Differential Geometry

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1 Preliminaries

1.1 Topology

Definition 1 A **topological space** is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} = \{U_\alpha \subseteq X : \alpha \in A\}$ is a family of subsets of X, called **open sets** satisfying the properties

(a) $X \in \mathcal{T}$, $\emptyset \in \mathcal{T}$

(b) For any subset $A' \subseteq A$, $\displaystyle\bigcup_{\alpha \in A'} U_\alpha \in \mathcal{T}$

(c) For any finite subset $A' \subseteq A$, $\displaystyle\bigcap_{\alpha \in A'} U_\alpha \in \mathcal{T}$.

A set $V \subseteq X$ is **closed** if $X \backslash V$ is open. □

It is customary to denote the topological space by just X, rather than $(X, \mathcal{T})$, the family $\mathcal{T}$ being understood.

Example 2 The **usual topology** on $\mathbb{R}$ is defined as follows: a set $U \subseteq \mathbb{R}$ is open if for each $x \in U$ there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq U$.

Definition 3 A map $f : X \to Y$ between topological spaces is **continuous** if for every open set $U \subseteq Y$, $f^{-1}(U) \subseteq X$ is open. □

Definition 4 A sequence $y : \mathbb{N} \to X$ **converges** to a limit $x \in X$ if for every open set U containing x there exists $m \in \mathbb{N}$ such that $n \geq m \Rightarrow x_n \in U$. □

Definition 5 A map $f : X \to Y$ is a **homeomorphism** if it is bijective and bicontinuous (both f and f^{-1} are continuous). Spaces X and Y are said to be **homeomorphic** if there exists a homeomorphism between them. Note that homeomorphism is an equivalence relation on the class of topological spaces. □

Definition 6 X is **Hausdorff** if given any distinct pair x_1, $x_2 \in X$, there exist open sets U_1, U_2 such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. □

The Hausdorff property is essential to guarantee that convergent sequences in X have a unique limit. Non-Hausdorff spaces are rather bizarre, and will not be of much interest to us.

Definition 7 Any nonempty subset Y of a topological space X inherits a natural topology called the **relative topology**:
A subset $U \subseteq Y$ is defined to be open if there exists an open set $\widetilde{U} \subseteq X$ such that $\widetilde{U} \cap Y = U$. □

Definition 8 The cartesian product $X \times Y$ of two topological spaces inherits a natural topology called the **product topology**:
A subset $U \subseteq X \times Y$ is open if $\forall (x, y) \in U$ there exist open sets $U_1 \subseteq X$ and $U_2 \subseteq Y$, containing x and y respectively, such that $U_1 \times U_2 \subseteq U$. □

In practice, most of the spaces we consider will be constructed from $\mathbb{R}$ with its usual topology, in conjunction with definitions 7, 8 and 15 (see below). It is easy to show that any subset of a Hausdorff space is Hausdorff (when equipped with the relative topology), as is the product of any pair of Hausdorff spaces (equipped with the product topology).

Definition 9 X is **connected** if there does not exist a pair of nonempty open subsets X_1 and X_2 such that $X = X_1 \sqcup X_2$ (that is, $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$). □

Definition 10 X is **path connected** if given any pair x_1,$x_2 \in X$ there exists a continuous map $f : [0, 1] \to X$ such that $f(0) = x_1$ and $f(1) = x_2$ (that is, a continuous path connecting the pair of points). □

Example 11 $\mathbb{R}$ is both connected and path connected, as is any interval in $\mathbb{R}$.

Example 12 The rationals $\mathbb{Q} \subset \mathbb{R}$, given the relative topology, are neither connected (e.g. $\mathbb{Q} = [\mathbb{Q} \cap (-\infty, \sqrt{2})] \sqcup [\mathbb{Q} \cap (\sqrt{2}, \infty)]$) nor path connected (intermediate value theorem).

Theorem 13 *Every path connected space is connected.*

Proof: Assume, to the contrary, that X is path connected, but not connected. Then there exist disjoint nonempty open sets X_1, X_2 such that $X = X_1 \cup X_2$. Let $x_1 \in X_1$ and $x_2 \in X_2$. Since X is path connected, there exists continuous $f : [0,1] \to X$ such that $f(0) = x_1$ and $f(1) = x_2$. Hence both $f^{-1}(X_1)$ and $f^{-1}(X_2)$ are nonempty subsets of $[0,1]$ and, by continuity of f, both are open. But

$$f^{-1}(X_1) \cup f^{-1}(X_2) = f^{-1}(X_1 \cup X_2) = f^{-1}(X) = [0,1]$$

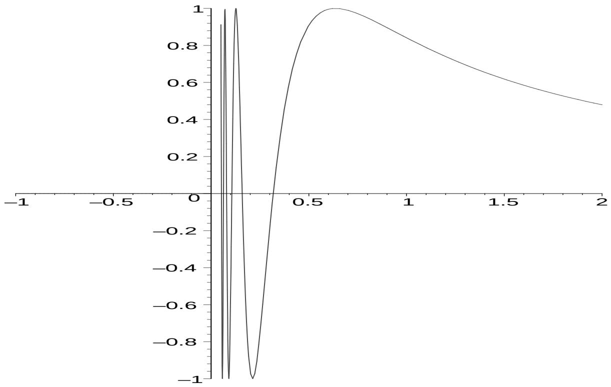
and

$$f^{-1}(X_1) \cap f^{-1}(X_2) = f^{-1}(X_1 \cap X_2) = f^{-1}(\emptyset) = \emptyset,$$

so $[0,1]$ is the disjoint union of nonempty open sets. But $[0,1]$ is connected, a contradiction. □

We will see later that, for manifolds, the converse is true. In general however connectedness does not imply path connectedness.

Counterexample 14 The union $\{(x, \sin(1/x)) : x > 0\} \cup \{(x, 0) : x \leq 0\} \subset \mathbb{R}^2$, equipped with the relative topology, is connected but not path connected.



Definition 15 Given an equivalence relation $\sim$ on a topological space X, the quotient set $X/\sim$ inherits a natural topology called the **quotient topology**: A subset $U \subseteq X/\sim$ is defined to be open if $\pi^{-1}(U) \subseteq X$ is open in X, where $\pi : X \to X/\sim$ is the projection map $\pi(x) = [x]$.
Note that π is continuous by definition when $X/\sim$ is equipped with the quotient topology. □

Exercise 16 Check that definition 15 really does define a topology on $X/\sim$, that is, the family of open sets defined above really does satisfy the axioms of definition 1.

Warning! Quotient spaces can be rather nasty. Unlike the relative and product topologies, the quotient topology does not necessarily preserve the Hausdorff property, that is $X/\sim$ might not be Hausdorff even if X is.

Counterexample 17 Let $X = \mathbb{R}$ and $\sim$ be the equivalence relation

$$x \sim y \quad \Leftrightarrow \quad \exists\, n \in \mathbb{Z} \text{ such that } x = 2^n y.$$

For any $\epsilon > 0$ and any $x \neq 0$, the interval $(-\epsilon, \epsilon)$ contains $2^n x$ for some n. Hence, the only open set in $X/\sim$ containing $[0] = \{0\}$ is $X/\sim$ itself! This certainly is not Hausdorff.

Definition 18 X is compact if every open cover (family of open sets $\{U_\alpha : \alpha \in A\}$ such that $\bigcup_{\alpha \in A} U_\alpha = X$) has a finite subcover. $\Box$

Definition 19 A **distance function** on a set X is a mapping $d : X \times X \to [0, \infty)$ which for all $x, y, z \in X$ satisfies

(a) $d(x, y) = d(y, x)$

(b) $d(x, y) = 0$ if and only if $x = y$

(c) $d(x, z) \leq d(x, y) + d(y, z)$.

One refers to $d(x, y)$ as the distance between the points x and y. $\Box$

Definition 20 A set X equipped with a distance function d inherits a natural topology called the **metric topology**:
A subset U is defined to be open if for every $x \in U$ there exists $r > 0$ such that $B_r(x) \subseteq U$, where $B_r(x) := \{y \in X : d(x, y) < r\}$ is the open ball of radius r.
The resulting toplogical space is called a **metric space**. $\Box$

Example 21 Natural topologies on $\mathbb{R}^n$ arise by

(a) Considering it to be a cartesian product of n copies of "usual" $\mathbb{R}$, endowed with the product topology.

(b) Equipping it with the metric topology arising from the distance function

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

In fact these two topologies are identical. Note that the usual topology on $\mathbb{R}$ itself is the metric topology with respect to $d(x, y) = |x - y|$.

1.2 Calculus in $\mathbb{R}^n$

Throughout this section U, V, W will denote open subsets of $\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^p$ respectively. The euclidean norm on $\mathbb{R}^n$ will be denoted $||\cdot||_n$, that is for all $x \in \mathbb{R}^n$,

$$||x||_n := \sqrt{\sum_{i=1}^{n} x_i^2}.$$

Definition 22 $f : U \to V$ is *differentiable* at $a \in U$ if there exists a linear map $df_a : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$||f(a+h) - f(a) - df_a h||_n = o(||h||_m).$$

The map df_a, if it exists, is unique and called the **differential** of f at a. $\square$

Being a linear map $\mathbb{R}^m \to \mathbb{R}^n$, df_a may be represented by a $n \times m$ matrix, called the Jacobian matrix J, once bases for $\mathbb{R}^m$ and $\mathbb{R}^n$ are chosen. Let these bases be $\{e_i : i = 1, 2, \ldots, m\}$ and $\{\epsilon_\alpha : \alpha = 1, 2, \ldots n\}$. Then, writing a general point x in U as

$$x = x^i e_i$$

(the sum over the repeated index i from 1 to m being understood) and decomposing f with respect to the basis in V,

$$f(x) = f^\alpha(x^1, x^2, \ldots, x^m)\epsilon_\alpha$$

(the sum over the repeated index α from 1 to n also being understood), one finds that the (β, j) component of the Jacobian matrix (β = row and j = column) is

$$J^\beta_{\ j} = \left.\frac{\partial f^\beta}{\partial x^j}\right|_{x=a}.$$

Note that this is the first instance where we have used **Einstein's**[1] **summation convention**: when dealing with expressions containing components of vectors and matrices with respect to given bases, we always assume that a twice repeated index is summed over. In fact, we will only sum over repeated indices if one is "upstairs" and one "downstairs," although the reason for this subtlety will not be apparent for some time.

The following lemma is very useful in practical calculations.

Lemma 23 *$f : U \to V$ is differentiable at $a \in U$ if and only if every partial derivative $\partial f^\alpha / \partial x^i$ exists and is continuous at a.* $\square$

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[1] Or Einstein's wife's, according to some revisionist historians of science.

Note that continuity of the partial derivatives *is essential* to guarantee differentiability. Existence is not enough.

Counterexample 24 The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{aligned} f(x,y) &= \frac{xy}{\sqrt{x^2+y^2}}, \qquad (x,y) \neq (0,0) \\ f(0,0) &= 0 \end{aligned}$$

is not differentiable at $(0,0)$. Both partial derivatives (with respect to x and y) exist at $(0,0)$ however.

Definition 25 $f : U \to V$ is k times continuously differentiable (is of class C^k where $k \in \mathbb{Z}^+$) if all partial derivatives of f up to and including order k exist and are continuous. $\square$

Definition 26 $f : U \to V$ is **smooth** (of class C^∞) if it is C^k for all $k \in \mathbb{Z}^+$. $\square$

Definition 27 $f : U \to V$ is a **diffeomorphism** if f is bijective and both f and f^{-1} are smooth. $\square$

Of course, not every smooth bijection is a diffeomorphism.

Counterexample 28 $f : x \mapsto x^3$ is a smooth bijection from $\mathbb{R}$ to itself, but not a diffeomorphism since f^{-1} is not smooth at 0.

Lemma 29 (Chain Rule) *If $f : U \to V$ is differentiable at a and $g : V \to W$ is differentiable at $f(a)$ then $g \circ f$ is differentiable at a, and*

$$d(g \circ f)_a = dg_{f(a)} \circ df_a$$

Theorem 30 (Inverse Function Theorem) *If $f : U \to V$ $(m = n)$ is C^k and $df_a : \mathbb{R}^n \to \mathbb{R}^n$ is injective, then there exist open sets $\widetilde{U}, \widetilde{V}$ containing a and $f(a)$ respectively, such that $f|_{\widetilde{U}}$ is invertible and $f^{-1}|_{\widetilde{V}}$ is C^k.*

2 Smooth Manifolds

2.1 Definitions

Idea: if we can break a topological space X into pieces, each homeomorphic to an open set in $\mathbb{R}^n$, we can use the homeomorphisms to transfer ordinary calculus onto X. If this is to be well-defined we need "agreement" on overlapping pieces, i.e. need the transitions from piece to piece to be smooth.

