphic is clear (since the two parts must have the same cardinality).

To prove existence of a core, we first remark that the relation $\sim$ on the set of parts given by $V_i \sim V_j$ if $i = j$ or $V_i \cup V_j$ is a perfect matching or its complement, is an equivalence relation on the set of parts (transitivity follows from Lemma 1.3). Then G' is obtained by choosing a representative of each $\sim$-class and taking the corresponding restriction. By Lemma 1.1, G' is homogeneous, and it is clearly reduced, and is a core of G. Suppose that G'' is another core. Then G'' must also comprise a choice of representatives of the $\sim$-classes, and it follows from Lemma 1.3 that it is isomorphic to G' in the sense stated (though the isomorphism or anti-isomorphism may need to be taken as the composition of several of those given by the lemma). Let Γ be a coding graph of G and θ the corresponding map onto Γ', and let G' be the core of G obtained by using the same choice as in the passage from Γ to Γ' (since by what we have just shown, any two choices of G' are isomorphic). Then by definition, Γ' is the reduced coding graph of G'.

Conversely, suppose that Γ, Γ', and G' are given. We can form G by adding in extra parts corresponding to the members of $\theta^{-1}\{v_i\}$ and the corresponding edges and non-edges are determined from the isomorphisms or anti-isomorphisms which are required to exist. □

4 Classification of countable homogeneous tripartite graphs

Now we move on to the classification of the countable homogeneous tripartite graphs. This will serve to illustrate some of the basic cases which can arise in more complicated situations, and suggest ways in which their treatment can be some extent streamlined. We shall use Lemmas 1.1, 1.2, 1.3 freely. In particular, if G is a homogeneous tripartite graph, then all the bipartite restrictions must themselves be homogeneous, and hence are already known.

Let us call the three parts V_0, V_1, and V_2. We can list all the possibilities for the three bipartite restrictions. By Lemma 1.3 we can restrict to the case of reduced graphs on at most 3 parts, where the relations are generic, empty, or complete. We could also eliminate 'complete' in favour of 'empty' by using Lemma 1.2, but we don't do this, since this is taken care of anyhow by the general strategy (viewing 'empty' as minimally omitting a single edge, and 'complete' as minimally omitting a single non-edge).

The first lemma illustrates a point which will recur throughout, so we prove the general case.

Lemma 4.1. *If A is a finite graph which is minimally omitted by some countable reduced homogeneous multipartite graph G, then all bipartite restrictions of A are empty or complete.*

Proof: Suppose that the bipartite restriction of A to V_iV_j is neither empty nor complete. First assume that every member of V_j is either joined to all members of V_i, or is joined to no member of V_i. Since V_iV_j is neither empty nor complete, there must be vertices of both kinds in V_j. Let $y \in V_j$ be joined to all members of V_i, and let $z \in V_j$ be joined to no member of V_i. Now pick any $x \in V_i$, and then xy is an edge, and xz is a non-edge. If the assumption is however incorrect, then there is some $x \in V_j$ joined to some member y of V_i, and not joined to some other member z of V_i. Thus in each case, xy is an edge, xz is a non-edge, and x lies in one of V_i, V_j, and y, z lie in the other. For ease we treat the case where $x \in V_i$.

Since A is minimally omitted, both $A - \{y\}$ and $A - \{z\}$ are realized in G. By homogeneity of G, they may be embedded so that the embeddings agree on $A - \{y, z\}$. Since y and z are differently joined to x, they are not identified in this embedding, and so A is embedded, which is a contradiction. □

Theorem 4.2. *A countable tripartite graph G is homogeneous if and only if all its bipartite restrictions are homogeneous, and it takes one of the following forms, or is obtained from one of them by replacing some or all of the relations by their complement:*

(i) at least two bipartite restrictions are empty,

(ii) all bipartite restrictions are perfect matchings and all cycles are of length 3 (so it is a disjoint union of triangles),

(iii) one bipartite restriction, V_0V_1 say, is a perfect matching, and the other two are generic, and for $a \in V_0$, $b \in V_1$, and $c \in V_2$ such that a and b are matched, ac is an edge if and only if bc is an edge,

(iv) the Fraïssé limit of the class of all finite tripartite graphs,

(v) the Fraïssé limit of the class of all finite tripartite graphs omitting a triangle with a vertex in each part,

(vi) the Fraïssé limit of the class of all finite tripartite graphs with V_iV_j empty, for some fixed $i \neq j$.

We remark that the list given in [8] and [14] appears rather longer, but the extra structures are obtained from the list given here by suitable complementation. Thus a tripartite graph with all parts of size 2 and which forms a 6-cycle (all bipartite restrictions perfect matchings) arises from (ii) by taking the complement of one of the bipartite restrictions (the point is that the complement of a perfect matching between 2-element sets is also a perfect matching, and this is the only cardinality where this applies). By taking complements in (v) we get a total of 8 possible graphs which may be omitted (triangles with any combinations of edges or non-edges). By taking complements in (iii) we get two versions, where the induced map from $V_0 \cup V_2$ to $V_1 \cup V_2$ is an isomorphism or anti-isomorphism, and by taking complements in (vi) we also get a Fraïssé limit where V_iV_j is complete.

Proof: We first remark that all graphs in the list are homogeneous. This follows by direct arguments (clause (i)), or by appeal to Lemma 1.3 (clauses (ii), (iii)) or Lemma 1.2 (the complemented versions) or Fraïssé's Theorem (clauses (iv), (v), (vi)). In these last cases we have to verify that the given class is an amalgamation class. Let A, B, C lie in the given class where $A = B \cap C$, and let the amalgam be $B \cup C$ with no new edges. Then in each case, $B \cup C$ lies in the corresponding class. For instance, for (v), any triangle of $B \cup C$ would already have to lie entirely in either B or C, since there are no new edges, contrary to these being triangle-free.

Conversely, let G be a countable homogeneous tripartite graph with parts V_0, V_1, and V_2, and we shall show that G is of one of the forms listed, possibly after replacing one or more of its relations by its complement. By Lemma 1.1, all bipartite restrictions are themselves homogeneous, so are empty, complete, a perfect matching or its complement, or generic, and by use of Lemma 1.2 we may suppose that complement of perfect matching does not arise.

In the first case we consider, there is a perfect matching, between V_0 and V_1 say. Then by Lemma 1.3, the map θ which agrees with the perfect matching on V_0 and fixes all points of V_2, is an isomorphism between $V_0 \cup V_2$ and $V_1 \cup V_2$ or its complement. By complementing if necessary we suppose that it is an isomorphism.

If $V_0 \cup V_2$ is empty or complete, then so is $V_1 \cup V_2$ and this is an instance of clause (i). If $V_0 \cup V_2$ is a perfect matching, then so is $V_1 \cup V_2$. Then for $x \in V_0$ and $z \in V_2$, x is joined to $z \Leftrightarrow \theta(x)$ is joined to z, and this ensures that G is of the form given in (ii). Similarly, if $V_0 \cup V_2$ is generic, then we obtain clause (iii).

Now suppose that there is no perfect matching, in which case G is reduced, meaning that the only bipartite restrictions are empty, complete, or generic. Let $O(G)$ be the family of minimally omitted finite graphs. By Lemma 1.4, two countable homogeneous m-partite graphs are isomorphic if and only if they minimally omit the same class of finite structures, so we must now investigate the possibilities for $O(G)$.

Any $A \in O(G)$ is defined on either two or three parts. If two, then let them be V_0 and V_1 say. Then the relation between V_0 and V_1 cannot be generic, so it is either empty or complete. By taking complements if necessary, we suppose it is empty. If V_0V_2 or V_1V_2 is also empty or complete, then (after possibly taking complements) this is an instance of (i). If both are generic then we show that G is the Fraïssé limit of the class of all finite tripartite graphs with V_0V_1 empty. For this, let B be any such. Then by genericity of the bipartite restrictions V_0V_2 and V_1V_2, the corresponding restrictions of B can both be embedded into G, and by homogeneity, we may suppose that the embeddings agree on V_2. Since V_0V_1 is empty, the union of the two embeddings is an embedding of B into G. This gives (vi) in the list.

Now suppose that any $A \in O(G)$ is defined on all of V_0, V_1, and V_2, and we show that there is some monic in $O(G)$. For this, we choose $A \in O(G)$ to have the smallest number of vertices, and we show that it is monic. Suppose for a contradiction that a_1 and a_2 are distinct members of some V_i^A, say $i = 0$. Let $b \in V_1^A$, $c \in V_2^A$, and let A' be formed from A by adding a new vertex b' to V_1 which is joined to a_1 but not a_2 and is joined to c if and only if b is *not* joined to c. Then $A' - \{a_1, b\}$ and $A' - \{a_1, b'\}$ are of size strictly less than A, so by minimality of $|A|$, they are each realized in G. Embed them both in G, and by homogeneity, assume that the embeddings agree on their intersection. Since b and b' are differently joined to c, their images under the embeddings must be distinct, so we have embedded $A' - \{a_1\}$ into G. Similarly we may embed $A' - \{a_2\}$ into G, and again we may suppose that the embeddings agree on $A' - \{a_1, a_2\}$. Since a_1 and a_2 are differently joined to b', they are not identified, so A' is embedded in G, and so is A, which is a contradiction.

Thus $O(G)$ contains a monic graph on three vertices (and none on two), which by taking suitable complements we may assume is a triangle, and we shall show that G is the Fraïssé limit of the class of all finite triangle-free tripartite graphs. Let B be finite triangle-free, and we show that B can be embedded in G. By Lemma 4.1, we may assume that each bipartite restriction of B is empty or complete. Since no finite bipartite graph is omitted, each bipartite restriction of G is generic. Hence if $V_i^B = \emptyset$ for some i, then B can be embedded in G, so from now on we suppose that $V_i^B \neq \emptyset$ for each i. Let $|V_i^B| = m_i$.