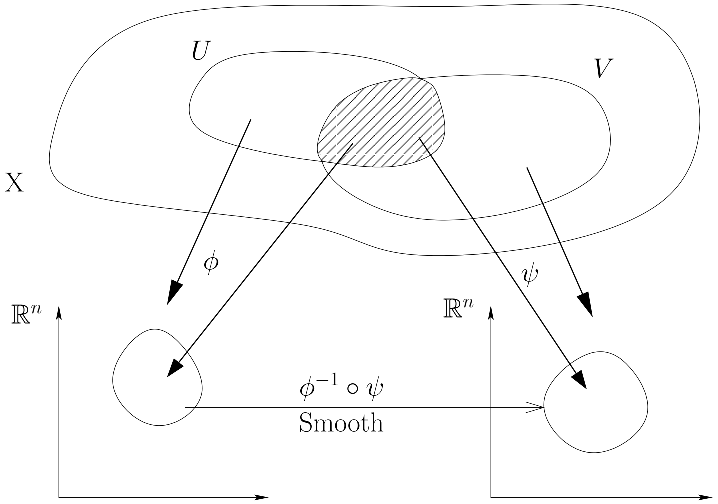


Figure 1: Two overlapping coordinate charts on $U, V \subset X$

Definition 31 An **atlas** on a Hausdorff space M consists of

(a) An open covering $\{U_i : i \in I\}$ of M

(b) For each i a homeomorphism $\phi_i : U_i \to \Omega_i$, an open set in $\mathbb{R}^n$

(c) For any $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, the function

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is smooth (ie. C^∞). □

Notes

(i) Each pair (U_i, ϕ_i) is called a coordinate chart and defines local coordinates on M, usually denoted x^i or y^i, etc. where $i \in \mathbb{Z}^+$.

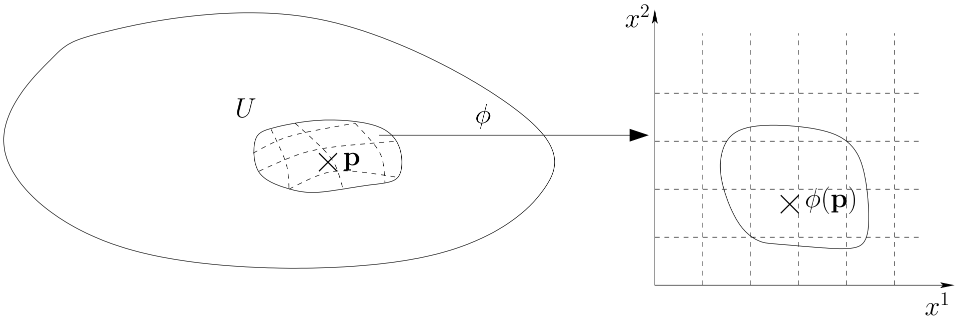


Figure 2: Local coordinates around **p**

(ii) Functions $\tau_{ij} = \phi_i \circ \phi_j^{-1}$ are called **transition functions**. One cannot put direct smoothness requirements on the ϕ_i, but we can on

$$\tau_{ij} : \left\{ \begin{matrix} \text{Open set} \\ \text{in } \mathbb{R}^n \end{matrix} \right\} \to \left\{ \begin{matrix} \text{Open set} \\ \text{in } \mathbb{R}^n \end{matrix} \right\}.$$

(iii) From the definition, τ_{ij} is bijective and $\tau_{ji} = \tau_{ij}^{-1}$ must be smooth. Hence each $\tau_{ij} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is a diffeomorphism.

(iv) If M is connected, each ϕ_i must inject into the same Euclidean space $\mathbb{R}^n$ ($\nexists$ a diffeomorphism between open sets $U \to V$ where U, V are open subsets of $\mathbb{R}^n, \mathbb{R}^m$ respectively with $n \neq m$). In this case, n is, by definition, the **dimension** of the manifold.

Definition 32 Two atlases $\{(U_i, \phi_i) : i \in I\}$ and $\{(V_j, \phi_j) : j \in J\}$ on M are said to be **equivalent** if their union is also an atlas for M. □

Exercise 33 Check this is really an equivalence relation.

Definition 34 A **differentiable structure** on M is an equivalence class of atlases on M. M together with a differentiable structure is called a **smooth manifold**. □

Notes

(i) In order to *define* a smooth manifold, all we need to do is define *one atlas* on a Hausdorff space. Given two *different atlases* on M they may or may not be equivalent, i.e. may or may not define the same manifold.

(ii) One can make other requirements on the transition functions $\tau_{ij} = \phi_i \circ \phi_j^{-1}$ rather than smoothness, e.g.

- none – then τ_{ij} are C^0 automatically ⇒ "topological manifold"
- C^k for $k < \infty \Rightarrow C^k$ manifold
- Real analytic ⇒ analytic manifold
- If n is even, $n = 2m$ then $\mathbb{R}^n \cong \mathbb{C}^m$ and we can demand that τ_{ij} are *holomorphic* ⇒ complex manifold.

The first two of these conditions are weaker than smoothness, while the final two are stronger. By far the most interesting of these variants is the final one (complex manifolds).

2.2 First Examples

Example 35 Any open subset U of $\mathbb{R}^n$, with the usual topology, is trivially a smooth manifold. We can define an atlas consisting of a single chart, namely (U, Id) (Id is clearly a homeomorphism). Clearly all transition functions are smooth (there are none!).

This is not the only possible differentiable structure on $\mathbb{R}^n$, as we shall see.

Example 36 The sphere $S^n \subset \mathbb{R}^{n+1}$ (with the relative topology)

$$S^n = \{x \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$

We can cover this with two charts $U_\pm = S^n \setminus \{(0, 0, \ldots, \pm 1)\}$ with homeomorphisms

$$\phi_\pm(\underbrace{x_1, x_2, \ldots, x_n}_{\mathbf{x}}, x_{n+1}) = \frac{\mathbf{x}}{1 \mp x_{n+1}} \in \mathbb{R}^n.$$

In this case the overlap is $U_+ \cap U_- = S^n \setminus \{(0, 0, \ldots, \pm 1)\}$.

$$\mathbf{p} = \phi_+(x_1, \ldots, x_{n+1}) = \frac{\mathbf{x}}{1 - x_{n+1}}$$

$$|\mathbf{p}|^2 = \frac{|\mathbf{x}|^2}{(1 - x_{n+1})^2} = \frac{1 - x_{n+1}^2}{(1 - x_{n+1})^2} = \frac{1 + x_{n+1}}{1 - x_{n+1}}$$

$$\Rightarrow x_{n+1} = \frac{|\mathbf{p}|^2 - 1}{|\mathbf{p}|^2 + 1}$$

$$\Rightarrow \mathbf{x} = \frac{2\mathbf{p}}{|\mathbf{p}|^2 + 1}$$

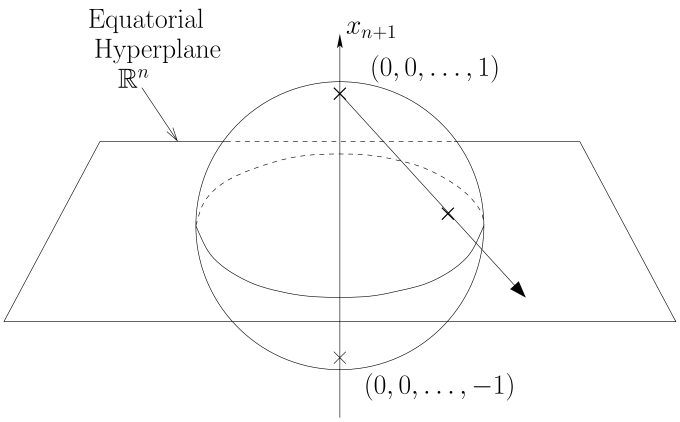


Figure 3: Projection from the "north pole"

$$\Rightarrow \phi_+^{-1}(\mathbf{p}) = \Bigg(\underbrace{\frac{2\mathbf{p}}{|\mathbf{p}|^2+1}}_{\mathbf{x}}, \underbrace{\frac{|\mathbf{p}|^2-1}{|\mathbf{p}|^2+1}}_{x_{n+1}}\Bigg)$$

Now,

$$\tau_{-+}(\mathbf{p}) = (\phi_- \circ \phi_+^{-1})(\mathbf{p}) = \frac{(2\mathbf{p})/(|\mathbf{p}|^2+1)}{1+\dfrac{|\mathbf{p}|^2-1}{|\mathbf{p}|^2+1}} = \frac{\mathbf{p}}{|\mathbf{p}|^2}$$

Clearly this is smooth on $\phi_+(U_+ \cap U_-) = \mathbb{R}^n \setminus \{\mathbf{0}\}$.
Finally,

$$\tau_{-+}^2(\mathbf{p}) = \tau_{-+}\left(\frac{\mathbf{p}}{|\mathbf{p}|^2}\right) = \frac{\mathbf{p}/|\mathbf{p}|^2}{|\mathbf{p}|^2/|\mathbf{p}|^4} = \mathbf{p}$$

$$ie. \quad \tau_{-+}^2 = \mathrm{Id} \Rightarrow \tau_{-+} = \tau_{-+}^{-1} = \tau_{+-}$$

so both transition functions are smooth. □

Example 37 Any topological space X homeomorphic to a smooth manifold M (ie. to a space possessing a differentiable structure) can be given the structure of a smooth manifold by carrying an atlas on M to X using the homeomorphism. For example, let X be the unit square in $\mathbb{R}^2$. Then radial projection $\pi : X \to S^1$ is a homeomorphism (see picture).

S^1 has "stereographic atlas" $\{U_\pm, \phi_\pm\}$ so we can define an atlas $\{(\pi^{-1}(U_\pm), \phi_\pm \circ \pi)\}$ on the square X.

$$\left.\begin{aligned} \tau_{+-} &= \phi_+ \circ \pi \circ (\phi_- \circ \pi)^{-1} = \phi_+ \circ \phi_-^{-1} \\ \tau_{+-} &= \phi_- \circ \pi \circ (\phi_+ \circ \pi)^{-1} = \phi_- \circ \phi_+^{-1} \end{aligned}\right\}\text{—smooth by Example 36.}$$

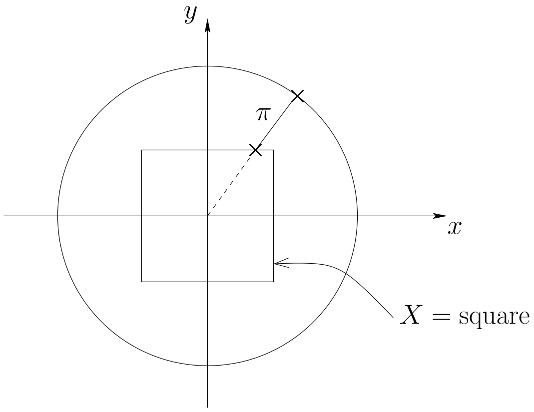


Figure 4: Radial projection of a square onto S^1

Another example: an infinite cone in $\mathbb{R}^3$ can be identified homeomorphically with the plane $\mathbb{R}^2$ by vertical projection (see picture).

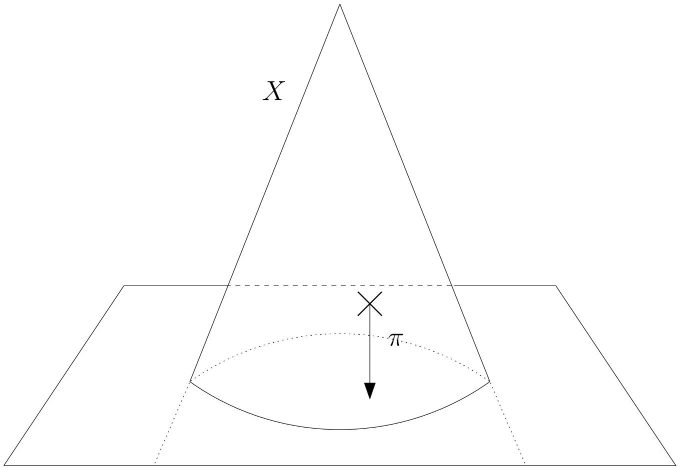


Figure 5: Vertical projection of a cone onto the plane

2.3 Connectedness

Recall that for any topological space X:

$$X \text{ path connected} \Rightarrow X \text{ connected}.$$

For manifolds, the converse is true too:

Theorem 38 *Every connected manifold M is path connected.*

Proof: Define an equivalence relation $\sim$ on M:

$$x \sim y \quad \Leftrightarrow \quad \begin{array}{c} \text{there exists a continuous map} \\ f : [0,1] \to M \text{ s.t. } f(0) = x \text{ and } f(1) = y \end{array}$$

(i.e. $x \sim y$ iff x is path connected to y). $M/\sim$ is a partition of M. We will show that each $[x] \in M/\sim$ is *open* in M.

Let $y \in [x]$ and (U, ϕ) be a chart containing y. Since $\phi(U)$ is open in $\mathbb{R}^n$, there exists $\varepsilon > 0$ such that $B_\varepsilon(\phi(y)) \subseteq \phi(U)$.

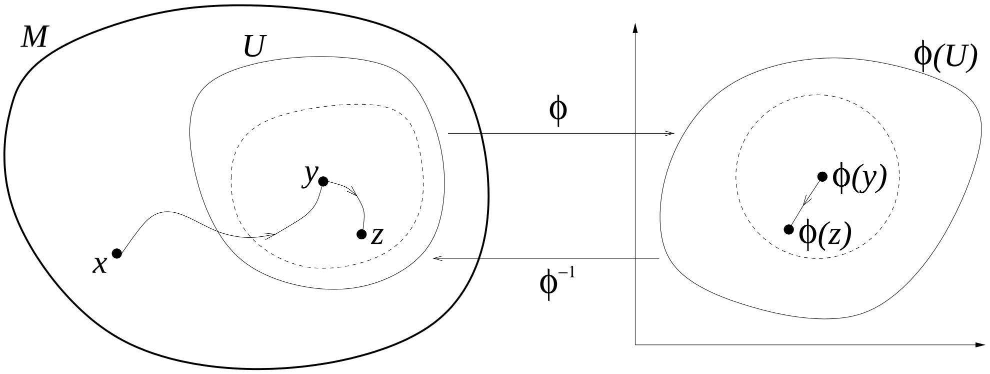


Figure 6: The open neighbourhood of $y \in [x]$

Clearly $z \sim y$ for every $z \in \phi^{-1}(B_\varepsilon(\phi(y))) \Rightarrow z \sim x$, since we can connect y to z by the image under ϕ^{-1} of a radial arc in $\phi(U)$. Hence $[x]$ contains the open set $z \in \phi^{-1}(B_\varepsilon(\phi(y))) \subseteq M$. But such an open set can be contructed at each $y \in [x]$, so $[x]$ is a uniion of open sets, hence itself open. Thus $M/\sim$ is an open partition of M, which must be trivial (i.e. $M/\sim = \{M\}$) since M is connected. Since $[x] = M$, we conclude that *all* points in M are path connected. $\square$

2.4 Orientation

An **orientation** of $\mathbb{R}^n$ is defined by choosing an *ordered* basis $(e_i) = (e_1, \ldots, e_n)$ for $\mathbb{R}^n$. Any other ordered basis, $(\overline{e}_i)$ say, must be related to (e_i) by an invertible matrix L:

$$\overline{e}_i = e_j L^j{}_i.$$

The set of such matrices is

$$\begin{aligned} GL(n, \mathbb{R}) &= \{n \times n \,\text{real-valued matrices with det} \neq 0\} \\ &\overset{\text{open}}{\subset} \mathbb{R}^{n^2} \end{aligned}$$

since we may identify the matrix with the ordered n^2-tuple consisting of its entries, and $\det : \mathbb{R}^{n^2} \to \mathbb{R}$ is continuous. Hence $GL(n, \mathbb{R})$ has a canonical differentiable structure (use matrix components $L^j{}_i \in \mathbb{R}^{n^2}$ as local coordinates). Clearly $GL(n, \mathbb{R})$ is *not* connected:

$$GL(n, \mathbb{R}) = G_+ \sqcup G_- = \det{}^{-1}(0, \infty) \sqcup \det{}^{-1}(-\infty, 0).$$

We say that $(\overline{e}_i)$ defines the same orientation as (e_i) if $\det L > 0$. Why? Then $L \in G_+ \Rightarrow L$ path connected to $\mathbb{I}_n \Rightarrow$ there exists a continuous deformation of $(\overline{e}_i)$ into (e_i) *through bases*.

Loosely speaking, a smooth manifold M is orientable if the local orientations on it defined by the charts can be "stitched together" consistently.

Definition 39 An atlas $\mathcal{A} = \{(U_i, \phi_i) : i \in I\}$ on a manifold M is an **orientation atlas** if the Jacobians of all transition functions have positive determinant. Two orientation atlases $\mathcal{A}_1, \mathcal{A}_{,2}$ are **equivalent** if $\mathcal{A}_1 \cup \mathcal{A}_2$ is an orientation atlas. An **orientation** on M is an equivalence class of orientation atlases. A manifold M is **orientable** if it (i.e. its differentiable structure) possesses an orientation atlas.

Note: Even if M is orientable, not very atlas will be an orientation atlas.

Example 40 The manifold S^n, as defined in Example 36, is orientable, although the stereographic atlas we defined there is *not* an orientation atlas :

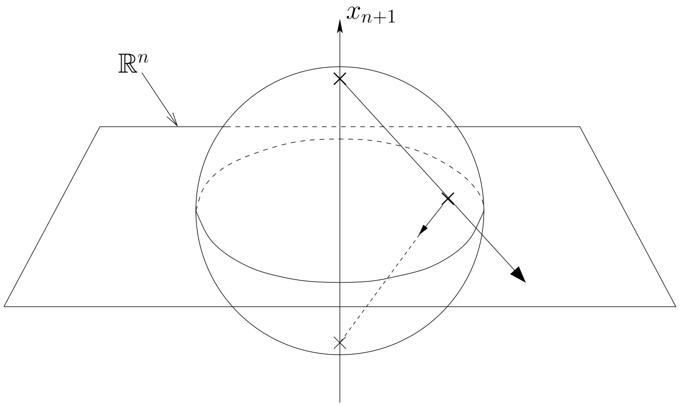


Figure 7: Stereographic projections of a point in S^n from the North and South poles.

Recall the transition function

$$\begin{aligned} \tau_{+-}: \quad \mathbb{R}^n \setminus \{0\} &\to \mathbb{R}^n \setminus \{0\} \\ \text{was} \qquad \mathbf{p} &\mapsto \frac{\mathbf{p}}{|\mathbf{p}|^2} \end{aligned}$$

The Jacobian of $\tau \equiv \tau_{+-}$ is

$$J(\mathbf{p}) = \begin{pmatrix} \frac{\partial \tau^1}{\partial p^1} & \frac{\partial \tau^1}{\partial p^2} & \cdots & \frac{\partial \tau^1}{\partial p^n} \\ \frac{\partial \tau^2}{\partial p^1} & \frac{\partial \tau^2}{\partial p^2} & \cdots & \frac{\partial \tau^2}{\partial p^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \tau^n}{\partial p^1} & \frac{\partial \tau^n}{\partial p^2} & \cdots & \frac{\partial \tau^n}{\partial p^n} \end{pmatrix} = \begin{pmatrix} |\mathbf{p}|^2 - 2p_1^2 & -2p_1p_2 & \cdots & -2p_1p_n \\ -2p_1p_2 & |\mathbf{p}|^2 - 2p_2^2 & \cdots & -2p_2p_n \\ \vdots & \vdots & \ddots & \vdots \\ -2p_1p_n & -2p_2p_n & \cdots & |\mathbf{p}|^2 - 2p_n^2 \end{pmatrix}$$

Since $\mathbb{R}^n \setminus \{0\}$ is connected and J must be invertible (because τ_{+-} is a diffeomorphism) it suffices to check $\det J(\mathbf{p})$ at just one point, e.g. $\mathbf{p}_0 = (1, 0, \ldots, 0)$

$$\det J(\mathbf{p}_0) = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} < 0$$

We can obtain an orientation atlas by composing ϕ_-, say, with any linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ with negative determinant, e.g. $L(p_1, p_2, \ldots, p_n) = (-p_1, p_2, \ldots, p_n)$. Then

$$\begin{aligned} \widetilde{\phi}_+(\underbrace{x_1, x_2, \ldots, x_n}_{\mathbf{x}}, x_{n+1}) &= \frac{\mathbf{x}}{1 - x_{n+1}} & \qquad (= \phi_+(\mathbf{x}, x_{n+1})) \\ \widetilde{\phi}_-(\underbrace{x_1, x_2, \ldots, x_n}_{\mathbf{x}}, x_{n+1}) &= \frac{(-x_1, x_2, x_3, \ldots, x_n)}{1 + x_{n+1}}. \end{aligned}$$

[*Exercise* : check this is an orientation atlas.] □

Theorem 41 *A connected orientable manifold M has precisely two orientations.*

Proof: Let $\mathcal{A} = \{(U_i, \phi_i) : i \in I\}$ be an orientation atlas. Then $\mathcal{A}' = \{(U_i, L \circ \phi_i) : i \in I\}$ (L as in Example 40 above) is another orientation atlas on M. Since $\mathcal{A} \cup \mathcal{A}'$ is clearly *not* an orientation atlas (transition function $(L \circ \phi_i) \circ \phi_i^{-1}$ has Jacobian L), M has at least two orientations.

Let $\overline{\mathcal{A}} = \{(V_j, \psi_j) : j \in J\}$ be another orientation atlas on M and define $M_+, M_- \subseteq M$ by

$$\begin{array}{rcl} p \in M_+ & \Leftrightarrow & \begin{array}{l} \exists \text{ charts } (U_i, \phi_i), (V_j, \psi_j) \text{ containing } p \\ \text{such that } \det(d[\phi_i \circ \psi_j^{-1}]_p) > 0 \end{array} \\ p \in M_- & \Leftrightarrow & \begin{array}{l} \exists \text{ charts } (U_i, \phi_i), (V_j, \psi_j) \text{ containing } p \\ \text{such that } \det(d[\phi_i \circ \psi_j^{-1}]_p) < 0 \end{array} \end{array}$$

Now, $M_+ \cup M_- = M$ because $\det[d(\phi \circ \psi^{-1})_p] \neq 0$ for all overlapping charts, and $M_+ \cap M_- = \emptyset$ because if $\det[d(\phi_i \circ \psi_j^{-1})_p] > 0$ and $\det[d(\phi_k \circ \psi_l^{-1})_p] < 0$ then, via the chain rule,

$$\begin{aligned} \det[d(\phi_i \circ \psi_k^{-1})_p] &= \det[d(\phi_i \circ \psi_j^{-1}) \circ (\psi_j \circ \psi_l^{-1}) \circ (\phi_k \circ \psi_l^{-1})_p^{-1}] \\ &= \det[d(\phi_i \circ \psi_j^{-1})_p \circ d(\psi_j \circ \psi_l^{-1})_p \circ d(\phi_k \circ \psi_l^{-1})_p^{-1}] \\ \Rightarrow \det[d(\phi_i \circ \phi_k^{-1})_p] &= \oplus \times \oplus \times \ominus < 0, \end{aligned}$$

which contradicts the assumption that $\mathcal{A}$ is an orientation atlas. So $M_+ \sqcup M_-$. Clearly M_+ is open in M (by continuity of det), as is M_-. But M is connected, so either $M_- = \emptyset$ (in which case $\overline{\mathcal{A}} \sim \mathcal{A}$) or $M_+ = \emptyset$ (in which case $\overline{\mathcal{A}} \sim \mathcal{A}'$). □

Example 42 (A non-orientable manifold) Let $X = \mathbb{R}^{n+1}\{0\}$ and define an equivalence relation, $\sim$, on X by $x \sim x' \Leftrightarrow \exists \lambda \in \mathbb{R}\backslash\{0\}$ such that $x' = \lambda x$. Then $X/\sim$ is the space of undirected straight lines through 0 in $\mathbb{R}^{n+1}$. We call this space, with the quotient topology, $\mathbb{R}P^n$. One can equip $\mathbb{R}P^n$ with an atlas (hence a differentiable structure) as follows.

Define open sets

$$U_i = \{[x] \in \mathbb{R}P^n : x_i \neq 0\} \qquad \text{for } i = 1, 2, \ldots, n$$

You should convince yourself these are open according to definition 15. Define homeomorphisms $\phi_i : U_i \to \mathbb{R}^n$ by

$$\phi_i([x]) = \left(\frac{x_1}{x_i}, \frac{x_2}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i} \right)$$

For example, U_1 consists of all lines through the origin which do not lie in the hyperplane $x_1 = 0$, and $\phi_1([x])$ can be identified with the intersection point of line $[x]$ with the plane $x_1 = 1$.
Exercise: check that all the transition functions τ_{ij} are smooth. Show that $\mathbb{R}P^2$ is not orientable. (In fact $\mathbb{R}P^n$ is orientable if and only if n is odd.)

Exercise 43 Let M, N be smooth manifolds. Show that $M \times N$ (with the product topology) inherits a canonical differentiable structure. Show that this product manifold is orientable if M and N are.

Example 44 It follows immediately that

(a) the n-torus $T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n\ times}$, and

(b) the cylinder $\mathbb{R} \times S^1$,

are orientable manifolds.

2.5 Calculus on Smooth Manifolds

Definition 45 A map $f : M \to N$ between smooth manifolds is **smooth at a point** $p \in M$ if there exist charts (U, ϕ) and (V, ψ) containing p and $f(p)$ respectively, such that $f(U) \subseteq V$ and the map

$$\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is smooth at $\phi(p)$ (in the sense of section 1.2). (see Figure 8)

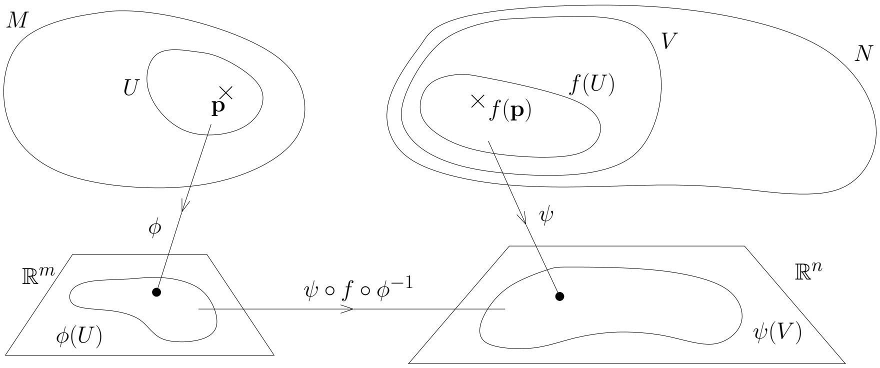


Figure 8: Smooth map

Note: Definition 45 would be useless if it depended on which charts (U, ϕ), (V, ψ) are chosen. Luckily it doesn't.

Let $(\widetilde{U}, \widetilde{\phi}), (\widetilde{V}, \widetilde{\psi})$ be alternative charts. Then

$$\widetilde{\psi} \circ f \circ \widetilde{\phi}^{-1} = (\widetilde{\psi} \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \widetilde{\phi}^{-1})$$

$\nwarrow$ $\nearrow$

diffeomorphisms of open sets
in $\mathbb{R}^n, \mathbb{R}^m$ respectively

$\Rightarrow \widetilde{\psi} \circ f \circ \widetilde{\phi}^{-1}$ is smooth at $\phi(p) \Leftrightarrow \psi \circ f \circ \phi^{-1}$ is smooth at $\phi(p)$.

Definition 46 $f : M \to N$ is **smooth** ($f \in C^\infty(M, N)$ if it is smooth at every $p \in M$. In particular, one writes $C^\infty(M)$ for $C^\infty(M, \mathbb{R})$. A bijection $f : M \to N$ is a **diffeomorphism** if both f and f^{-1} are smooth.