Case 1: B has no edges.

By genericity of V_0V_1 there are distinct $x_1, \ldots, x_{m_1} \in V_0$ and distinct $y_1, \ldots, y_{m_2}, y_{m_2+1} \in V_1$ such that x_iy_j is a non-edge for each i, and $j \leq m_2$ and $x_iy_{m_2+1}$ is an edge for each i. Similarly there are distinct $y'_1, \ldots, y'_{m_2}, y'_{m_2+1} \in V_1$ and distinct $z_1, \ldots, z_{m_3} \in V_2$ such that y'_jz_k is a non-edge for $j \leq m_2$ and $y'_{m_2+1}z_k$ is an edge for each k. By homogeneity of G we may suppose that $y_j = y'_j$ for each j. Then as $x_iy_{m_2+1}$ and $y_{m_2+1}z_k$ are edges, it follows since G is triangle-free that x_iz_k is a non-edge for each i, k. Hence $\{x_1, \ldots, x_{m_1}, y_1, \ldots, y_{m_2}, z_1 \ldots z_{m_3}\} \cong B$.

Case 2: Just one bipartite restriction of B is complete, $V_0^BV_1^B$ say.

This time find distinct $x_1, \ldots, x_{m_1}$ and $y_1, \ldots, y_{m_2}, y_{m_2+1}$ such that all x_iy_j are edges, and $y'_j = y_j$ and z_k so that y'_jz_k are non-edges for $j \leq m_2$ and $y'_{m_2+1}z_k$ are edges. Since $x_iy_{m_2+1}$ and $y_{m_2+1}z_k$ are edges, x_iz_k are non-edges, so $\{x_1, \ldots, x_{m_1}, y_1, \ldots, y_{m_2}, z_1 \ldots z_{m_3}\} \cong B$.

Case 3: Just one bipartite restriction of B is empty, $V_0^BV_2^B$ say.

Choose $x_i \in V_0$ and $y_j \in V_1$ so that x_iy_j are all edges and $y'_j = y_j \in V_1$ and $z_k \in V_2$ so that all y_jz_k are edges. Since G is triangle-free, no x_iz_k is an edge, so again $\{x_1, \ldots, x_{m_1}, y_1, \ldots, y_{m_2}, z_1 \ldots z_{m_3}\} \cong B$.

Thus we have clause (v) on the list.

Finally suppose that $O(G)$ is empty. Then no finite graphs are minimally omitted, so none are omitted at all, and hence all are realized. Hence the age of G equals the class of all finite tripartite graphs, and G is the Fraïssé limit of the class of all finite tripartite graphs, and this is clause (iv) in the list. □

5 Countable homogeneous quadripartite graphs

In this section we consider the countable homogeneous quadripartite graphs. This introduces all the main features which are used in analyzing the general case of multipartite graphs, and it turns out that the essential new complication over the tripartite case, the so-called 'omission quartets', represents all that can happen, even when the number of parts is increased beyond 4. Following what we have shown in section 3, we restrict to reduced graphs.

Now guided by the tripartite case we can see that we at once expect there to be analogues of the various types of generic graph that arose there obtained by omitting a monic tripartite graph. Specifically, for any monic quadripartite graph, the finite graphs omitting it will form an amalgamation class, and this gives us 108 possibilities straight away corresponding to the monic graphs on 2, 3, or 4 vertices.

To analyze the more general situation, if G is reduced m-partite graph, we can consider the family $O(G)$ of all the finite graphs that it minimally omits. In the cases just mentioned, $O(G)$ has just one member, and in the tripartite case, as we saw, that is all that can happen. When we have more parts, $O(G)$ may have more than one member. Now we shall show in section 6 that for a reduced graph, all members of $O(G)$ are monic, so in our discussion now we restrict to the monic case. In the simplest situation where $O(G)$ has more than one member, any two members of its members will agree on any shared edge type. This means that if they are both have vertices in V_i and V_j then they either both have an edge between V_i and V_j , or they both have a non-edge. (Note that this makes sense since they are assumed monic.) For instance, they could both be triangles on sets of parts having just two parts in common. It is easy to verify that for any family F of pairwise non-embeddable monics such that any two agree (on edge types where they are both defined) the family of finite m-partite graphs omitting all members of F is an amalgamation class, and the class of graphs minimally omitted in the corresponding Fraïssé limit is precise equal to F. Even this situation cannot occur for less then 4 parts, but it is nevertheless straightforward, and easy to describe.

The more interesting case is when there are members of $O(G)$ that differ on some shared edge type, and this section begins with an important example where this happens. This configuration is referred to as an 'omission quartet', and it comprises 4 triangles defined on distinct sets of parts, such that any two differ on their shared edge type. The existence of omission quartets is the first hint that a generalization of the classification to an arbitrary number of parts may not be entirely straightforward.

We have already explained what it means for two graphs to 'differ' on some edge type. Related to this we also have the following definition. Let G be a reduced graph which minimally omits a set of finite graphs $O(G)$. An omitted graph $A \in O(G)$ is said to be *differing* if it differs from some (other) member B of $O(G)$. This means that there is some bipartite restriction V_iV_j where A and B are both defined and one has an edge and the other has a non-edge. Otherwise A is called *non-differing.* We note that since we are in the monochromatic case, the edge type of a graph $A \in O(G)$ on a bipartite restriction is either 'edge' or 'non-edge'. Since the definition of differing is restricted to minimally omissible graphs, all the graphs have only complete or empty bipartite restrictions (as we proved in Lemma 4.1) and thus the edge type is unambiguous for any bipartite restriction. (Actually the definition is restricted to monics, but we don't officially know that yet—see Theorem 6.2.)

An *omission quartet* in a reduced m-partite graph G is a family of four monic tripartite graphs, such that there are four parts $V_{i_0}, V_{i_1}, V_{i_2}, V_{i_3}$ for which the tripartite graphs are defined on $V_{i_0}V_{i_1}V_{i_2}$, $V_{i_0}V_{i_1}V_{i_3}$, $V_{i_0}V_{i_2}V_{i_3}$ and $V_{i_1}V_{i_2}V_{i_3}$, respectively, which agree (if defined) on $V_{i_0}V_{i_1}$, $V_{i_1}V_{i_2}$, $V_{i_2}V_{i_3}$, $V_{i_3}V_{i_0}$, and differ on $V_{i_0}V_{i_2}$ and $V_{i_1}V_{i_3}$. See figure 1.

Lemma 5.1. *The family of quadripartite graphs omitting all members of an omission quartet forms an amalgamation class.*

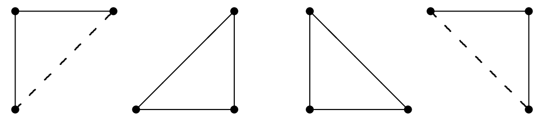

Figure 1: Example of an omission quartet.

Proof: For ease, suppose that the omission quartet is on V_0, V_1, V_2, V_3 and the members of the quartet agree on V_0V_1, V_1V_2, V_2V_3, V_3V_0, and differ on V_0V_2 and V_1V_3. Let the quartet be $\{T_0, T_1, T_2, T_3\}$ defined on $V_0V_1V_2$, $V_0V_1V_3$, $V_0V_2V_3$, and $V_1V_2V_3$, and for definiteness, suppose that in T_0 V_0V_2 has an edge but in T_2 it has a non-edge, and in T_1 V_1V_3 has an edge but T_3 has a non-edge.

To verify the amalgamation property, let A, B, and C be quadripartite graphs such that $A = B \cap C$, and each of B and C is a 1-point extension of A (which we know suffices to establish amalgamation in general). Let $B = A \cup \{x\}$ and $C = A \cup \{y\}$. The amalgam is taken to be $B \cup C$, and we just have to say how to join x and y so that no element of the quartet is realized. Let $x \in V_i$ and $y \in V_j$.

In the first case, $i = j$, in which case no decision is required. The next easiest case is where the T_r agree on the edge type of V_iV_j, in which case we decide to join x and y in the 'opposite' way (that is, we join x and y if there is an edge between V_i and V_j in the T_r, and otherwise we do not join them). Suppose therefore that two members of the quartet differ on V_iV_j. Without loss of generality, suppose that $i = 0$ and $j = 2$. First try joining them in $B \cup C$. If this omits all members of the quartet, then we have finished. If not, then some T_r becomes realized on making this decision, and the only one with an edge between V_0 and V_2 is T_0, so there is $z \in V_1$ such that $\{x, y, z\}$ is a copy of T_0. We next try not joining x and y, and this succeeds unless we thereby realize a copy of T_2, in which case there is $t \in V_3$ such that $\{x, y, t\}$ is a copy of T_2. Since $B \cap C = A$, $z, t \in A$. Hence $x, z, t \in B$, and the edge types of xz and xt are as agreed in the quartet. Since T_1 is omitted by B, zt is a non-edge in B, and hence also in A. Applying a similar argument to $y, z, t \in C$, we find that as T_3 is omitted, zt must be a non-edge in C, and hence also in A. This contradiction proves the result. $\square$

In the remainder of this section we present the classification of the countable homogeneous quadripartite graphs. This is a special case of the classification of the countable homogeneous m-partite graphs, and we concentrate on the particular features of the quadripartite case, quoting two crucial results, Theorem 6.2 and Lemma 6.5 from the next section, to avoid unnecessary repetition.