Diffeomorphism is the natural notion of equivalence in the class of smooth manifolds, just as homeomorphism is for topological spaces.

Example 47 We may define (infinitely many) inequivalent atlases on $\mathbb{R}$, hence infinitely many different differentiable structures. However, they are all diffeomorphic. For example, $\mathbb{R}_{\text{standard}}$ has atlas $\{(\mathbb{R}, \text{Id})\}$. We could define $\mathbb{R}_{\text{fake}}$ to have atlas $\{(\mathbb{R}, \phi)\}$ where $\phi(p) = p^3$. $\mathbb{R}_{\text{standard}}$ and $\mathbb{R}_{\text{fake}}$ are not the *same* manifold since $\{(\mathbb{R}, \text{Id}), (\mathbb{R}, \phi)\}$ is not an atlas:

$$(\text{Id} \circ \phi^{-1})(x) = \text{Id}(x^{\frac{1}{3}}) = x^{\frac{1}{3}} \quad \text{is not smooth.}$$

Nevertheless, they *are* diffeomorphic. Here's a diffeomorphism:

$$f : \mathbb{R}_{\text{fake}} \to \mathbb{R}_{\text{standard}}, \quad f : p \to p^3.$$

In local coordinates, this map is

$$(\text{Id} \circ f \circ \phi^{-1})(x) = (\text{Id} \circ f)(x^{\frac{1}{3}}) = \text{Id}(x) = x$$

which is clearly smooth, with smooth inverse. So $\mathbb{R}_{\text{standard}}$ and $\mathbb{R}_{\text{fake}}$ are different but equivalent smooth manifolds. $\square$

Bizarre Fact 48 Every $\mathbb{R}^n$ has precisely one differentiable structure up to diffeomorphism, except $\mathbb{R}^4$, which has *uncountably infinitely many* inequivalent differentiable structures!

3 The Tangent Bundle

3.1 Tangent Space T_pM

Tangent vectors to a smooth surface $X \subseteq \mathbb{R}^n$ at $\mathbf{p} \in X$ are easy to define: they are the velocity vectors of curves in X passing through $\mathbf{p}$, that is

$$\dot{c}(0) = \lim_{h \to 0} \frac{c(h) - c(0)}{h}.$$

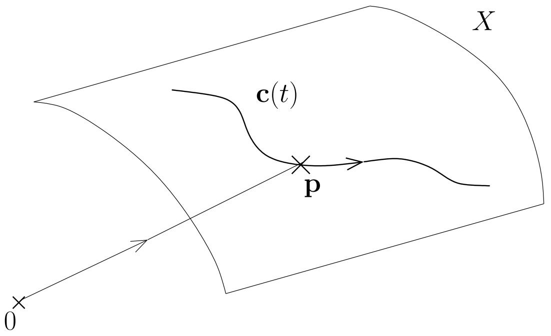


Figure 9: Defining a tangent vector to a smooth surface in R^n.

This definition relies on the linearity of the ambient space $\mathbb{R}^n$, and *does not* generalize directly to abstract smooth manifolds:
what does $\lim_{h \to 0} \frac{c(h) - c(0)}{h}$ mean ?

Definition 49 A **smooth curve through a point** $p \in M$ is a smooth map $c : I \to M$ where I is an open interval in $\mathbb{R}$ containing 0 and $c(0) = p$. $\Box$

Given any chart (U, ϕ) around p,

$$\phi \circ c : I \to \phi(U) \subseteq \mathbb{R}^m$$

has a well defined derivative ("velocity vector") at 0: $(\phi \circ c)'(0) \in \mathbb{R}^m$.

Definition 50 Two curves c_1, c_2 through p are said to be equivalent, $c_1 \sim c_2$, if there exists a chart (U, ϕ) around p such that

$$(\phi \circ c_1)'(0) = (\phi \circ c_2)'(0) \qquad \Box$$

Crucial point: it doesn't matter which chart one chooses to check equivalence. Let (V, ψ) be another chart around p. Then

$$\begin{aligned} (\psi \circ c_i)'(0) &= ((\psi \circ \phi^{-1}) \circ (\phi \circ c_i))'(0) \\ &= d(\psi \circ \phi^{-1})_{\phi(p)} \circ (\phi \circ c_i)'(0) \qquad \text{(Chain rule)} \end{aligned}$$

$\Rightarrow (\psi \circ c_1)'(0) = (\psi \circ c_2)'(0) \Leftrightarrow (\phi \circ c_1)'(0) = (\phi \circ c_2)'(0)$.

Definition 51 A **tangent vector** at p is an equivalence class $[c]$ of curves through p. The set of all such equivalence classes is called the **tangent space**, T_pM. Given a tangent vector v any curve in v is called a **generating curve** of v. $\square$

To justify the names “vector” and “space” we must equip T_pM with a canonical linear structure over $\mathbb{R}$, i.e. we will make T_pM a vector space isomorphic to $\mathbb{R}^m$ ($m = \dim M$).

Definition 52 Given a chart (U, ϕ) around p, define

$$\mathrm{Vec}_{(U,\phi)} : T_pM \to \mathbb{R}^m, \quad \mathrm{Vec}_{(U,\phi)}\big([c]\big) = (\phi \circ c)'(0) \ . \qquad \square$$

$\mathrm{Vec}_{(U,\phi)}$ is a bijection: it is injective by definition of $\sim$. It is surjective since, for any $\mathbf{v} \in \mathbb{R}^m$, let $c(t) = \phi^{-1}(\mathbf{x}_p + \mathbf{v}t)$ where $\mathbf{x}_p = \phi(p) \in \mathbb{R}^m$. Then $\mathrm{Vec}_{(U,\phi)}\big([c]\big) = \mathbf{v}$ by definition. Therefore, one can use $\mathrm{Vec}_{(U,\phi)}$ to transfer the linear structure of $\mathbb{R}^m$ to T_pM.

i.e. $\alpha[c_1] + \beta[c_2] := \mathrm{Vec}^{-1}_{(U,\phi)}\big(\alpha \mathrm{Vec}_{(U,\phi)}\big([c_1]\big) + \beta \mathrm{Vec}_{(U,\phi)}\big([c_2]\big)\big)$

But this only makes sense if it is chart independent. (Note that the identification

$$[c] \overset{\mathrm{Vec}_{(U,\phi)}}{\longrightarrow} \mathbf{v} \in \mathbb{R}^m$$

certainly is *not*.) It is, by previous work: if $\mathbf{v} = \mathrm{Vec}_{(U,\phi)}\big([c]\big)$ and $\widetilde{\mathbf{v}} = \mathrm{Vec}_{(U,\psi)}\big([c]\big)$, then

$$\widetilde{\mathbf{v}} = \underbrace{d(\psi \circ \phi^{-1})_{\phi(p)}}_{\text{isomorphism of } \mathbb{R}^m} \mathbf{v} \quad = J\mathbf{v} \tag{1}$$

where J is the Jacobian matrix of $\psi \circ \phi^{-1}$ at p.

Associated to the chart (U, ϕ) around $p = \phi^{-1}(x_p)$, there is a natural basis for T_pM called the **coordinate basis**, namely

$$\left.\frac{\partial}{\partial x^i}\right|_p := \mathrm{Vec}^{-1}_{(U,\phi)}(e_i)$$

where $e_1, \ldots, e_m$ denotes the usual basis for $\mathbb{R}^m$. A natural generating curve for $\partial/\partial x^i|_p$ is

$$\phi^{-1}(x_p^1, x_p^2, \ldots, x_p^i + t, \ldots, x_p^m).$$

Thinking of ϕ as associating local coordinates $x^1, \ldots, x^m$ to points in $U \subset M$, this is the curve along which all coordinates except x^i are held fixed, and x^i increases at “unit speed”. The reason for the notation, which may seem odd at first, will become clear shortly.

Example 53 Let $M = \mathbb{R}P^2$ and $p = [1, 2, 0]$. This is in charts (U_1, ϕ_1) and (U_2, ϕ_2), with respect to which it has local coordinates $y_p = \phi_1(p) = (2, 0)$ and $z_p = \phi_2(p) = (\frac{1}{2}, 0)$ respectively. Let us express the coordinate basis vectors $\partial/\partial z^1|_p, \partial/\partial z^2|_p$ for chart 2 in terms of $\partial/\partial y^1|_p, \partial/\partial y^2|_p$, the coordinate basis vectors for chart 1. As a generating curve for $\partial/\partial z^1|_p$ we can take

$$c_1(t) = \phi_2^{-1}(\frac{1}{2} + t, 0) = [\frac{1}{2} + t, 1, 0]$$

The image of this under ϕ_1 is

$$\phi_1(c_1(t)) = \phi_1([\frac{1}{2} + t, 1, 0]) = (\frac{1}{\frac{1}{2} + t}, 0)$$

whose derivative at $t = 0$ is

$$(\phi_1 \circ c_1)'(0) = (-4, 0)$$

Hence

$$\left.\frac{\partial}{\partial z^1}\right|_p = -4 \left.\frac{\partial}{\partial y^1}\right|_p$$

Similarly, $\partial/\partial z^2|_p$ is generated by

$$c_2(t) = \phi_2^{-1}(\frac{1}{2}, t) = [\frac{1}{2}, 1, t],$$

whose image under ϕ_1 is

$$\phi_1(c_2(t)) = (2, 2t).$$

Now

$$(\phi_1 \circ c_2)'(0) = (0, 2)$$

so

$$\left.\frac{\partial}{\partial z^2}\right|_p = 2 \left.\frac{\partial}{\partial y^2}\right|_p$$

3.2 Directional Derivatives

Definition 54 Let $f \in C^\infty(M)$ and $\mathbf{v} \in T_pM$.The **directional derivative** of f with respect to v, denoted $v[f]$, is $(f \circ c)'(0) \in \mathbb{R}$, where c is any generating curve for v.

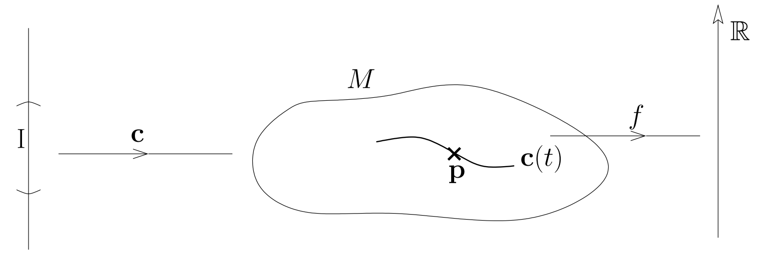


Figure 10: $f \circ c$

We should check that $v[f]$ is independent of the choice of generating curve $c \in v$. So, let $\bar{c}$ be another generating curve for v, and (U, ϕ) be a chart around p. Then

$$\begin{aligned}(f \circ \bar{c})'(0) &= (f \circ \phi^{-1} \circ \phi \circ \bar{c})'(0) \\ &= d(f \circ \phi^{-1})_{\phi(p)}(\phi \circ \bar{c})'(0) \qquad (\clubsuit) \\ &= d(f \circ \phi^{-1})_{\phi(p)}(\phi \circ c)'(0) = (f \circ c)'(0)\end{aligned}$$

since $\bar{c} \sim c$.

Proposition 55 *The map $T_pM \times C^\infty(M) \to \mathbb{R}$, $(v, f) \to v[f]$ is $\mathbb{R}$ linear. That is, for all $u, v \in T_pM$, $f, g \in C^\infty(M)$ and $a, b \in \mathbb{R}$,*

$$(au + bv)[f] = au[f] + bv[f], \qquad u[af + bg] = au[f] + bu[g]$$

Proof: Exercise. It follows immediately from formula ($\clubsuit$) above, and the definition of the linear structure on T_pM. $\square$

Example 56 Let $f : \mathbb{R}P^2 \to \mathbb{R}$ such that

$$f([x_1, x_2, x_3]) = \frac{x_1 x_2}{x_3^2 + x_2^2},$$

$p = [1, 2, 0]$ and $v = \left.\frac{\partial}{\partial y_1}\right|_p$ (as in example 53). What is $v[f]$? Pick a representative curve, e.g. $c_1(t) = [1, 2 + t, 0]$. Then

$$(f \circ c_1)(t) = \frac{2 + t}{0 + (2 + t)^2} \quad \Rightarrow v[f] = (f \circ c)'(0) = -\frac{1}{4} \qquad \square$$

Let (U, ϕ) be a chart around $p \in M$, and denote by $x^1, \ldots, x^m$ the corresponding local coordinates. Recall that $\partial/cdx^i|p$ is the tangent vector generated by the curve $c_i(t) = \phi^{-1}(x_p^1, x_p^2, \ldots, x_p^i + t, \ldots, x_p^m)$. So, for any $f \in C^\infty(M)$,

$$\left.\frac{\partial}{\partial x^i}\right|_p [f] = (f \circ c_i)'(0) = \left.\frac{d}{dt}\right|_{t=0} (f \circ \phi^{-1})(x_p^1, x_p^2, \ldots, x_p^i + t, \ldots, x_p^m) = \frac{\partial}{\partial x^i} \widehat{f}(x^1, x^2, \ldots, x^m)\Big|_{x=x_p}$$

where $\widehat{f} = f \circ \phi^{-1}$ is the function f "expressed in local coordinates". This explains why we denote the coordinate basis vectors by $\partial/\partial x^i$.