Lemma 5.2. *In any countable reduced homogeneous quadripartite graph G, any two differing members of $O(G)$ belong to an omission quartet, and if this arises, $O(G)$ has size 4 (and so consists precisely of that omission quartet).*

Proof: We begin by remarking that no bipartite member A of $O(G)$ can differ from another member B of $O(G)$. For if this happens, suppose that A is defined on V_0V_1, and let b_0 and b_1 be the elements of B lying in V_0 and V_1. As B is minimally omitted, $B - \{b_0\}$ and $B - \{b_1\}$ both embed in G, and by homogeneity they can be embedded so that the embeddings agree on their intersection. Since B is omitted, under this embedding, b_0b_1 cannot agree with B, so it agrees with A, and A is realized after all (using Theorem 6.2, which tells us that A is monic, or Theorem 2.2 in fact suffices).

Next, by Lemma 6.5, no differing member of $O(G)$ can be defined on all 4 parts. So if we suppose that A and B are differing members of $O(G)$, it follows that they are 3-monics. From the tripartite case we know that it is not possible for two omitted 3-monics to be defined on the same parts. Therefore, there can be at most four different omitted 3-monics. Suppose that A and B are defined on $V_0V_1V_2$ and $V_0V_2V_3$ respectively, so that they differ on V_0V_2. We define a certain (monic) quadripartite graph H having vertices x_0, x_1, x_2 and x_3 in the corresponding parts. The edge type x_0x_2 is as yet undefined. Apart from this, the edge types in $\{x_0, x_1, x_2\}$ and $\{x_0, x_2, x_3\}$ are as in A and B respectively, and it remains to decide x_1x_3.

Suppose first that any two members of $O(G)$ that are defined on V_1V_3 agree there. Then we can let x_1x_3 disagree with all members of $O(G)$. It follows that the restrictions of H to $V_0V_1V_3$ and

$V_1V_2V_3$ (which *are* defined, even though H is not yet defined on V_0V_2) are realized in G. Supposing that they have both been embedded, by homogeneity, we may suppose that the embeddings agree on $\{x_1, x_3\}$, and so H has been embedded. Under this embedding, x_0x_2 is decided, and so either A or B has been embedded in G after all, which is a contradiction.

The conclusion is that there must be two members C and D of $O(G)$ which differ on V_1V_3. As we have remarked, they cannot be defined on 2 or 4 parts, so they are both 3-monics, and are defined on $V_0V_1V_3$ and $V_1V_2V_3$. We have shown that there are members of $O(G)$ defined on each of $V_0V_1V_2$, $V_0V_1V_3$, $V_0V_2V_3$ and $V_1V_2V_3$. It follows that $O(G)$ contains no 4-monics, for any such could not differ from any member of $O(G)$, as already shown, so would contain each of the omitted 3-monics, and so would not be *minimally* omitted. We deduce that $O(G) = \{A, B, C, D\}$, and to show that $\{A, B, C, D\}$ is an omission quartet, it remains to verify that the *only* pairs that differ are A/B and C/D. For this it suffices, as a typical case, to show that A and C agree on V_0V_1. Again we build a suitable quadripartite graph K having vertices y_0, y_1, y_2 and y_3 in the corresponding parts. We leave y_0y_2 as yet undefined. Apart from this, $y_0y_1y_2$ is as in A, $y_0y_2y_3$ is as in B, and y_1y_3 is as in C. Then as C and D differ on V_1V_3, y_1y_3 differs from D, and it follows that $K - \{y_0\}$ can be embedded in G (since it is a 3-monic defined on the same parts as D but differing from D). Suppose for a contradiction that $K - \{y_2\}$ can also be embedded in G. Then using homogeneity we may assume that the two embeddings agree on their intersection, and this then embeds the whole of K into G. Since the edge type of y_0y_2 is now determined, we must have embedded either A or B, which is impossible. The conclusion is that $K - \{y_2\}$ cannot be embedded in G, so it is omitted. Since it is defined on the same parts as C, it must therefore agree with C, and hence A and C agree on V_0V_1 as desired. □

Theorem 5.3. *A countable quadripartite graph G is homogeneous if and only if it takes one of the following forms, or is obtained from one of them by renumbering its parts and/or replacing some or all of the relations by their complements:*

(i) V_0V_1 *is a perfect matching* θ*, and* θ *induces an isomorphism between the restrictions to* $V_0V_2V_3$ *and* $V_1V_2V_3$*, and these are in the tripartite list (from Theorem 4.2),*

(ii) all bipartite restrictions are empty, complete, or generic, and there is a family $\mathcal{F}$ *of monic graphs which is either an omission quartet, or no two of its elements differ, such that G is the Fraïssé limit of the class of all finite quadripartite graphs omitting all members of* $\mathcal{F}$*.*

We remark that in a sense this classification is less explicit than that given in Theorem 4.2. On the other hand, it could easily be made so, and would then be very long. As it stands, we have a version which it is possible to generalize to a greater number of parts.

6 Non-monic realization and non-complication theorems

This section is devoted to the proofs of the two main theorems, conjectured in [8] and proved in [14], which enable us to complete the classification. These are referred to as the 'non-monic realization theorem' and the 'non-complication theorem'.

It is clear from the work presented on the tripartite and quadripartite cases, that a key point is to discuss which finite structures can be minimally omitted in some reduced homogeneous structure. We saw that in those two cases, any such minimally omitted graph is monic. It was conjectured that this continues to be true on more parts, and this is proved in Theorem 6.2.

The other thing is to establish that the situation with regard to omission does not get any worse than it already is in the quadripartite case, where we have seen that there can be omission quartets. This is called the non-complication theorem, Theorem 6.9. The very rough intuition is that for more than 4 parts, there is enough 'room' to amalgamate the relevant structures so as to omit what is required, except for configurations that live in some quadripartite restriction.

It turns out that omission quartets can arise in more complicated configurations, which we call *omission families*, on more than four parts, as explained in Theorem 7.3; their existence does not however impede the proof of the main result, and they only play a minor part in the proof of Lemma 6.8, so we only prove directly in Lemma 6.8 what is actually required, and give a more thorough treatment in section 7, since they are needed for a fuller understanding of the possible structure of $O(G)$.

The following result is needed in the proof of the non-monic realization theorem.

Lemma 6.1. *If all monic graphs are realized in a countable reduced homogeneous m-partite graph, then all finite graphs are realized.*

Proof: The result for bipartite graphs is clear from the results of section 3, where they were all classified, so we suppose that $m \geq 3$, and suppose for a contradiction that there is some m-partite reduced homogeneous graph G in which all monics are realized, but not all finite graphs on the m parts. Let $\mathcal{X}$ be the set of finite graphs on the m parts which are omitted by G, and which have the smallest possible number of vertices subject to this (which in particular implies that they are minimally omitted). Since G realizes all monics, all members of $\mathcal{X}$ are non-monic, so have some over-populated part (by which is meant a part having more than one vertex). Let H be a member of $\mathcal{X}$ having the largest over-populated part, V_0 say. We consider two cases, each of which leads to a contradiction.

Case 1: H has a second overpopulated part V_1 say. Let x and y be distinct members of V_1^H, and let H^+ be formed from H by adding a new vertex z to V_0 joined to x but not to y (and arbitrarily joined to the other points). Let H_1 and H_2 be obtained from H^+ by removing x and y respectively. Then $|H_1| = |H_2| = |H|$, and H_1 and H_2 each have one more point in V_0 than H does, so by choice of H, $H_1, H_2 \notin \mathcal{X}$ and so they are realized in G. Thus we may suppose that they are subgraphs of G, and by homogeneity, we may also suppose that they agree on their intersection. Since x and y are differently joined to z, they are unequal under this identification, and so we have embedded H^+ in G. Hence H also embeds in G, which is a contradiction.

Case 2: V_0 is the only over-populated part of H. This time pick distinct x and y in V_0^H, and let H^+ be formed from H by adding a new vertex z to some other part V_1, which in H has just one element u say. Let z be joined to x and not to y, and for some other part V_2 (which exists since $m \geq 3$), let z be joined to its (unique) member v in the opposite way to u. Thus z is joined to v if and only if u is *not* joined to v. Again let $H_1 = H^+ - \{x\}$ and $H_2 = H^+ - \{y\}$, which have the same size as H. Then $H_1 - \{u\}$ has size less than $|H|$, so is realized in G, and so is $H_1 - \{z\}$. By homogeneity they may be embedded in G to agree on their intersection, and because u and z are differently joined to v, they are distinct under this embedding. So H_1 is embedded in G. Similarly H_2 can be embedded in G, and the same argument as before shows that their union H^+, and hence H, is embedded in G, which is a contradiction. $\Box$

Theorem 6.2 (non-monic-realization)**.** *In a countable reduced homogeneous m-partite graph, all minimally omitted graphs are monic.*

Proof: Suppose not, and take the least m for which there is a counter-example, G say. Let A be one of the non-monics minimally omitted by G. Then if A is defined on fewer than m parts, the restriction of G to those parts minimally omits A, contrary to minimality of m. Hence all minimally omitted non-monics are defined on all m parts. By Lemma 6.1, some monic B is also omitted. By Lemma 4.1, all bipartite restrictions of A are empty or complete. Note that as A is minimally omitted, B must differ from A on some bipartite restriction.

Now choose a non-monic minimally omitted graph A, and a monic minimally omitted graph B, such that B differs from A on the least possible number of edge types, and subject to this, some bipartite restriction of A on which it differs from B, has the least possible number of vertices. For ease let us number the parts so that this bipartite restriction is V_0V_1, and so that B is defined on all V_i for $0 \leq i < r$. We remark that $r \geq 3$. For if $r = 2$, B just consists of a single edge or non-edge between V_0 and V_1, suppose the former. Since A and B differ on V_0V_1, the restriction of A to V_0V_1 is empty. By minimality of A, both $A - V_0^A$ and $A - V_1^A$ are realized in G, and by homogeneity we may suppose that they are embedded to agree on their intersection. Since B is omitted, there is no edge between V_0^A and V_1^A under this embedding, which agrees with how they are related in A, and we have succeeded in embedding A in G, which is a contradiction.

We now describe a graph H defined from A and B. This has the same vertices as A, together with an extra vertex in each V_i for $2 \le i < r$, written as b_i where this is the (unique) member of V_i^B. We also choose $a_0 \in V_0^A$ and $a_1 \in V_1^A$. The edge types of H are chosen so that $a_i b_j$ for $0 \le i \le 1$, $2 \le j < r$, and $b_i b_j$ for $2 \le i < j < r$ agree with B; all other edge types agree with A, except for $a_0 a_1$, which is undefined (the idea being that a_0 and a_1 will play the role of members of either A or B, and so will have ambiguous status, until it is decided how to embed H in G).

Let $H_0 = H - \{a_0\}$ and $H_1 = H - \{a_1\}$. Since H_0 and H_1 each exclude one point of the undefined edge type, they are both completely defined m-partite graphs. We aim to show that both H_0 and H_1 embed in G. Suppose not, and for instance that G omits H_0. Then there is a subgraph A' of H_0 which is minimally omitted by G. First we show that A' is non-monic. For if A' is monic, then on any edge type where A' differs from A, it must agree with B (since all edge types of H_0, and hence also of A', agree with either A or B). Between V_0 and V_1 however, A and B differ, but in H_0 (and hence in A') all edge types between V_0 and V_1 agree with A. Hence A' differs from A on strictly fewer edge types than B does, contrary to the choice of B.

Hence A' is non-monic. In particular this means that it is defined on all m parts, and all its bipartite restrictions are empty or complete. If A' does not differ from A on any edge type, then it differs from B on exactly the same edge types as A does, and so if we start with A' instead of A, we can take the same B, but as $a_1 \notin A'$, $|V_0^{A'} \cup V_1^{A'}| < |V_0^A \cup V_1^A|$, contrary to the choice of A. Hence A' differs from A on some edge type. Since every edge type of H_0, and hence also A', agrees with A or B, A' differs from B on strictly fewer edge types than A does, which again contradicts the choice of A.

Our conclusion is that G realizes H_0. Similarly it realizes H_1. By homogeneity we may embed them both into G so that the embeddings agree on $H_0 \cap H_1$. Under this embedding, a_0 and a_1 are either joined or not joined. One of these options agrees with A, and the other agrees with B, and it follows that either A or B embeds in G, which is a contradiction. □

The last and most challenging task is to verify that we cannot minimally omit any differing graphs, apart from omission quartets. By Theorem 6.2, we can restrict ourselves to monics. To prepare for the main 'non-complication theorem' 6.9, we give a series of lemmas.

Let us say that a reduced homogeneous m-partite graph is *minimal* if two members of $O(G)$ differ and do not lie in an omission quartet, and m is the least number for which this can hold.

Lemma 6.3 (all sets proposition)**.** *If G is a minimal countable reduced homogeneous m-partite graph, and $A, B \in O(G)$ differ and do not lie in the same omission quartet, then*
(i) all m parts have points in either A or B,
(ii) $|A| + |B| \ge m + 2$.

Proof: If V_i is a part on which neither A nor B is defined, then $G - V_i$ is reduced homogeneous $(m-1)$-partite, contrary to the minimality of G, giving (i).

For (ii), since A and B differ on some edge type, there are two parts they are both defined on. Since between them they are defined on all parts, $|A| + |B| \ge m + 2$. □

In what follows, we shall usually argue by contradiction, and show that if what we are trying to establish fails, then the multipartite graph we are considering is not homogeneous. For this purpose, a 'partially defined' finite multipartite graph H is produced, from given differing A, B, in the same style as in the proof of Theorem 6.2. By this we mean that although we know what the *vertices* of H are, not all its edges or non-edges are yet determined. During the proof, all except one of the edges/non-edges will be chosen (the one not chosen is the edge type on $V_0 V_1$), thus forming a partially defined graph H^* extending H and on the same vertex set. The subgraphs $H_0 = H^* - V_1$ and $H_1 = H^* - V_0$ will be totally defined. A key point will be to ensure that the edges or non-edges are chosen to form H^* in such a way that all graphs in $O(G)$ are omitted by both H_0 and H_1. However, this only needs to be done explicitly for certain members of $O(G)$, because some will be omitted *automatically*, and so need not be considered. In particular, we do not need to consider graphs defined on both V_0 and V_1. The most important of these to consider are those which lie in omission quartets not defined on either V_0 or V_1, or which are defined on V_0 but not V_1 (which are said to be of *type 0*), or which are defined on V_1 but not V_0 (which are *type 1* graphs). We write

$O(G)_H$ for the set of members of $O(G)$ not defined on both V_0 and V_1, and which are *compatible* with H, meaning that there is *some* choice of H^* they embed into.

Whenever we refer to 'type 0' or 'type 1', it is assumed that we have some choice of A and B in mind, and that $H = H(A, B)$ is given as above, and these notions and $O(G)_H$ are given relative to this particular H.

Lemma 6.4 (same type proposition)**.** *If G is a countable minimal reduced homogeneous m-partite graph, then members of $O(G)_H$ that have the same type cannot differ if they do not belong to the same omission quartet.*

Proof: Suppose for instance that they are of type 0. Then they are not defined on V_1, and so by Lemma 6.3 and the minimality of G, cannot differ, unless they belong to the same omission quartet. □

To set the scene a little more, let us suppose that $A, B \in O(G)$ differ on the edge type V_0V_1, and between them they are defined on all m parts, and we define $H = H(A, B)$ to have vertices c_0, c_1, c_i^A, c_i^B for each $i \geq 2$ such that both A and B are defined on V_i, a_i for i such that just A is defined on V_i, and b_i for i such that just B is defined on V_i. The subscript tells us which part each element lies in, and apart from c_0c_1, edge types involving c_0, c_1, c_i^A, a_j are as in A, and those involving c_0, c_1, c_i^B, b_j are as in B. All other edge types are so far undefined. We let $\mathcal{V}_A$, $\mathcal{V}_B$ be the sets of parts that just A, or just B, is defined on respectively, and parts other than V_0 and V_1 that they are both defined on are called *overlapping*. If $C \in O(G)_H$, we also write $\mathcal{C}_A$ and $\mathcal{C}_B$ for the sets of parts in $\mathcal{V}_A$, $\mathcal{V}_B$ respectively on which C is also defined.

There are three kinds of edge type which need to be decided in order to define H^* from H. They are of the forms a_ib_j, called *free* edge types, $c_i^A b_j$ or $c_i^B a_j$, called *semifree*, and $c_i^A c_j^B$ for $i \neq j$, called *odd*. Most of the time, it suffices to assign the free edge types; in two cases (Lemma 6.7 Cases 2 and 3B) we also need semifree edge types; but odd edge types are never needed, so can be assigned arbitrarily. Free edge types a_ib_j are the most useful, since this is the only edge or non-edge between V_i and V_j. For semifree edge types, $c_i^A b_j$ or $c_i^B a_j$ are not the only edge or non-edges, since we already have $c_i^B b_j$, $c_i^A a_j$, so any argument about these has to focus on the points in the parts rather than just the parts themselves.

If a type 0 graph not lying in any omission quartet is defined on a free edge type, then by Lemma 6.4 any two such type 0 graphs agree there, and so they can all be omitted by choosing the opposite edge type. We say that we choose the edge type to be *type 0 omitting*. Similarly we can talk about choosing an edge type as *type 1 omitting*.

As the construction proceeds, some of the graphs in $O(G)_H$ will have been omitted, others not. We use the notation $O(G)_H^c$ for the family of members of $O(G)_H$ which have still not been omitted at some intermediate stage ('c' for 'current'). A key idea is that the situation can change for members of $O(G)_H$ during the construction, even for ones which remain in $O(G)_H^c$. For instance, it is possible that $C \in O(G)_H^c$ differs as a member of $O(G)_H$, but not as a member of $O(G)_H^c$ (as all graphs that it differed from may have been removed).

One type of assignment of edge types can be carried out at any stage without any detriment (meaning that such an assignment will never reduce the possibilities in the future for omitting graphs by choosing edge types—the only difficult decisions are where two members of $O(G)_H^c$ *disagree* about an edge type). Namely, if there is a free edge type a_ib_j that has not yet been decided, and at least one member of $O(G)_H^c$ is defined on V_iV_j, and any two such members of $O(G)_H^c$ agree on V_iV_j, then these can all be omitted by choosing the opposite edge type to all of them. So we make such decisions for all such edge types, and this reduces the set $O(G)_H^c$. Making these choices may ensure that other undecided members of $O(G)_H$ which previously differed from some other undecided member no longer differ and so we may be able to repeat the process, though only finitely many times. We describe this process as *omitting agreed edge types*, which will be done at various stages in the construction.

Lemma 6.5 (m-monic omission)**.** *Suppose that G is a countable reduced homogeneous m-partite graph with $m \geq 4$. Then any differing $A \in O(G)$ has $< m$ vertices.*

Proof: Suppose for a contradiction that $A \in O(G)$ having m vertices differs from some member B of $O(G)$. Choose such B which differs from A on the least number of edge types, and let A and B differ on V_0V_1 (possibly elsewhere) and suppose that B is defined on parts $V_0, V_1, \ldots, V_{r-1}$. Consider $H = H(A, B)$ as defined above, having $m+r-2$ (distinct) vertices $c_0, c_1, c_2^A, \ldots, c_{r-1}^A, c_2^B, \ldots, c_{r-1}^B, a_r \ldots,$ a_{m-1}, where c_0c_1 is as yet undefined, and apart from that, all edge types of $c_0, c_1, c_2^B, \ldots, c_{r-1}^B$ are as in B, and all edge types of $c_0, c_1, c_2^A, \ldots, c_{r-1}^A, a_r, \ldots, a_{m-1}$ are as in A. To form H^* we just let all so far undefined edge types except for that of c_0c_1 be as in A.