Example 56 revisited Working in the chart (U_1, ϕ_1), we see that

$$f([x_1, x_2, x_3]) = \frac{(x_2/x_1)}{(x_3/x_1)^2 + (x_2/x_1)^2}$$

that is

$$\widehat{f}(y^1, y^2) = \frac{y^1}{(y^2)^2 + (y^1)^2}$$

and $p = [1, 2, 0]$ has local coordinates $y_p = (2, 0)$. So

$$\left.\frac{\partial}{\partial y^1}\right|_p f = \frac{\partial}{\partial y^1} \frac{y^1}{(y^2)^2 + (y^1)^2}\Big|_{y=(2,0)} = -\frac{1}{4}$$

in agreement with our previous calculation.

3.3 The differential mapping

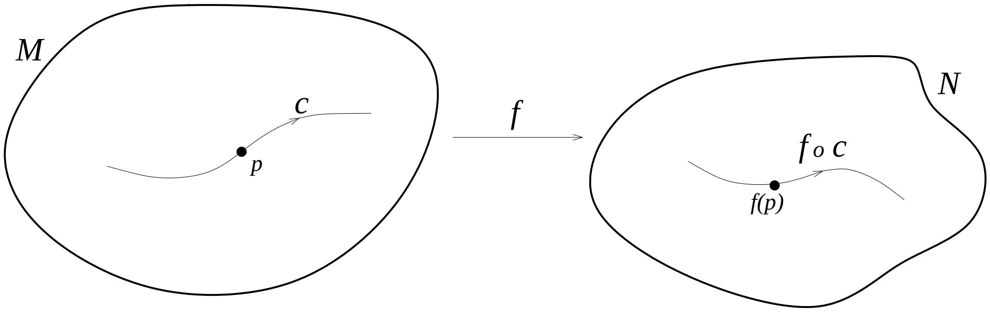


Figure 11: Function composition defines a natural map $T_pM \to T_{f(p)}N$

Any smooth map $f : M \to N$ maps a curve c through p to a curve $f \circ c$ through $f(p)$. Let $\widetilde{c} \sim c$ Choose charts (U, ϕ), (V, ψ) around p, $f(p)$ respectively. Then

$$(\phi \circ \widetilde{c})'(0) = (\phi \circ c)'(0)$$

by definition of $\sim$. Hence

$$\begin{aligned}(\psi\circ f\circ\widetilde{c})'(0) &= (\psi\circ f\circ\phi^{-1}\circ\phi\circ\widetilde{c})'(0)\\ &= d(\psi\circ f\circ\phi^{-1})_{\phi(p)}(\phi\circ\widetilde{c})'(0)\\ &= d(\psi\circ f\circ\phi^{-1})_{\phi(p)}(\phi\circ c)'(0)\\ &= (\psi\circ f\circ c)'(0), \qquad (2)\end{aligned}$$

that is $f\circ\widetilde{c}\sim f\circ c$, the image curves are also equivalent! So composition with f maps *equivalence classes* of curves through $p\in M$ to *equivalence classes* of curves through $f(p)\in N$.

Definition 57 The **differential map** of f at p is $df_p : T_pM \to T_{f(p)}N$ defined by $df_p : [c]\mapsto [f\circ c]$. $\Box$

Another interpretation of the last equality in (2) is

$$\begin{aligned}\mathrm{Vec}_{(V,\psi)}\big(df_p[c]\big) &= d(\psi\circ f\circ\phi^{-1})_{\phi(p)}\,\mathrm{Vec}_{(U,\phi)}\big([c]\big)\\ \Rightarrow\quad df_p &= \mathrm{Vec}^{-1}_{(V,\psi)}\circ d(\psi\circ f\circ\phi^{-1})_{\phi(p)}\circ\mathrm{Vec}_{(U,\phi)}.\end{aligned}$$

Recall that the linear structures on T_pM, $T_{f(p)}N$ are *defined* by transferring those on $\mathbb{R}^m$, $\mathbb{R}^n$ using $\mathrm{Vec}_{(U,\phi)}$, $\mathrm{Vec}_{(V,\psi)}$ respectively. Hence df_p is **linear** by linearity of $d(\psi\circ f\circ\phi^{-1})_{\phi(p)} : \mathbb{R}^m\to\mathbb{R}^n$.

Proposition 58 (Chain Rule) *If $f : M\to N$ is smooth at p and $g : N\to P$ is smooth at $f(p)$ then $g\circ f$ is smooth at p and*

$$d(g\circ f)_p = dg_{f(p)}\circ df_p.$$

Proof: Check smoothness of $g\circ f$ in charts. Then for any $[c]\in T_pM$,

$$\begin{aligned}d(g\circ f)_p[c] &= [(g\circ f)\circ c] = [g\circ(f\circ c)]\\ &= dg_{f(p)}[f\circ c]\\ &= dg_{f(p)}\big(df_p[c]\big)\ . \qquad\Box\end{aligned}$$

So the Chain Rule follows immediately from associativity of function composition! Note, however, that we needed the “old” chain rule (for maps between open sets in $\mathbb{R}^k$) to prove that df_p is well defined – we haven’t got something for nothing. In fact, relative to the coordinate bases defined by (U,ϕ), (V,ψ), df_p *is* the usual differential map $d\widehat{f}_{x_p}$ where $\widehat{f}=\psi\circ f\circ\phi^{-1}$ and $x_p=\phi(p)$. (Often in defining f one actually specifies $\psi\circ f\circ\phi^{-1}$ rather than f itself.)

Example 59 Let $f : \mathbb{R}P^2 \to \mathbb{R}P^3$ such that

$$f([x_1, x_2, x_3]) = [x_1^3, x_2^3, x_3^3, x_1x_2x_3].$$

Define the charts

$$\begin{aligned} U = \{[x] \in \mathbb{R}P^2 \ : \ x_1 \neq 0\}, &\qquad V = \{[x] \in \mathbb{R}P^3 \ : \ x_1 \neq 0\} \\ \phi([x]) = \big(\frac{x_2}{x_1}, \frac{x_3}{x_1}\big) &\qquad \psi([x]) = \big(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_1}\big). \end{aligned}$$

Then the coordinate expression of f in these charts is

$$\begin{aligned} \widehat{f}(y^1, y^2) &= (\psi \circ f\phi^{-1})(y^1, y^2) = (\psi \circ f)([1, y^1, y^2]) \\ &= \psi([1, (y^1)^3, (y^2)^3, y^1y^2]) = ((y^1)^3, (y^2)^3, y^1y^2)) \end{aligned}$$

Clearly

$$d\widehat{f}_{(y^1,y^2)} = \begin{bmatrix} 3(y^1)^2 & 0 \\ 0 & 3(y^2)^2 \\ y^2 & y^1 \end{bmatrix}.$$

So the image of $\partial/\partial y^1|_{[1,1,2]}$ under $df_{[1,1,2]}$ is, relative to the (V, ψ) coordinate basis,

$$d\widehat{f}_{(1,2)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 12 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix},$$

that is,

$$df_{[1,1,2]} \frac{\partial}{\partial y^1}\Big|_{[1,1,2]} = 3\frac{\partial}{\partial z^1}\Big|_{f([1,1,2])} + 2\frac{\partial}{\partial z^3}\Big|_{f([1,1,2])} = 3\frac{\partial}{\partial z^1}\Big|_{[1,1,8,2]} + 2\frac{\partial}{\partial z^3}\Big|_{[1,1,8,2]}. \quad \square$$

3.4 The Tangent Bundle

At each $p \in M$ we have a vector space T_pM. The **tangent bundle** is the union of all these tangent spaces:

$$TM = \bigsqcup_{p \in M} T_pM$$

Note: we usually denote a point v in TM by an ordered pair (p, v) (where $v \in T_pM$) to emphasize which “fibre” v lies in. TM should not (necessarily) be thought of as $M \times \mathbb{R}^m$, however.

Define the projection map $\pi : TM \to M$ by

$$\pi : (p, v) \mapsto p$$

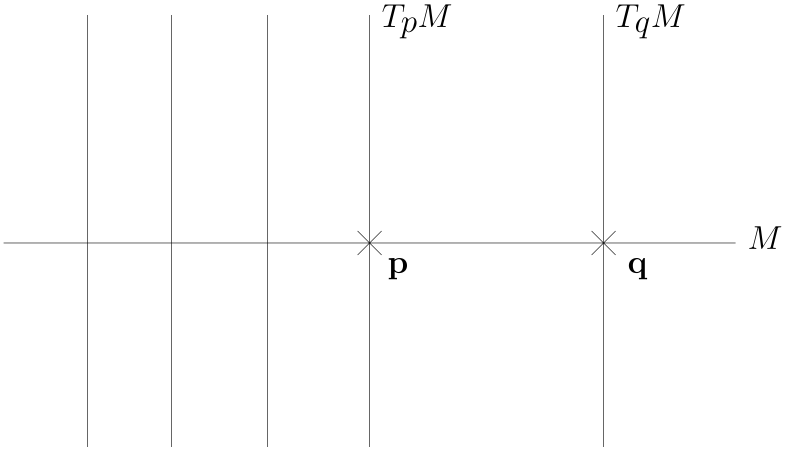


Figure 12: The tangent bundle.

Proposition 60 *Let M be a smooth manifold of dimension m. Then TM has a canonical differentiable structure with respect to which it is an orientable smooth manifold of dimension $2m$.*

Proof: Let $\mathcal{A} = \{(U_i, \phi_i) : i \in I\}$ be an atlas on M. Define an "atlas" $T\mathcal{A}$ on TM as follows

$$T\mathcal{A} = \{(TU_i, \Phi_i) : i \in I\}$$

where $TU_i = \pi^{-1}(U_i)$ and Φ_i is the bijection

$$\Phi_i(p,v) = (\phi_i(p), \mathrm{Vec}_{(U_i,\phi_i)}(v)) \in \phi_i(U_i) \times \mathbb{R}^m$$

Clearly $T\mathcal{A}$ is a covering of TM (open, assuming π is continuous) and $\Phi_i(TU_i)$ are all open subsets of $\mathbb{R}^{2m}$.

Claim: $\exists$ a natural topology on TM with respect to which Φ_i are homeomorphisms (has basis $\Phi_i^{-1}(B)$, B open in $\phi_i(U_i) \times \mathbb{R}^m$: the details are technical).

It remains to check smoothness of transition functions:
Let $\xi = (p,v) \in TU_i \cap TU_j$ and

$$\Phi_i(p,v) = (\phi_i(p), \mathrm{Vec}_{(U_i,\phi_i)}(v)) = (x, \widetilde{v}) \in \mathbb{R}^{2m}$$

Then

$$(\Phi_j \circ \Phi_i^{-1})(x, \widetilde{v}) = ((\phi_j \circ \phi_i^{-1})(x), (\mathrm{Vec}_{(U_j,\phi_j)} \circ \mathrm{Vec}^{-1}_{(U_i,\phi_i)})(\widetilde{v}))$$

$$= \left.\Big(\underset{\substack{\uparrow \qquad \uparrow \\ C^\infty \text{ w.r.t. } x \\ \text{trivially } C^\infty \text{ w.r.t. } \widetilde{v}}}{(\phi_j \circ \phi_i^{-1})(x)}\ , \qquad \underset{\substack{\uparrow \qquad \uparrow \\ C^\infty \text{ w.r.t. } x \\ C^\infty \text{ w.r.t. } \widetilde{v}}}{d(\phi_j \circ \phi_i^{-1})_x(\widetilde{v})} \Big)\right\} \textit{smooth}$$

Hence $T\mathcal{A}$ is an atlas. To prove the differentiable structure on TM is *canonical*, we should really check that $\mathcal{A} \sim \mathcal{A}'$ implies $T\mathcal{A} \sim T\mathcal{A}'$. It does.

Now consider the differential at $(x, \widetilde{v})$ of $(\Phi_j \circ \Phi_i^{-1})$

$$d(\Phi_j \circ \Phi_i^{-1})_{(x,\widetilde{v})} = \left(\begin{array}{c|c} d(\phi_j \circ \phi_i^{-1})_x & 0 \\ \hline \begin{array}{c}\text{2nd}\\ \text{derivative}\\ \text{stuff}\end{array} & d(\phi_j \circ \phi_i^{-1})_x \end{array}\right) : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$$

So

$$\det[d(\Phi_j \circ \Phi_i^{-1})_{(x,\widetilde{v})}] = \left[\det[d(\phi_j \circ \phi_i^{-1})_x]\right]^2 > 0$$

Hence $T\mathcal{A}$ is automatically an orientation atlas (irrespective of whether M is orientable). $\square$

Definition 61 TM is **trivial** if there exists a diffeomorphism $f : TM \to M \times \mathbb{R}^m$ preserving fibres, i.e. such that $f : \underbrace{\pi^{-1}(p)}_{T_pM} \to \{p\} \times \mathbb{R}^m$ is a vector space isomorphism. $\square$

Trivial: Any 1-dimensional manifold, $U \subset \mathbb{R}^n$, any compact orientable 3-dimensional manifold (*not* obvious!), any Lie Group (more later), n-torus T^n.
Non-trivial: S^2, any non-orientable M.

3.5 Vector fields

Definition 62 A **vector field** on M is a smooth map $X : M \to TM$ such that $\pi \circ X = \mathrm{Id}_M$ where $\pi : TM \to M$ is the projection map (i.e. $X(p) \in T_pM$ for all $p \in M$). The set of all vector fields on M is denoted $\Gamma(TM)$. $\square$

Example 63 Consider S^2 with stereographic coordinate $\boldsymbol{\rho} = (\rho^1, \rho^2)$ as before.
Now, $X = \rho^2 \dfrac{\partial}{\partial \rho^1} - \rho^1 \dfrac{\partial}{\partial \rho^2}$ is a smooth vector field on $S^2 \setminus \{(0,0,1)\}$. Can it be extended to a smooth vector field on all S^2?