Now although H^* is not fully defined, its restrictions H_0 and H_1 *are*. Suppose that both of these graphs are realized in G. Then they can both be embedded, and by homogeneity of G, we may suppose that the embeddings agree on their intersection. Under the resulting embedding, the edge type of c_0c_1 becomes defined. Since there are only two possibilities for it (edge or non-edge), it must agree with either A or B (which differ there). This realizes either A or B in G as $c_0, c_1, c_2^A, \ldots, c_{r-1}^A, a_r, \ldots, a_{m-1}$ or as $c_0, c_1, c_2^B \ldots, c_{r-1}^B$, which is a contradiction. Hence either H_0 or H_1 is omitted by G, suppose the former for instance. Let $C \subseteq H_0$ be minimally omitted. Since C is defined on at most $m-1$ parts, and A is *minimally* omitted, C cannot be isomorphic to a subgraph of A. Hence C differs from A on some edge type. However, on any edge type where A and C differ, C must agree with B (since all edge types were in agreement with either A or B) and so A and B also differ there. Hence the set of edge types where A and C differ is a proper non-empty subset of the set of those where A and B differ. This is contrary to the choice of B as differing from A on the least number of edge types. □

Lemma 6.6. *If G is a minimal reduced homogeneous m-partite graph, and $C \in O(G)_H$, then C is defined on members of both $\mathcal{V}_A$ and $\mathcal{V}_B$.*

Proof: Suppose that whenever C is defined on V_i, then so is B. By Lemma 6.5, $|B| \leq m-1$, so by Lemma 6.3, B and C cannot differ (since they clearly cannot be differing members of an omission quartet). It follows that C is a subgraph of B, but since C is not defined on both V_0 and V_1, it is a proper subgraph, contrary to B being *minimally* omitted. We deduce that C is defined on some part where B is not defined, and as A and B are between them defined on all parts, it follows that C is defined on some member of $\mathcal{V}_A$. Similarly it is defined on some member of $\mathcal{V}_B$. □

Lemma 6.7 ($(m-1)$-monic omission for $m > 4$)**.** *If $m > 4$, no minimal countable reduced homogeneous m-partite graph G can minimally omit a differing $(m-1)$-monic.*

Proof: Suppose the result is false. By Lemma 6.5, there is no differing member of $O(G)$ on m vertices. We shall choose a differing $(m-1)$-monic graph $A \in O(G)$, and $B \in O(G)$ that differs from A, subject to certain conditions given below. By renumbering the parts if necessary, we suppose that A and B differ on V_0V_1. In terms of these, the partially defined m-partite graph $H = H(A, B)$ is as given above. Since A is an $(m-1)$-monic, $|\mathcal{V}_B| = 1$.

To avoid using too many subsidiary inductions, and redefinitions, we suppose that A fulfils the following special conditions:

(i) some $B \in O(G)$ differs from A on the least number of edge types,

(ii) subject to (i), some $B \in O(G)$ that differs from A on the least number of edge types is as small as possible,

(iii) subject to (i) and (ii), if $C \in O(G)_H$ is a differing $(m-1)$-monic, then we may number the parts (possibly interchanging V_0 and V_1) so that A is defined on $V_0, V_1, \ldots, V_{m-2}$ and C is defined on $V_0, V_2, \ldots, V_{m-1}$ and C does not differ from any 3-monic in $O(G)_H$ defined on V_1, V_2, V_{m-1} where $H = H(A, B)$.

The third condition is rather technical, but it cuts out a 'relabelling' step in one crucial case, and so makes the overall structure of the proof a little easier.

We have to see first that it is possible to choose such A and B. By our assumption, there is a differing $(m-1)$-monic $A \in O(G)$. By definition of 'differing', there is $B \in O(G)$ which differs from A, and we may choose B to differ from A on the least number of edge types, and subject to that to have the least number of vertices. Thus (i) and (ii) are fulfilled. Let A and B differ on V_0V_1. Suppose that clause (iii) is violated, and let C be an $(m-1)$-monic corresponding to its failure. By interchanging V_0 and V_1 if necessary, we suppose that C is a type 0 graph. Then A and C are between them defined on all m parts, so by renumbering, we may suppose that A is defined on $V_0, V_1, \dots, V_{m-2}$ and C is defined on $V_0, V_2, V_3, \dots, V_{m-1}$. By the failure of (iii), for every i between 2 and $m-2$ there is a 3-monic $D_i \in O(G)_H$ defined on V_1, V_i, V_{m-1} which differs from C. Let $A' = C$, $B' = D_2$, and let $H' = H(A', B')$. Then A' is an $(m-1)$-monic, and $B' \in O(G)$ differs from A' on the least number of edge types (just 1) and subject to that, B' is as small as possible (it cannot have size less than 3), so (i) and (ii) are fulfilled for A' and B', from which it follows that B must have been a 3-monic too, and hence $r = 2$ and H is defined on $\{c_0, c_1, a_2, \dots, a_{m-2}, b_{m-1}\}$.

We now see that clause (iii) is also satisfied by A' and B'. To set things up in a similar enumeration to that used for A and B, note that A' is defined on $V_2, V_{m-1}, V_0, V_3, \dots, V_{m-2}$, and B' is defined on V_2, V_{m-1}, V_1. Since we are assuming that we cannot fulfil conditions (i), (ii), and (iii), they must fail also for A' and B', and so there is some $(m-1)$-monic $C' \in O(G)_{H'}$ which is defined on $V_2, V_0, V_3, \dots, V_{m-2}, V_1$, or on $V_{m-1}, V_0, V_3, \dots, V_{m-2}, V_1$, and for every $j \in \{0, 3, \dots, m-2\}$, C' differs from a 3-monic D'_j in $O(G)_{H'}$ defined on V_{m-1}, V_j, V_1 (V_2, V_j, V_1 respectively). The second possibility cannot occur, and this is because if it did, as $m \geq 4$, $D_3 \in O(G)$ is defined on V_1, V_3, and V_{m-1}, so by Lemma 6.3 must agree with C', but then would be a subgraph of C', contrary to C' being *minimally* omitted. Hence C' is defined on $V_2, V_0, V_3, \dots, V_{m-2}, V_1$, which is the same as A, so by Lemma 6.3 again A and C' cannot differ. Hence $C' = A$. Using $m > 4$ again, 3 is a possible value for both i and j, so we may consider D_3 and D'_3. Now A agrees with D_3 on V_1V_3 (since D_3 is compatible with H) and differs from D'_3 there (since $A = C'$), whereas A' agrees with D'_3 on V_3V_{m-1} and differs from D_3 there. It follows that the 3-monics D_3 and D'_3 differ on (at least) two edge types, which is contrary to Lemma 5.2.

This contradiction shows that a suitable A fulfilling the three conditions exists, and we let B correspond to this choice of A. The main task remaining is to show that the so far undefined edge types in H (apart from c_0c_1) can be chosen in such a way that each of $H_0 = H^* - V_1$ and $H_1 = H^* - V_0$ is realized in G. Assuming this can be done, we should then be able to embed each of them in G, and by homogeneity assume that the embeddings agree on their intersection. Under the resulting embedding, the edge type of c_0c_1 is now determined, but this would mean that either A or B was embedded in G, giving a contradiction.

So we examine in turn the members of $O(G)_H$, and ensure that the (so far undefined) edge types of H are chosen so that all members of $O(G)$ are omitted in both H_0 and H_1. If H_i is omitted in G, then there is some minimally omitted subgraph, which by construction cannot be the case (since it would clearly lie in $O(G)_H$), and this would establish that in fact H_i is embedded.

We first remark that $O(G)_H$ contains no omission quartets. For by Lemma 6.6, any differing member of $O(G)_H$ is defined on some member of $\mathcal{V}_B = \{V_{m-1}\}$, so if there is an omission quartet C_1, C_2, C_3, C_4, all C_i would have to be defined on V_{m-1}, contrary to the definition of 'omission quartet'.

We begin by omitting all agreed edge types. This means that from now on, we only need to arrange that members of $O(G)^c_H$ are omitted, which differ from some other member of $O(G)^c_H$.

Consider then some differing $C \in O(G)^c_H$ of greatest possible size. By definition of $O(G)_H$, C is not defined on both V_0 and V_1, and without loss of generality, we suppose that it is not defined on V_1.

Case 1: $|C| = m - 1$ and $|B| = 3$.

Thus C is defined on $V_0, V_2, \dots, V_{m-1}$ and B is defined on V_0, V_1, and V_{m-1}. Therefore A and B do not overlap. Since (i)-(iii) are fulfilled, for some i between 2 and $m-2$, C does not differ from any 3-monic in $O(G)_H$ defined on V_1, V_i, V_{m-1}. We can therefore define a new edge or non-edge extending H by saying that a_ib_{m-1} is joined in the opposite way to C (type 0 omitting). This ensures that C will be omitted. In addition we let all other free edge types be type 1 omitting (or arbitrary if no type 1 graph is defined on them). This stipulation is unambiguous, since by Lemma 6.4, no two members D and E of $O(G)$ of type 1 differ, unless they are members of an omission quartet, which as remarked above does not happen in this case.

Since $|C| = m - 1$, C is the only type 0 graph in $O(G)_H$, and it is omitted. If D is a type 1 graph in $O(G)_H$, then either $|D| = 3$, so it has been omitted explicitly, or $|D| \geq 4$, in which case D is defined on some V_jV_{m-1} for $j \neq i$, $2 \leq j \leq m-2$, and then the construction also omits D.

Case 2: $|C| = m-1$ and $|B| \geq 4$.

This time we have at least one part V_2 say, that A, B, and C are all defined on. We choose the semifree edge types $c_2^A b_{m-1}$ and $c_2^B a_{m-2}$ to be opposite to C, and then C cannot be realized as there is no choice for its vertex in V_2. In this case, C is again the only type 0 graph in $O(G)_H$, so we just have to omit all type 1 members of $O(G)_H$, which we do by choosing all free edge types to be type 1 omitting. By Lemma 6.4 these choices are unambiguous, and by Lemma 6.6 this ensures that all type 0 and type 1 members of $O(G)_H$ are omitted. Any edge types still undefined may be assigned arbitrarily.

Case 3: $|C| < m-1$.

We first remark that by Lemma 6.3, and the fact that there are no omission quartets, all differing members of $O(G)_H$ have at least 4 vertices. By Lemma 6.6, all members of $O(G)_H$ are defined on V_{m-1}, and furthermore, by Lemma 6.3, any two differing members of $O(G)_H$ are between them defined on all V_i for $2 \leq i \leq m-2$.

Case 3A: $|C| < m-1$ and $|B| = 3$.

We just have the free edge types $a_i b_{m-1}$ to decide for $2 \leq i \leq m-2$. For each such i for which C is not defined on V_i (of which there is at least one since $|C| < m-1$) let $a_i b_{m-1}$ be type 1 omitting, and for any j such that C is defined on V_j, let $a_j b_{m-1}$ be chosen to differ from C.

We now show that all differing members of $O(G)_H$ are thereby omitted. First, if differing $D \in O(G)_H$ is of type 1, not omitted, then it cannot be defined on any V_i for $i \geq 2$ on which C is not defined. By Lemma 6.3, D does not differ from C. By Lemma 6.6, there is some j such that $2 \leq j \leq m-2$ and D is defined on V_j. Then D agrees with C on $V_j V_{m-1}$, so has been omitted by construction. If however differing $D \in O(G)_H$ is of type 0, not omitted, then it cannot be defined on any V_j for $2 \leq j \leq m-2$ on which C is defined. Let D differ from $E \in O(G)_H$. Then E is of type 1 and is defined on all such V_j where C is defined, and on V_1, V_{m-1}, and also on some extra part V_i (otherwise it cannot differ from D), and thus $|E| \geq |C|+1$, contrary to maximality of $|C|$.

Case 3B: $|C| < m-1$ and $|B| \geq 4$.

If no differing type 0 graph in $O(G)_H$ is defined on all overlapping parts, then we let all semifree edge types be type 1 omitting, and all free edge types be type 0 omitting. Then all type 0 graphs in $O(G)_H$ are omitted by virtue of the free edge types (and using Lemma 6.6). Let $D \in O(G)_H$ be differing of type 1, and let it differ from E of type 0. Then there is an overlapping part V_i on which E is not defined, so by Lemma 6.3, D *is* defined on V_i, and also (by Lemma 6.6) on some V_j for $r \leq j \leq m-2$ and on V_{m-1}. Hence the choice of semifree edge types also omits D.

A similar argument applies with type 1 in place of type 0.

So now we assume that there are differing type 0 and type 1 graphs D and D' respectively defined on all overlapping parts, and we choose such D and D' of greatest size. Suppose without loss of generality that $|D| \geq |D'|$. We choose all semifree edge types to be type 0 omitting. If $r \leq i \leq m-2$ and D is defined on V_i, let $a_i b_{m-1}$ be chosen opposite to D. Otherwise we let $a_i b_{m-1}$ be type 1 omitting. This omits all differing type 1 graphs E in $O(G)_H$. For by Lemma 6.6, E is defined on V_i and V_{m-1} for some i with $r \leq i \leq m-2$. If E is not yet omitted, then D must also be defined on all such V_i, but as $|D| \leq |C| < m-1$, there is j such that $r \leq j \leq m-2$ and D is not defined on V_j. By Lemma 6.3 we deduce that D and E do not differ, and so E has been omitted by choice of the edge type of $a_i b_{m-1}$ after all.

Finally suppose that E is a differing type 0 graph in $O(G)_H$ which is not omitted. Since it cannot differ from D, it can only be defined on V_{m-1}, and on parts that D is *not* defined on. For if it is defined on some V_i with $r \leq i \leq m-2$ that D is also defined on, then it would be omitted by choice of $a_i b_{m-1}$, and if it is defined on some overlapping part V_i, then it is omitted by choice of the semifree edge types (and appeal to Lemma 6.6). Therefore any type 1 graph differing from E is defined on all parts that D is defined on, apart from V_0. Since it is defined on V_1 and on at least one part other than V_{m-1} where E is defined (to differ from E), it is defined on more parts than E, and on all overlapping parts. This contradicts maximality of D and concludes the proof. □

Lemma 6.8 (3- to $(m-2)$-monic omission)**.** *Any minimal countable homogeneous reduced m-partite* *graph G cannot minimally omit any differing k-monic, for $3 \leq k \leq m-2$, apart from a set of 3-monics which belong to an omission quartet.*

Proof: We again take A to be a largest differing monic in $O(G)$, differing from B. By Lemmas 6.5 and 6.7, $3 \leq |B| \leq |A| \leq m-2$. If $|A| = 3$ then the only differing members of $O(G)$ are 3-monics, and we may see from this that they all lie in omission quartets. For, still calling the two differing monics A and B, by Lemma 1.1, the restriction of G to the set of parts arising in A or B is homogeneous and is tripartite or quadripartite (since A and B share an edge type). It cannot be tripartite by Theorem 4.2, so it is quadripartite, and by Lemma 5.2, A and B lie in an omission quartet.

From now on we therefore suppose that $4 \leq |A| \leq m-2$. By Lemma 6.3, $|B| \geq 4$ too. Applying Lemma 6.3 again, A and B are between them defined on all m parts, and it follows that $|\mathcal{V}_A|, |\mathcal{V}_B| \geq 2$. This extra 'room for manoeuvre' gives us enough space to omit all members of $O(G)_H$ in defining H^* from H.

We first deal with any differing monics in $O(G)_H$ which do not lie in any omission quartet.

Case 1: Some such differing $C \in O(G)_H$ is not defined on either V_i or V_j, where V_i is some member of $\mathcal{V}_A$ and V_j is some member of $\mathcal{V}_B$. Suppose without loss of generality that C is a type 0 graph. Then any member of $O(G)_H$ differing from C is a type 1 graph, and by Lemma 6.3 is defined on V_i and V_j. We choose a_ib_j to be type 1 omitting, and we choose all other free edge types between members of $\mathcal{C}_A$ and $\mathcal{C}_B$ to be type 0 omitting. Finally we omit any agreed edge types of (the new value of) $O(G)^c_H$.

To see that this omits all differing members of $O(G)_H$ not lying in any omission quartet, let D be such. If it differs from C, then as remarked above, it is defined on V_iV_j and is of type 1, so has been omitted by the choice of a_ib_j. If it does not differ from C, and shares a free edge type with C, then it has been omitted by the second part of the definition. Otherwise by the final part of the definition, it must lie in $O(G)^c_H$, and differ from some $E \in O(G)^c_H$. By Lemma 6.3, one of D and E is of type 0, D say, and then E is of type 1. Since the free edge types in $\mathcal{C}_A \cup \mathcal{C}_B$ were chosen type 0 omitting, and D is not yet omitted, D is not defined on members of both $\mathcal{C}_A$ and $\mathcal{C}_B$. If E is defined on members of both $\mathcal{C}_A$ and $\mathcal{C}_B$, then as it is not yet omitted, it differs from C there, and so it is omitted by the choice of a_ib_j after all, which is a contradiction. We deduce that neither D nor E is defined on members of both $\mathcal{C}_A$ and $\mathcal{C}_B$. By Lemma 6.3, D is defined on all members of $\mathcal{C}_A$ and none of $\mathcal{C}_B$ (or the other way round, assume the former). Let $V_{i'} \in \mathcal{C}_A$. Then D is defined on $V_{i'}$, and by Lemma 6.6 it is also defined on some $V_{j'} \in \mathcal{V}_B$, and this must lie in $\mathcal{V}_B - \mathcal{C}_B$. By the final clause of the definition, D differs from some $E' \in O(G)^c_H$ on $V_{i'}V_{j'}$. But since D is not defined on any member of $\mathcal{C}_B$, E' must be defined on all members of $\mathcal{C}_B$, as well as on $V_{i'}$, so differs from C on a free edge type and was omitted by the choice of a_ib_j at the first stage after all, giving a contradiction.

Case 2: Every differing $C \in O(G)_H$ is either defined on all of $\mathcal{V}_A$ and at least one member of $\mathcal{V}_B$, or all of $\mathcal{V}_B$ and at least one member of $\mathcal{V}_A$. Since $|\mathcal{V}_B| \leq |\mathcal{V}_A|$, we may choose a_ib_j for certain i, j with $r \leq i < s \leq j < m$ so that each i occurs exactly once and each j occurs at least once (for example, a_rb_s, $a_{r+1}b_{s+1}, \ldots, a_{m-s+r-1}b_{m-1}, \ldots, a_{s-1}b_{m-1}$), and we let these edge types be type 0 omitting, and all other free edge types be type 1 omitting. We have to see that this omits all members C of $O(G)_H$ not lying in any omission quartet. If C is defined on all of $\mathcal{V}_A$ and at least one member of $\mathcal{V}_B$, let it be defined on V_j say. Then there are i and i' such that a_ib_j was chosen and $a_{i'}b_j$ was not, and so whether C is of type 0 or type 1, it is omitted, and similarly if it defined on all of $\mathcal{V}_B$ and at least one member of $\mathcal{V}_A$.