Recall that the stereographic coordinates projected from the South pole $(0,0,-1)$, $\boldsymbol{\sigma} = (\sigma^1, \sigma^2)$ are related to (ρ^1, ρ^2) by $\boldsymbol{\sigma} = \dfrac{\boldsymbol{\rho}}{|\boldsymbol{\rho}|^2}$ and $\boldsymbol{\rho} = \dfrac{\boldsymbol{\sigma}}{|\boldsymbol{\sigma}|^2}$.

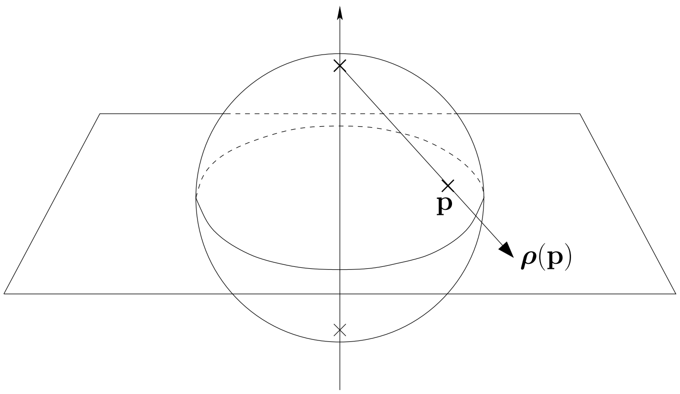


Figure 13: Stereographic projection from $(0, 0, 1)$

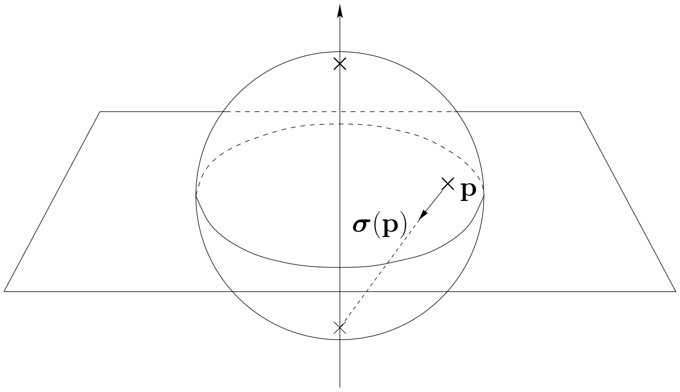


Figure 14: Stereographic projection from $(0, 0, -1)$

Now,

$$\begin{aligned}
\frac{\partial}{\partial \rho^1} &= \frac{\partial \sigma^i}{\partial \rho^1}\frac{\partial}{\partial \sigma^i} = \frac{\partial \sigma^1}{\partial \rho^1}\frac{\partial}{\partial \sigma^1} + \frac{\partial \sigma^2}{\partial \rho^1}\frac{\partial}{\partial \sigma^2} \\
&= \frac{|\boldsymbol{\rho}|^2 - 2(\rho^1)^2}{|\boldsymbol{\rho}|^4}\frac{\partial}{\partial \sigma^1} + \frac{(-2\rho^1\rho^2)}{|\boldsymbol{\rho}|^4}\frac{\partial}{\partial \sigma^2} \\
&= \frac{(\rho^2)^2 - (\rho^1)^2}{|\boldsymbol{\rho}|^4}\frac{\partial}{\partial \sigma^1} + \frac{(-2\rho^1\rho^2)}{|\boldsymbol{\rho}|^4}\frac{\partial}{\partial \sigma^2} \\
&= \frac{(\rho^2)^2 - (\rho^1)^2}{|\boldsymbol{\rho}|^4}\frac{\partial}{\partial \sigma^1} + \frac{(-2\rho^1\rho^2)}{|\boldsymbol{\rho}|^4}\frac{\partial}{\partial \sigma^2} \\
&= [(\sigma^2)^2 - (\sigma^1)^2]\frac{\partial}{\partial \sigma^1} - 2\sigma^1\sigma^2\frac{\partial}{\partial \sigma^2}
\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial}{\partial\rho^2} &= -\frac{2\rho^1\rho^2}{|\boldsymbol{\rho}|^4}\frac{\partial}{\partial\sigma^1} + \frac{(\rho^1)^2-(\rho^2)^2}{|\boldsymbol{\rho}|^4}\frac{\partial}{\partial\sigma^2} \\ &= -2\sigma^1\sigma^2\frac{\partial}{\partial\sigma^1} + [(\sigma^1)^2-(\sigma^2)^2]\frac{\partial}{\partial\sigma^2}\end{aligned}$$

$$\begin{aligned}\Rightarrow X = &\frac{\sigma^2}{|\boldsymbol{\sigma}|^2}\Big\{[(\sigma^2)^2-(\sigma^1)^2]\frac{\partial}{\partial\sigma^1} - 2\sigma^1\sigma^2\frac{\partial}{\partial\sigma^2}\Big\} \\ &-\frac{\sigma^1}{|\boldsymbol{\sigma}|^2}\Big\{-2\sigma^1\sigma^2\frac{\partial}{\partial\sigma^1} + [(\sigma^1)^2-(\sigma^2)^2]\frac{\partial}{\partial\sigma^2}\Big\} \\ =&\,\sigma^2\frac{\partial}{\partial\sigma^1} - \sigma^1\frac{\partial}{\partial\sigma^2}\end{aligned}$$

So we can extend X smoothly to $p=(0,0,1)$ ($\boldsymbol{\sigma}=(0,0)$). Namely $X(0,0,1)=0$. □

Definition 64 A **derivation** on $C^\infty(M)$ is an $\mathbb{R}$-linear map $\delta : C^\infty(M) \to C^\infty(M)$ satisfying the Leibniz rule:

$$\forall\, f,g \in C^\infty(M) \qquad \delta[fg] = \delta[f]g + f\delta[g].$$ □

Every vector field X defines a derivation δ_X, by directional derivatives:

$$\big(\delta_X[f]\big)(p) = X(p)[f] = \text{directional derivative of } f \text{ w.r.t. } X(p)$$

We will henceforth denote this derivation $X[f]$. If, in local coordinates, $X(p) = X^i(x_p)\frac{\partial}{\partial x^i}\Big|_p$ and $f = f(x^1,\dots,x^m)$ then

$$\big(X[f]\big)(p) = X^i(x_p)\frac{\partial f}{\partial x^i}\Big|_{x_p}$$

It follows immediately that this map $C^\infty(M)\to C^\infty(M)$ satisfies the Leibniz rule, by the product rule for partial differentiation.

In fact, the converse is also true: given a derivation δ, there exists a unique vector field $X_\delta \in \Gamma(TM)$ such that $X_\delta[f] = \delta[f]$ for all $f\in C^\infty(M)$. (Construct it by considering the images of $f(x^1,\dots,x^m)=x^i$, $i=1,\dots,m$, under δ.) Thought of this way, it makes sense to *compose* vector fields:

$$\begin{aligned}\forall\, X,Y\in\Gamma(TM) \quad &XY : C^\infty(M)\to C^\infty(M) \\ &(XY)[f] = X\big[Y[f]\big]\end{aligned}$$

Note, however, that XY is *not* a derivation since $(XY)[fg] \neq (XY)[f]g + f(XY)[g]$).

However, $(XY - YX) : C^\infty(M) \to C^\infty(M)$ *is* a derivation:

$$\begin{aligned}(XY-YX)[fg] &= X\big[Y[f]g + fY[g]\big] - \big(Y\big[X[f]g + fX[g]\big]\big)\\ &= (XY)[f]g + Y[f]X[g] + X[f]Y[g] + f(XY)[g]\\ &\quad -\big((YX)[f]g + Y[f]X[g] + X[f]Y[g] + f(YX)[g]\big)\\ &= (XY-YX)[f]g + f(XY-YX)[g]\end{aligned}$$

In local coordinates: $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^i \frac{\partial}{\partial x^i}$,

$$XY - YX = \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\right)\frac{\partial}{\partial x^i}$$

Definition 65 The **Lie bracket** of $X, Y \in \Gamma(TM)$ is

$$[X, Y] = XY - YX.$$

Note: The map $[,] : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$,

(i) is antisymmetric: $[X, Y] \equiv -[Y, X]$

(ii) is $\mathbb{R}$-linear: $\forall\, \alpha, \beta \in \mathbb{R} \quad [\alpha X_1 + \beta X_2, Y] \equiv \alpha[X_1, Y] + \beta[X_2, Y]$

(iii) satisfies the Jacobi identity

$$\big[[X,Y],Z\big] + \big[[Y,Z],X\big] + \big[[Z,X],Y\big] \equiv 0$$

So $\Gamma(TM)$ is a **Lie algebra** with bracket $[,]$. Clearly, it is infinite dimensional.

Example 66 Let $M = S^1 \times \mathbb{R}$, use local coordinates $(\theta, z) \in (0, 2\pi) \times \mathbb{R}$. Then the Lie bracket of

$$X = \frac{\partial}{\partial \theta} + \frac{\partial}{\partial z}, \quad Y = \sin\theta \frac{\partial}{\partial z}$$

is

$$[X, Y] = \left[\frac{\partial}{\partial\theta} + \frac{\partial}{\partial z}, \sin\theta\frac{\partial}{\partial z}\right] = \left[\frac{\partial}{\partial\theta}, \sin\theta\frac{\partial}{\partial z}\right] = \cos\theta\frac{\partial}{\partial z} \ \square$$

Definition 67 Let $X \in \Gamma(TM)$. The **flow** of X is the one parameter family (parametrized by $p \in M$) of smooth curves $c_p : I_p \to M$, where I_p is some open interval containing 0, which satisfy the differential equation

$$\dot{c}_p(t) = X(c_p(t)) \tag{3}$$

with initial data $c_p(0) = p$. $\square$

Notes:

(i) Given a curve $c : I \to M$, $\dot{c}(t)$ means, for each fixed $t \in I$, the tangent vector in $T_{c(t)}M$ generated by the curve $a(s) = c(t+s)$ through $c(t)$, i.e. the vector $[a]$. This is how we interpret the l.h.s. of equation (3), not as the derivative of $c(t)$ with respect to t, which, as it stands, makes no sense. An alternative interpretation is

$$\dot{c}(t) = dc_t \frac{d}{dt}\bigg|_t$$

where $d/dt|_t$ is the coordinate basis vector on I.

(ii) We can express equation (3) locally as a system of $m = \dim M$ coupled (in general, nonlinear) ordinary differential equations, by introducing a coordinate chart about the point p:

$$\dot{x}^i = X^i(x^1, \ldots, x^m), \quad \text{where} \quad X = X^i \frac{\partial}{\partial x^i}.$$

The right hand side of this system is smooth, hence Lipschitz, so the usual existence and uniqueness theory for solutions to ODE systems applies, and we can be sure that each $c_p(t)$ exists locally (in time), is unique, and varies smoothly with p.

(iii) The maximal interval of existence I_p may be finite (the system is nonlinear) and depends on p in general. If X has compact support (that is, $X = 0$ outside some compact subset of $K \subseteq M$ – this is vacuously true if M itself is compact) then $I_p = \mathbb{R}$ for all $p \in M$. The point is that K can be covered by a *finite* collection of coordinate charts, so we may define a *global* Lipschitz constant for the r.h.s. of (3) by taking the maximum of the Lipschitz constants on this finite collection, and hence a global interval $I = [-T, T]$ on which the flow exists for all p. We may then apply the flow iteratively to obtain a solution for all time whatever the initial position p.

(iv) If X does not have compact support (so, necessarily, M must be noncompact), it is possible that

$$\bigcap_{p \in M} I_p = \{0\}$$

so that there is no time interval on which the flow is well defined for all p.

Example 68 On $M = \mathbb{R}$ compute the flow of $X = x^2\, d/dx$. The whole of $\mathbb{R}$ is covered by a single chart. The flow equation is

$$\begin{array}{rcl} \dot{x}(t) & = & x(t)^2, \qquad x(0) = x_0 \\ \Rightarrow \quad \dfrac{d}{dt}\left(\dfrac{1}{x(t)}\right) & = & 1 \\ \Rightarrow \quad -\dfrac{1}{x(t)} & = & t + \kappa = t - \dfrac{1}{x_0} \\ \Rightarrow \quad x(t) & = & \dfrac{x_0}{1 - x_0 t} \\ \text{Hence} \quad c_x(t) & = & \dfrac{x}{1 - xt} \\ \Rightarrow \quad I_x & = & \left\{ \begin{array}{cl} (-\infty, x^{-1}), & x > 0 \\ (x^{-1}, \infty), & x < 0 \\ \mathbb{R}, & x = 0 \end{array} \right. \\ \Rightarrow \quad \displaystyle\bigcap_{x \in \mathbb{R}} I_x & = & \{0\} \end{array}$$

All point except $x = 0$ (which is a fixed point of the flow) get swept by the flow to $\pm\infty$ in finite time, and that time shrinks to 0 as $|x| \to \infty$. □

Exercise 69 Compute the flows of the vector fields in Example 66, i.e. solve for curves $c : I \to M$ satisfying

$$\dot{c}(t) = X(c(t))$$

$\big($or $Y\big(c(t)\big)$, $[X, Y]\big(c(t)\big)\big)$ with general initial data. □

If the flow of X is globally defined, that is, $c_p(t)$ exists for all $t \in \mathbb{R}$ for all $p \in M$, then we can use it to define a one parameter family of diffeomorphisms of X, namely, for each $t \in \mathbb{R}$ we define

$$\theta_t : M \to M \quad \text{such that} \quad \theta_t : p \mapsto c_p(t).$$

The map θ_t is smooth by the smooth dependence of solutions of a (smooth) ODE system on initial data, and has (smooth) inverse $\theta_t^{-1} = \theta_{-t}$. In fact, $\theta_{t_1} \circ \theta_{t_2} = \theta_{t_1 + t_2}$, so $\{\theta_t \,:\, t \in \mathbb{R}\}$ is a subgroup of the group of diffeomorphisms of M, homomorphic to the additive reals. For a more detailed treatment, see Gallot, Hulin and Lafontaine, Riemannian Geometry, pp23-26.