Finally, we have to show how to omit all members of $O(G)_H$ which lie in an omission quartet. We remark that the edge types which have been chosen so far will not 'interfere' with the task of omitting graphs in omission quartets. This is because all the edge types were chosen because they differed from an edge type of some member of $O(G)_H$. Now if C and D differ and lie in $O(G)_H$, and one of them, C say, lies in an omission quartet, then there is an omission quartet containing both C and D. For if not, by Lemma 6.3, $|D| \geq m + 2 - |C| = m - 1$, contrary to Lemmas 6.5 and 6.7. So this means that if an edge type occurring in a member C of an omission quartet has been chosen, then the effect is to omit C. Furthermore, if this has happened, then the edge type must be an agreed edge type of any omission quartet containing C, as if not we can apply the same argument to the member of the quartet which differs from C on that edge type.

Next we show that if C_0, C_1, C_2, C_3 are the members of an omission quartet in $O(G)_H$, then the agreed edge types of this omission quartet are also agreed in $O(G)_H$. For suppose not, and let the parts that this quartet is defined on be $V_{i_0}, V_{i_1}, V_{i_2}, V_{i_3}$, where $V_{i_0}V_{i_1}$, $V_{i_1}V_{i_2}$, $V_{i_2}V_{i_3}$, $V_{i_3}V_{i_0}$ are agreed, and let $D \in O(G)_H$ differ from the graph C_2 defined on $V_{i_0}, V_{i_1}, V_{i_3}$ on the edge type $V_{i_0}V_{i_3}$. By the previous paragraph, C_2 and D lie in an omission quartet, and the same applies to D and the graph C_1 defined on $V_{i_0}, V_{i_2}, V_{i_3}$ (since C_1 and C_2 agree on $V_{i_0}V_{i_3}$). Let D be defined on $V_{i_0}, V_{i_3}, V_{i_4}$ (since D differs from C_1 and C_2, $i_4 \neq i_1, i_2$). This gives us 3-monics on $V_{i_0}, V_{i_1}, V_{i_4}$, $V_{i_1}, V_{i_3}, V_{i_4}$, $V_{i_0}, V_{i_2}, V_{i_4}$, and $V_{i_2}, V_{i_3}, V_{i_4}$. Now the edge types on $V_{i_0}V_{i_2}$ in the 3-monic E on $V_{i_0}, V_{i_2}, V_{i_4}$ and C_1 agree (being an agreed edge type of the omission quartet on $V_{i_0}, V_{i_2}, V_{i_3}, V_{i_4}$) but differ from that edge type in C_3 on $V_{i_0}, V_{i_1}, V_{i_2}$. Hence E and C_3 differ on V_{i_0}, V_{i_2}, so lie in an omission quartet on $V_{i_0}, V_{i_1}, V_{i_2}, V_{i_4}$. Similarly there is an omission quartet on $V_{i_1}, V_{i_2}, V_{i_3}, V_{i_4}$. In summary, there is a (unique) omission quartet in $O(G)_H$ on every 4-element subset of $V_{i_0}, V_{i_1}, V_{i_2}, V_{i_3}, V_{i_4}$. By Lemma 6.6, each member of $O(G)_H$ is defined on a member of both $\mathcal{V}_A$ and $\mathcal{V}_B$. Then *every* V_{i_j} lies in $\mathcal{V}_A$ or $\mathcal{V}_B$. For if V_{i_0} does not, for instance, two of V_{i_1}, V_{i_2}, V_{i_3} say, lie in the same one of $\mathcal{V}_A$ and $\mathcal{V}_B$, $V_{i_1}, V_{i_2} \in \mathcal{V}_A$ say. But then no part of $V_{i_0}V_{i_1}V_{i_2}$ lies in $\mathcal{V}_B$, which gives a contradiction. Since there are now five parts $V_{i_0}, V_{i_1}, V_{i_2}, V_{i_3}, V_{i_4}$, three of them must lie in the same one of $\mathcal{V}_A$ and $\mathcal{V}_B$, which would give a member of $O(G)_H$ having no part in the other one, again giving a contradiction, as required.

Now applying similar reasoning to C_0, C_1, C_2, C_3, each of $\mathcal{V}_A$ and $\mathcal{V}_B$ occurs exactly twice, so this means that we can have essentially just the following cases (after relabelling): $V_{i_0}, V_{i_1} \in \mathcal{V}_A$, $V_{i_2}, V_{i_3} \in \mathcal{V}_B$, or $V_{i_0}, V_{i_2} \in \mathcal{V}_A$, $V_{i_1}, V_{i_3} \in \mathcal{V}_B$. In each case, the edge types on $V_{i_1}V_{i_2}$ and $V_{i_3}V_{i_0}$ are free edge types, so all members of the omission quartet can be omitted by choosing these opposite to the edge types that these agree on, and as we have shown that these edge types are agreed with all other members of $O(G)_H$, this is compatible with all other such choices made. □

We can now sum up the results of this section in the main theorem.

Theorem 6.9 (non-complication theorem)**.** *Let G be a countable homogeneous reduced m-partite graph. If two minimally omitted monic graphs in G differ on some bipartite restriction, then they lie in the same omission quartet.*

Proof: This follows from the previous results. For, if the result is false, we take the smallest possible value of m, and now our graph is 'minimal' in the sense introduced just before Lemma 6.3. By Theorem 4.2, $m \geq 4$. By Lemma 5.2, $m \geq 5$. By Lemma 6.5, the greatest size of a member of $O(G)$ is at most $m - 1$, and by Lemma 6.7, it is at most $m - 2$. Finally, Lemma 6.8 shows that it does not have any value between 3 and $m - 2$, giving a contradiction, and proving the theorem. □

7 Final classification and concluding remarks

In this section we put together the results from the previous sections to achieve a complete classification of the homogeneous m-partite graphs. We begin with the reduced ones, and then modify this using the material in section 3 to deal with all countable homogeneous m-partite graphs. We recall that in section 3 we introduced the notions of 'reduced coding graph' (for reduced homogeneous graphs) and 'coding graph' (for all the countable homogeneous multipartite graphs).

Theorem 7.1 (classification of the countable reduced homogeneous m-partite graphs)**.** *For any countable reduced homogeneous m-partite graph G, there is a reduced coding graph Γ which encodes G. Conversely, for any reduced coding graph Γ of size m, there is a countable reduced homogeneous m-partite graph G, uniquely determined up to isomorphism, encoded by Γ.*

Proof: Let Γ have vertices $0, 1, \ldots, m - 1$. It has vertex labels given by the cardinalities of the corresponding parts of G, and we take $\mathcal{G}$ to be $O(G)$. The fact that Γ is a reduced coding graph follows from Theorem 6.9.

Conversely, let Γ be a reduced coding graph on m vertices v_i for $0 \leq i < m$ with vertex labelling function φ and family $\mathcal{G}$ of graphs on subsets of $\{0, 1, \ldots, m - 1\}$ fulfilling the stated conditions in the definition of 'reduced coding graph'. We show that the class $\mathcal{C}$ of all finite m-partite graphs whose parts obey the cardinality restrictions given by φ, and which omit all members of $\mathcal{G}$, is an amalgamation class, and it will follow that there is a corresponding Fraïssé limit, which is the desired (uniquely determined) homogeneous m-partite graph classified by Γ. As usual we let $A = B \cap C$ where $B = A \cup \{x\}$, $C = A \cup \{y\}$ lying in $\mathcal{C}$ and we have to amalgamate B and C over A by choosing how x and y are joined. If they are in the same part V_i, then they will not be joined, and we at once obtain the amalgamated structure. The only problem comes about if we thereby exceed the value of $\varphi(i)$, which must therefore be finite. In this case, we identify x and y, and we remark that this is possible because of condition (ii) in the definition of 'reduced coding graph'. For this implies that the relation between V_i and any other V_j in each of B and C is empty or complete. Thus if $z \in V_j$ in B and C, and hence in A, either x and y are both joined to z, or they are both not joined to z, so in each case there is no clash in identifying x and y. If x and y are in different parts V_i and V_j, then there will be no problem about the value of φ, since the constraint is obeyed by both B and C, so we just have to show that we can join or not join x and y so as to omit all members of $\mathcal{G}$. If no members of $\mathcal{G}$ have an edge on V_iV_j then amalgamate by choosing xy to be an edge, and similarly if no members of $\mathcal{G}$ have a non-edge there. So the only difficulty comes about if there are D and E in $\mathcal{G}$ such that D has an edge on V_iV_j but E has a non-edge. By assumption, D and E lie in an omission quartet, and the proof is concluded as in the proof of Lemma 5.1. □

Theorem 7.2 (classification of the countable homogeneous m-partite graphs)**.** *For any countable homogeneous m-partite graph G, there is a coding graph Γ which encodes G. Conversely, for any coding graph Γ of size m, there is a countable homogeneous m -partite graph G, uniquely determined up to isomorphism, encoded by Γ.*

Proof: This follows from Theorem 7.1 by appealing to Theorem 3.1. □

We now remark that the classification achieved is 'effective', in the following sense. Given m, there are only countably many countable homogeneous m-partite graphs, and they can be effectively listed. To do this (in rather a crude fashion) one may list all the sets of monic graphs on at most m vertices, and see which ones are possible values of $O(G)$. For this, the condition is that no two should differ unless they are members of an omission quartet. This procedure enables us to list all the reduced countable homogeneous k-partite graphs for $k \leq m$. In the typical case, there are only finitely many possibilities, but when one includes the possible finite structures (which only arise in the trivial cases of a part's being related to all others by an empty or complete relation) then there will be infinitely many. Finally, for any possible value of $O(G)$ on $k \leq m$ parts, one can then list the possibilities for the m-partite countable homogeneous but not necessarily reduced graphs having G as core.