4 Vector Bundles

4.1 Basic definition

The tangent bundle is an example (in some sense, the most important example) of a more general structure called a vector bundle. Loosely speaking, a vector bundle is a family of vector spaces indexed in a "smooth" way by a point taking values in a manifold M. The important point is that the family as a whole is a smooth manifold, which locally looks like $U \times \mathbb{R}^k$, where U is an open subset of M.

Definition 70 A **real vector bundle of rank** k is a triple (π, E, M) where E and M are manifolds and $\pi : E \to M$ is a smooth surjection such that

(a) there exists an open cover $\{U_i \,:\, i \in \, I\}$ of M and diffeomorphisms

$$h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$$

such that $h_i(\pi^{-1}(x)) = \{x\} \times \mathbb{R}^k$, and such that

(b) for $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, the diffeomorphisms

$$h_i \circ h_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^k \to (U_i \cap U_j) \times \mathbb{R}^k$$

are of the form

$$h_i \circ h_j^{-1}(x, v) = (x, g_{ij}(x)v)$$

where $g_{ij} : U_i \cap U_j \to GL(k, \mathbb{R})$ is smooth.

Notes:

- We often denote the bundle just by E, rather than the triple (π, E, M).
- The maps $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$ are called **local trivializations** of the bundle E.
- For each $x \in M$ we refer to $E_x = \pi^{-1}(x) \subset E$ as the **fibre above** x. Given a point x, we can choose $i \in I$ such that $x \in U_i$, and then

$$h_i : E_x \to \{x\} \times \mathbb{R}^k$$

provides a bijective correspondence between elements of the fibre and elements of the vector space $\mathbb{R}^k$. So we can use h_i to equip each E_x with $x \in U_i$ with the structure of a real vector space of dimension k: for all $\xi_i, \xi_2 \in E_x$ and $a, b \in \mathbb{R}$, we set

$$a\xi_1 + b\xi_2 := h_i^{-1}(a h_i(\xi_1) + b h_i(\xi_2)).$$

What if we choose a different local trivialization $h_j : \pi^{-1}(U_j) \to U_j \times \mathbb{R}^k$ to define the linear structure on E_x? By property (b) of the definition, we are guaranteed that the laws of vector addition and scalar multiplication will be identical as those defined by h_i:

$$\begin{aligned}
a\xi_i + b\xi_2 &:= h_j^{-1}(ah_j(\xi_1) + bh_j(\xi_2)) \\
&= h_j^{-1}[a(h_j \circ h_i^{-1} \circ h_i)(\xi_1) + b(h_j \circ h_i^{-1} \circ h_i)(\xi_2)] \\
&= h_j^{-1}[ag_{ji}(x)h_i(\xi_1) + bg_{ji}(x)h_i(\xi_2)] \\
&= h_j^{-1}g_{ji}(x)[ah_i(\xi_1) + bh_i(\xi_2)] \\
&= h_i^{-1}(ah_i(\xi_1) + bh_i(\xi_2))
\end{aligned}$$

So each fibre inherits a **canonical** linear structure.

Definition 71 A **section** of a vector bundle (π, E, M) is a smooth map $V : M \to E$ such that $\pi \circ V = \mathrm{Id}_M$. The set of sections of E is denoted $\Gamma(E)$.

Example 72 The tangent bundle TM is a vector bundle of rank m (the dimension of M). π is the usual projection map. $E_x = T_xM$. Local trivializations are provided by the atlas defined in the proof of Proposition 60. That is, given a chart (U, ϕ) for M we define

$$h : \pi^{-1}(U) \to U \times \mathbb{R}^m$$

by

$$h([c]) = (x, (\phi \circ c)'(0))$$

for each $[c] \in T_xM$.

Example 73 $E = M \times \mathbb{R}^k$ is a trivial bundle. Our collection of local trivializations is just $\{(M, \mathrm{Id})\}$. Sections of E are just smooth mappings $M \to \mathbb{R}^k$.

One can define nontrivial vector bundles over a manifold M by breaking it into pieces and specifiying the overlap functions $g_{ij} : U_i \cap U_j \to GL(k, \mathbb{R})$ directly. (They can't be just any old functions: clearly we must have $g_{ij} = g_{ji}^{-1}$, and they must also satisfy a so-called "co-cycle" condition.) The bundles of interest to us, however, will be defined globally, by certain constructions on TM, so we won't pursue this line of thought further.

Definition $73\frac{1}{2}$ (π, E, M) is **trivial** if there exists a diffeomorphism $\phi : E \to M \times \mathbb{R}^k$ such that, for each $x \in M$, $\phi| : E_x \to \{x\} \times \mathbb{R}^k$ is linear.

Remark If E is trivial, there exist k pointwise linearly independent sections of E, for example

$$\xi_i(x) = \phi^{-1}(x, e_i)$$

where $e_1, \ldots, e_k$ is the standard basis for $\mathbb{R}^k$.

Conversely, if E has k pointwise linearly independent sections $\xi_1, \ldots, \xi_k$, we can decompose any $v \in E$ as

$$v = v^i(x)\xi_i(x), \qquad \text{where } x = \pi(v).$$

Then

$$\phi : E \to M \times \mathbb{R}^k, \qquad v \mapsto (x, v^1(x), \ldots, v^k(x))$$

is a diffeomeorphism which maps E_x isomorphically to $\{x\} \times \mathbb{R}^k$. Hence E is trivial.

Example $73\frac{3}{4}$ The normal bundle of $S^2 \subset \mathbb{R}^3$, NS^2 is a rank 1 real vector bundle (a real line bundle) over S^2. It has a global nonvanishing section, $\xi(x) = x$. Hence NS^2 is a trivial bundle.

4.2 Linear algebra recap: tensor products

Given any finite dimensional vector space V over $\mathbb{R}$, it has a **dual** vector space V^*, the space of linear maps $V \to \mathbb{R}$. Linear structure: for all $a, b \in \mathbb{R}$ and $\omega_1, \omega_2 \in V^*$,

$$a\omega_1 + b\omega_2 := [u \mapsto a\omega_1(u) + b\omega_2(u), \quad \text{for all } u \in V.]$$

V and V^* are isomorphic vector spaces. Let e_i, $i = 1, \ldots, m$, be a basis for V. The corresponding basis for V^* is θ^i, where

$$\theta^i(e_j) := \delta^i_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Then the map $u^i e_i \mapsto u^i \theta^i$ defines a (basis dependent) isomorphism $V \to V^*$.

The dual space V^* also has a dual, V^{**} which is *canonically* isomorphic to V: to a vector u we associate the linear map

$$u^{**} : V^* \to \mathbb{R}, \qquad u^{**}(\omega) = \omega(u), \text{ for all } \omega \in V^*.$$

Henceforth u, u^{**} will both be denoted u.

Given two vector spaces V, W, the tensor product of their dual spaces $V^* \otimes W^*$ is the space of bilinear maps $V \times W \to \mathbb{R}$ with the obvious linear structure:

$$a\omega_1 + b\omega_2 := [(v, w) \mapsto a\omega_1(v, w) + b\omega_2(v, w), \quad \text{for all } v \in V, w \in W.]$$

Given $\omega \in V^*$ and $\eta \in W^*$, we define

$$\omega \otimes \eta \in V^* \otimes W^*, \qquad (\omega \otimes \eta)(v, w) = \omega(v)\eta(w).$$

If $\{\theta^1, \ldots, \theta^m\}$, $\{\xi^1, \ldots, \xi^n\}$ are bases for V^*, W^* then $\{\theta^i \otimes \xi^a \ : \ 1 \leq i \leq m, 1 \leq a \leq n\}$ is a basis for $V^* \otimes W^*$. Hence $\dim(V^* \otimes W^*) = \dim(V)\dim(W)$. Since V, W are canonically isomorphic to V^{**}, W^{**}, we may similarly define $V \otimes W$ as the space of bilinear maps $V^* \times W^* \to \mathbb{R}$. Similarly, given any finite collection of vector spaces $V_1, \ldots, V_k$, we can define

$$V_1^* \otimes \cdots \otimes V_k^* = \{\text{multilinear maps } V_1 \times \cdots \times V_k :\to \mathbb{R}\}.$$

It has dimension $\dim(V_1)\dim(V_2)\cdots\dim(V_k)$.

Let $\mathrm{End}(V)$ denote the space of linear maps (endomorphisms) $V \to V$. There is a canonical isomorphism $\mathrm{End}(V) \to V \otimes V^*$ which associates to a linear map $L : V \to V$ the map

$$V^* \times V \to \mathbb{R}, \qquad (\omega, v) \mapsto \omega(Lv).$$

In the reverse direction it associates to $v \otimes \omega \in V \otimes V^*$ the linear map

$$L_{v\otimes\omega} : u \mapsto \omega(u)v.$$

This allows one to define a useful construction called **contraction**:

$$\begin{array}{rcl} W \otimes V \otimes V^* \to W \otimes \mathrm{End}(V) & \to & W \\ w \otimes v \otimes \omega \mapsto w \otimes L_{v\otimes\omega} & \mapsto & \mathrm{tr}\,(L_{v\otimes\omega})w \end{array}$$

Example 74 Given $R \in V \otimes V^* \otimes V^* \otimes V^*$ we can define three different contractions of R. One of them is $\rho \in V^* \otimes V^*$ defined by

$$\rho(u, v) = R(\theta^i, u, e_i, v)$$

where e^i is any basis for V and θ^i is its dual basis (and we sum over the repeated index i, as usual). Note that this is independent of the choice of basis.

4.3 Tensor bundles

All the previous linear algebra can be applied in the case $V = T_xM$, the tangent space at x to the manifold M. In particular, the dual space to T_xM is called the **cotangent** space. For each choice of non-negative integers p, q we can construct a bundle over M whose fibre at x is the space

$$\begin{array}{rcl} T^{p,q}_x M & = & \text{space of multilinear maps } T^*_xM \times \cdots \times T^*_xM \times T_xM \times \cdots \times T_xM. \\ & = & \underbrace{T_xM \otimes \cdots \otimes T_xM}_{p} \otimes \underbrace{T^*_xM \otimes \cdots \otimes T^*_xM}_{q} \end{array}$$

This bundle is denoted $T^{p,q}M$ and inherits a canonical differentiable structure from the differentiable structure on M (just copy the construction in Proposition 60). The triple $(\pi, T^{(p,q)}M, M)$, where π is the obvious projection, is a vector bundle.

Let us examine the case $p = 0, q = 1$ in detail. This bundle is usually denoted T^*M and called the **cotangent bundle**. Let $\mathcal{A}$ be an atlas on M and (U, ϕ) be a chart from $\mathcal{A}$. Denote by $\partial/\partial x^i$ be the associated coordinate basis vector fields. The dual basis, for T^*_xM is denoted dx^i. By definition

$$dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j.$$

Let $T^*U = \bigcup_{x\in U} T^*_xM$ and define $\Phi : T^*U \to \phi(U) \times \mathbb{R}^m$ by

$$\Phi(x, \omega) = (\phi(x), \omega_1, \ldots, \omega_m)$$

where $\omega_i = \omega_x(\frac{\partial}{\partial x^i})$ (i.e. the components of ω in the dual basis). One can check that this defines an atlas $T^*\mathcal{A}$ on T^*M, and that equivalent atlases on M produce equivalent atlases on T^*M. As for TM, we denote by $\pi : T^*M \to M$ the projection which assigns to $\omega \in T_xM$ its base point $x \in M$. To each chart (U, ϕ) one has a local trivialization $h : T^*U \to U \times \mathbb{R}^m$ by

$$h(x, \omega) = (x, \omega_1, \ldots, \omega_m).$$

The collection $\{(T^*U, h) : U \in \mathcal{A}\}$ defines a vector bundle structure on T^*M. Sections of T^*M are called **covector fields** or **one forms**.

Exercise 75 Note that given a smooth function $f : M \to \mathbb{R}$, there is an associated one-form $\mathrm{d}f$,

$$\mathrm{d}f : T_pM \to \mathbb{R}, \qquad v \mapsto v[f].$$

Given a chart (U, ϕ) denote by x^i the (local) function $U \to \mathbb{R}$ which assigns to each point p the ith entry of $\phi(p)$. Show that

$$\mathrm{d}(x^i) = dx^i$$

where the one-form on the RHS is the ith coordinate cobasis vector. This explains the notation for the cobasis vectors.

Exercise 76 Let (U, ϕ) and (V, ψ) be overlapping charts on M and denote the corresponding local coordinates $(x^1, \ldots, x^m)$ and $(y^1, \ldots, y^m)$. Show that, at any point $p \in U \cap V$,

$$dy^i|_p = \left.\frac{\partial y^i}{\partial x^j}\right|_{\phi(p)} dx^j|_p.$$

This also explains the notation for the cobasis vectors.

4.4 Tensor fields

In general, a (p, q) **tensor field** is a section of $T^{(p,q)}M$. Tensors of type $(p, 0)$ are called **contravariant**. Tensors of type $(0, q)$ are called **covariant**.