The verification that a particular list of monic graphs on m vertices fulfils the conditions to be a possible value of $O(G)$ is potentially long, and it can be illuminated by giving further information on how omission quartets can interact. As hinted above, it is possible for the agreed edge types of an omission quartet to differ from other members of $O(G)$, but of course only at the expense of there being larger and more complicated families of omitted 3-monics. We avoided the full details of this in the proof of Lemma 6.8 by showing that we could rule out the existence of such families on five vertices. But this was only in the context of the inductive assumption (that G was 'minimal') and this will not be true in general. We conclude by giving details about general 'omission families' and how they can arise inside $O(G)$. The definition is as follows: Let $4 \leq r \leq m$. An *r-omission family* is a family $\mathcal{F}$ of 3-monic graphs on r of the m parts such that the parts on which members of $\mathcal{F}$ are defined may be enumerated as $\{V_i : i < r\}$ in such a way that

(i) for every 3-element subset P of $\{0, 1, \ldots, r-1\}$ there is a unique member X_P of $\mathcal{F}$ defined on $\{V_i : i \in P\}$,

(ii) for every consecutive $i, j < r$ (including $i = r-1$, $j = 0$), any two members of $\mathcal{F}$ defined on V_iV_j have the same edge type there, (and we call these the *agreed edge types* of the family),

(iii) the members of $\mathcal{F}$ defined on any four parts form an omission quartet.

An *omission family* is a family which is an r-omission family for some r.

We remark that an omission quartet is just the special case of this for $r = 4$.

Theorem 7.3. *(i) In any reduced homogeneous m-partite graph G, any omission quartet contained in $O(G)$ is a subset of a uniquely determined omission family $\mathcal{F} \subseteq O(G)$ whose agreed edge types are also agreed in $O(G)$.*

(ii) For every r such that $4 \le r \le m$ there is an r-omission family.

(iii) For any r-omission family $\mathcal{F}$, the family of m-partite graphs omitting $\mathcal{F}$ is an amalgamation class.

Proof: (i) Since as remarked above, an omission quartet is a 4-omission family, for this it suffices to show that if $\mathcal{F}$ is an r -omission family having an agreed edge type which is not agreed in $O(G)$ then $\mathcal{F}$ can be extended to an $(r+1)$-omission family contained in $O(G)$. Let $\mathcal{F}$ have agreed edge types on V_0V_1, $V_1V_2, \ldots, V_{r-2}V_{r-1}, V_{r-1}V_0$, and suppose that the edge type on $V_{r-1}V_0$ is not agreed with $O(G)$. Thus there is $X \in O(G)$ defined on $V_{r-1}V_0$ and differing from the member Y_i of $\mathcal{F}$ defined on $V_{r-1}V_0V_i$ there for each i between 1 and $r-2$. By Theorem 6.9, X and Y_i lie in an omission quartet, so in particular, X is a 3-monic. Let the third part on which X is defined be V_r. This gives us the other two 3-monics of an omission quartet, on $V_{r-1}V_iV_r$ and $V_0V_iV_r$.

So far we know that there are omission quartets on $V_iV_jV_kV_l$ where $i < j < k < l$ and $l \le r-1$, and also on $V_0V_iV_{r-1}V_r$ for $1 \le i \le r-2$. Next consider $V_0V_iV_jV_r$ where $i \le i < j \le r-2$. Now $\mathcal{F}$ has an omission quartet on $V_0V_iV_jV_{r-1}$, and in this omission quartet, there is an agreed edge type on either V_0V_i or V_0V_j, suppose V_0V_i. Then the edge types on V_0V_j differ between the 3-monics on $V_0V_iV_j$ and $V_0V_jV_{r-1}$. But this edge type is agreed between the 3-monics on $V_0V_jV_{r-1}$ and $V_0V_jV_r$, and so it differs between the 3-monics on $V_0V_iV_j$ and $V_0V_jV_r$. Hence there is an omission quartet on $V_0V_iV_jV_r$ contained in $O(G)$. A similar argument shows that there is an omission quartet on $V_iV_jV_kV_r$ for $0 < i < j < k < r$, and thus the family of 3-monics on $V_0, V_1, \ldots, V_r$ is an $(r+1)$-omission family contained in $O(G)$.

(ii) We may use an inductive construction based on the proof in (i). It suffices to show how to build an $(r+1)$-omission family from an r -omission family $\mathcal{F}$. Let $\mathcal{F}$ be defined on parts $V_0, \ldots, V_{r-1}$ with agreed edge types on V_iV_{i+1}, and we add a new part V_r and a new 3-monic on $V_0V_{r-1}V_r$ which differs from $\mathcal{F}$ on $V_{r-1}V_0$. We now choose 3-monics on $V_0V_iV_r$ and $V_iV_{r-1}V_r$ for each i between 1 and $r-2$ which differ on V_iV_r to ensure that there is an omission quartet on $V_0V_iV_{r-1}V_r$. Finally, if $1 \le i, j \le r-2$ are such that the edge types on V_0V_i agree between the 3-monics on $V_0V_iV_j$ and $V_0V_iV_{r-1}$, we choose a 3-monic on $V_iV_jV_r$ in which the edge types on V_iV_j, V_jV_r, and V_iV_r are as in the 3-monics on $V_0V_iV_j$, $V_0V_jV_r$, and opposite to that on $V_0V_iV_r$ respectively.

This defines a unique 3-monic on each 3-element set of the $r+1$ parts, so it remains to show that it provides an omission quartet on every set of four parts. We already have omission quartets on all 4-element sets not containing V_r (by the induction hypothesis), and on sets containing V_0, V_{r-1}, and V_r, so it remains to consider V_0, V_i, V_j, V_r for $1 \le i < j \le r-2$ and V_i, V_j, V_k, V_r for $1 \le i < j < k \le r-1$.

For V_0, V_i, V_j, V_r we note that V_0V_r is automatically an agreed edge type, and so are V_iV_j and V_jV_r and the edge type on V_iV_r differs in $V_iV_jV_r$ and $V_0V_iV_r$ by definition. The edge type on V_0V_i agrees between $V_0V_iV_j$ and $V_0V_iV_{r-1}$ by assumption, and between $V_0V_iV_{r-1}$ and $V_0V_iV_r$ by construction of the omission quartet on $V_0V_iV_{r-1}V_r$, and hence between $V_0V_iV_j$ and $V_0V_iV_r$, and for a similar reason, the edge type on V_0V_j differs between $V_0V_iV_j$ and $V_0V_jV_r$, giving an omission quartet on V_0, V_i, V_j, V_r.

For V_i, V_j, V_k, V_r we already know what all the 3-monics must be, since any three of these parts lie in one of V_0, V_i, V_j, V_k; V_0, V_i, V_j, V_r; V_0, V_i, V_k, V_r; or V_0, V_j, V_k, V_r, on each of which we already know that we have an omission quartet, so we just have to verify that these four 3-monics do indeed form an omission quartet on V_i, V_j, V_k, V_r. Since the edge type on V_0V_r is agreed, either the edge type on V_0V_i agrees between $V_0V_iV_j$ and $V_0V_iV_r$ and the edge type on V_0V_j differs between $V_0V_iV_j$ and $V_0V_jV_r$, or the other way round, with similar statements for the pairs i, k and j, k. There appear to be 8 cases to consider, but by considerations of symmetry, we only have these:

the edge type on V_0V_i agrees between $V_0V_iV_j$, $V_0V_iV_k$, and $V_0V_iV_r$ and the edge type on V_0V_j agrees between $V_0V_jV_k$ and $V_0V_jV_r$,

the edge type on V_0V_i agrees between $V_0V_iV_j$, $V_0V_iV_r$, the edge type on V_0V_k agrees between $V_0V_iV_k$ and $V_0V_kV_r$, and the edge type on V_0V_j agrees between $V_0V_jV_k$ and $V_0V_jV_r$ (with corresponding differing assertions).

The latter possibility cannot actually arise, since then the edge type on V_0V_i agrees between $V_0V_iV_j$, $V_0V_iV_r$, but differs between $V_0V_iV_k$, $V_0V_iV_r$, and hence differs between $V_0V_iV_j$ and $V_0V_iV_k$. But then the edge types on V_0V_j agree in the omission quartet on V_0, V_i, V_j, V_k, that is, in the 3-monics $V_0V_iV_j$ and $V_0V_jV_k$, and hence the edge types on V_0V_j in $V_0V_iV_j$ and $V_0V_jV_r$ agree, which is a contradiction.

So we are left with just the former case. Here we have enough information to find all the edge types on the 3-monics in V_i, V_j, V_k, V_r, and verify that this is an omission quartet. This working may be set out as follows:

In the omission quartet on V_0, V_i, V_j, V_k the differing edge types are V_0V_j and V_iV_k (the rest are agreed).

In the omission quartet on V_0, V_i, V_j, V_r the differing edge types are V_0V_j and V_iV_r.

In the omission quartet on V_0, V_i, V_k, V_r the differing edge types are V_0V_k and V_iV_r.

In the omission quartet on V_0, V_j, V_k, V_r the differing edge types are V_0V_k and V_jV_r.

Hence the edge type of V_iV_j is agreed between $V_0V_iV_j$ and $V_iV_jV_k$, and also between $V_0V_iV_j$ and $V_iV_jV_r$. Hence it is agreed between $V_iV_jV_k$ and $V_iV_jV_r$. Similarly, the edge types of V_iV_r, V_jV_k, V_kV_r are agreed on their 3-monics in V_i, V_j, V_k, V_r, and those of V_iV_k and V_jV_r differ, which shows that we have an omission quartet on these four parts.

(iii) This follows at once from the proof of Theorem 7.1, since the omission family $\mathcal{F}$ fulfils the conditions required of $\mathcal{G}$ in the definition of 'reduced coding graph'. □

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