We can shift from a pointwise to a more global perspective by thinking of $\Gamma(TM)$ and $\Gamma(T^*M)$ as $C^\infty(M)$ modules (loosely speaking, "vector spaces" over the ring $C^\infty(M)$). It is clear that given a $(1, 2)$ tensor field τ say, there is an associated map $\hat{\tau} : \Gamma(T^*M) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$,

$$(\hat{\tau}(\omega, X, Y))(p) = \tau(\omega(p), X(p), Y(p)).$$

This map is clearly $\mathbb{R}$ multilinear:

$$\hat{\tau}(\omega, aX_1 + bX_2, Y) = a\hat{\tau}(\omega, X_1, Y) + b\hat{\tau}(\omega, X_2, Y) \quad \text{etc}$$

In fact, it is $C^\infty(M)$ multilinear since, for example, given any $f \in C^\infty(M)$,

$$\begin{aligned} (\hat{\tau}(\omega, fX, Y))(p) &= \tau(\omega(p), f(p)X(p), Y(p)) = f(p)\tau(\omega(p), X(p), Y(p)) \\ \text{so} \qquad \hat{\tau}(\omega, fX, Y) &= f\hat{\tau}(\omega, X, Y). \end{aligned}$$

So a $(1,2)$ tensor field uniquely determines a $C^\infty(M)$ multilinear map

$$\Gamma(T^*M) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(M).$$

In fact, the converse holds too, that is, a $C^\infty(M)$ multilinear map

$$\hat{\tau} : \Gamma(T^*M) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$$

uniquely determines a $(1,2)$ tensor field. Let $p \in M$, $\omega \in T_p^*M$ and $X, Y \in T_pM$. Then let $\widetilde{\omega} \in \Gamma(T^*M)$ and $\widetilde{X} \in \Gamma(TM)$ be extensions of ω, X, Y, that is, smooth fields with $\widetilde{\omega}(p) = \omega$, $\widetilde{X}(p) = X$, $\widetilde{Y}(p) = Y$. Then define $\tau_p : T_p^*M \times T_pM \times T_pM \to \mathbb{R}$ by

$$\tau_p(\omega, X, Y) = (\hat{\tau}(\widetilde{\omega}, \widetilde{X}, \widetilde{Y}))(p).$$

This is independent of the choice of extensions by $C^\infty(M)$ multilinearity. To see this, let $X_1, \ldots, X_m$ be a smooth local frame for TM. Then, on a neighbourhood of p, the extension $\widetilde{X}$ can be written

$$\widetilde{X} = f_1 X_1 + \cdots + f_m X_m$$

for some $f_1, \ldots, f_m \in C^\infty(M)$. Any other extension, $\widehat{X}$ say, can likewise be written

$$\widehat{X} = \widehat{f}_1 X_1 + \cdots + \widehat{f}_m X_m$$

with $\widehat{f}_i(p) = f_i(p)$ (since $\widehat{X}(p) = X = \widetilde{X}(p)$). Hence

$$\begin{aligned} (\hat{\tau}(\widetilde{\omega}, \widehat{X}, \widetilde{Y}))(p) &= \sum_i (\hat{\tau}(\widetilde{\omega}, \widehat{f}_i X_i, \widetilde{Y}))(p) = \sum_i (\widehat{f}_i \hat{\tau}(\widetilde{\omega}, X_i, \widetilde{Y}))(p) \\ &= \sum_i \widehat{f}_i(p)(\hat{\tau}(\widetilde{\omega}, X_i, \widetilde{Y}))(p) = \sum_i f_i(p)(\hat{\tau}(\widetilde{\omega}, X_i, \widetilde{Y}))(p) \\ &= (\hat{\tau}(\widetilde{\omega}, \widetilde{X}, \widetilde{Y}))(p). \end{aligned}$$

Similar arguments work for the ω, Y entries. So an alternative (and actually rather useful) characteriztion of a (p,q) tensor field is a $C^\infty(M)$ multilinear map

$$\underbrace{\Gamma(T^*M) \times \cdots \times \Gamma(T^*M)}_{p} \times \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{q} \to C^\infty(M).$$

Example 77 The Lie bracket $[\cdot,\cdot] : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ determines a $\mathbb{R}$ multilinear map $B : \Gamma(T^*M) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$, namely,

$$B(\omega, X, Y) = \omega([X, Y]).$$

Is B a $(1,2)$ tensor? No! It fails to be $C^\infty(M)$ linear with respect to its second and third entries, e.g.

$$B(\omega, X, fY) = \omega([X, fY]) = \omega(X[f]Y + f[X, Y]) = X[f]\omega(Y) + fB(\omega, X, Y).$$

Exercise 78 Let $\omega \in \Gamma(T^*M)$ and consider the $\mathbb{R}$ bilinear map $\mu : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ defined by

$$\mu(X, Y) = X[\mu(Y)] - Y[\mu(X)] - \mu([X, Y]).$$

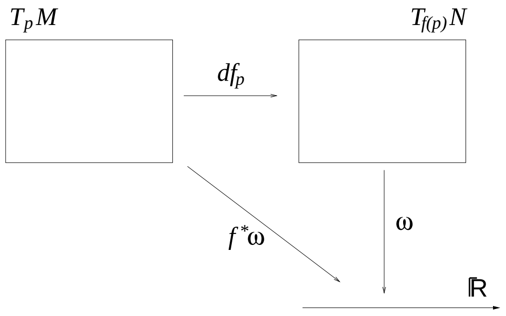
Show that μ is $C^\infty(M)$ bilinear, and hence is a $(0,2)$ tensor field.

4.5 Pullback

Recall that to any smooth map $f : M \to N$, there is associated, for each $p \in M$, a linear map $df_p : T_pM \to T_{f(p)}N$, the differential mapping. We use this to make the following definition:

Definition 79 For any $f : M \to N$ smooth at $p \in M$, define the **pullback** $f^* : T^*_{f(p)}N \to T^*_pM$ such that for all $v \in T_pM$,

$$(f^*\omega)(v) := \omega(df_p\, v). \qquad \square$$



We can extend the pullback pointwise so that it maps any $\omega \in \Gamma(T^*N)$ to $f^*\omega \in \Gamma(T^*M)$ defined such that for all $X \in \Gamma(TM)$,

$$((f^*\omega)(p))(X(p)) := (\omega(p))(df_pX(p))$$

Proposition 80 *Let $f : M \to N$ and $g : N \to P$ be smooth. Then*

$$(g \circ f)^* = f^* \circ g^*$$

Proof: For all $\omega \in T^*_{(g\circ f)(p)}P$ and $v \in T_pM$ we have

$$((g\circ f)^*\omega)(v) = \omega(d(g\circ f)_p v) = \omega(dg_{f(p)}\circ df_p v) = (g^*\omega)(df_p v) = (f^*(g^*\omega))(v) \qquad \square$$

Example 81 Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the map $f(x^1, x^2, x^3) = (x^1x^2, x^2 + x^3)$ and $\omega = y^1dy^2 - y^2dy^1 \in \Gamma(T^*\mathbb{R}^2)$. What is $f^*\omega$? It's a one-form on $\mathbb{R}^3$, so

$$f^*\omega = a_1dx^1 + a_2dx^2 + a_3dx^3$$

where $a_i = f^*\omega(\partial/\partial x^i) = \omega(df\,\partial/\partial x^i)$. So

$$\begin{aligned}
a_1 &= = \omega(df\,\partial/\partial x^1) = \omega(x^2 \left.\frac{\partial}{\partial y^1}\right|_{(x^1x^2,x^2+x^3)}) = x^2(-y^2)|_{(x^1x^2,x^2+x^3)} = -x^2(x^2+x^3) \\
a_2 &= \omega(df\,\partial/\partial x^2) = \omega(x^1 \left.\frac{\partial}{\partial y^1}\right|_{(x^1x^2,x^2+x^3)} + \left.\frac{\partial}{\partial y^2}\right|_{(x^1x^2,x^2+x^3)}) \\
&= x^1(-y^2))|_{(x^1x^2,x^2+x^3)} + (y^1)|_{(x^1x^2,x^2+x^3)} = -x^1(x^2+x^3) + x^1x^2 = -x^1x^3 \\
a_3 &= = \omega(df\,\partial/\partial x^3) = \omega(\left.\frac{\partial}{\partial y^2}\right|_{(x^1x^2,x^2+x^3)}) = (y^1)|_{(x^1x^2,x^2+x^3)} = x^1x^2.
\end{aligned}$$

Hence

$$f^*\omega = -x^2(x^2+x^3)dx^1 - x^1x^3dx^2 + x^1x^2dx^3.$$

We can extend the notion of pullback to *covariant* tensors of any rank (but not mixed or contravariant tensors) in the obvious way.

Definition 82 Given $f : M \to N$ and $\tau \in \Gamma(T^{(0,q)}N)$ we define $f^*\tau \in \Gamma(T^{(0,q)}M)$ by

$$f^*\tau(X_1, \dots, X_q) = \tau(df\,X_1, \dots, df\,X_q)$$

Proposition 80 ($(g \circ f)^* = f^* \circ g^*$) generalizes immediately to this setting.

5 Connexions on vector bundles

5.1 Basic definition

Given a vector bundle E over M, we must treat fibres E_x and E_y with $x \neq y$ as distinct vector spaces. There is no canonical way to identify elements in these two spaces. Hence, to say that a section ξ of E is "constant" has no well-defined meaning. We will associate to any given smooth curve c connecting x and y an identification $E_x \equiv E_y$, but to do so will require us to equip E with extra structure – a connexion – and the identification will (in general) depend on the whole curve c, not just its endpoints.

Definition 83 A **connexion** on a vector bundle (π, E, M) is a mapping

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

which associates to any vector field X and section ξ another section $\nabla_X \xi$ such that

(i) $\nabla_{X_1+X_2}\xi \equiv \nabla_{X_1}\xi + \nabla_{X_2}\xi$

(ii) $\nabla_X(\xi_1 + \xi_2) \equiv \nabla_X\xi_1 + \nabla_X\xi_2$

(iii) $\forall\, f \in C^\infty(M), \quad \nabla_{fX}\xi \equiv f\nabla_X\xi$

(iv) $\forall\, f \in C^\infty(M), \quad \nabla_X f\xi \equiv X[f]\xi + f\nabla_X\xi.$

Notes:

(a) Axioms (i) and (ii) $\Rightarrow \nabla_X\xi$ is $C^\infty(M)$ linear with respect to X. So $(\nabla_X\xi)(p)$ depends on $X(p)$, but not on the values of X away from p.

(b) Axioms (ii) and (iv) $\Rightarrow \nabla_X\xi$ is $\mathbb{R}$-linear, but *not* $C^\infty(M)$ linear with respect to ξ. Hence $\nabla_X\xi(p)$ does not only depend on $\xi(p)$, but the values of ξ on a neighbourhood of p.

(c) Given the axioms (i)-(iv), we can reconstruct ∇ once we know its action on any set of $m = \dim M$ (pointwise) linearly independent vector fields, and $k = \mathrm{rank}(E)$ linearly independent sections, see below.

We will think of $(\nabla_X\xi)(p)$ as "the directional derivative of the section ξ with respect to $X(p)$." With this in mind:

Example 84 There is one vector bundle over M which *does* come ready equipped with a canonical connexion, namely the trivial bundle $E = M \times \mathbb{R}$. Sections of E are just elements of $C^\infty(M)$, and we can define

$$\nabla_X f = X[f] = \mathrm{d}f\, X \qquad (1).$$

In general, to specify a connexion (locally), we can choose a basis of sections $e_1, \cdots, e_k$ for E, and a basis of vector fields $E_1, \ldots, E_m$ on M, then specify the mk smooth sections $\nabla_{E_i} e_a \in \Gamma(E)$. Each of these sections can be expanded in the basis $e_1, \ldots, e_k$,

$$\nabla_{E_i} e_a = \Gamma^b_{ia} e_b$$

so this amounts to specifying mk^2 smooth (local) functions Γ^b_{ia} on M. Once these **connexion coefficients** are fixed, the connexion itself is uniquely defined by the axioms it satisfies. That is, given any $X \in \Gamma(TM)$ and $\xi \in \Gamma(E)$, we expand $X = X^i E_i$ and $\xi = \xi^a e_a$ so that

$$\nabla_X \xi = \nabla_{X^i E_i}(\xi^a e_a) = X^i \nabla_{E_i}(\xi^a e_a) = X^i \left\{ E_i[\xi^a] e_a + \xi^a \Gamma^b_{ia} e_b \right\}.$$

In the case where $E = TM$, we only need to choose a basis for TM, for example the coordinate basis associated with a particular coordinate chart.

Example 85 On $M = \mathbb{R}^2$ let's use the obvious basis $\partial/\partial x^1$, $\partial/\partial x^2$, and define the connexion ∇ on TM to have connexion coefficients $\Gamma^1_{22} = x^1$, all others zero. Then

$$\nabla_X \frac{\partial}{\partial x^1} = 0$$

for all $X \in \Gamma(TM)$, while

$$\nabla_X \frac{\partial}{\partial x^2} = X^2 x^1 \frac{\partial}{\partial x^1}.$$

5.2 New connexions from old

Given vector bundles E, F over M, one defines the tensor product bundle $E \otimes F$ to be the bundle whose fibre over x is $E_x \otimes F_x$. If we have connexions ∇^E, ∇^F on E, F respectively, $E \otimes F$ inherits a canonical connexion, defined by requiring that for all $X \in \Gamma(TM)$, $\xi \in \Gamma(E)$ and $\nu \in \Gamma(F)$,

$$\nabla^{E\otimes F}_X (\xi \otimes \mu) = (\nabla^E_X \xi) \otimes \mu + \xi \otimes (\nabla^F_X \mu) \qquad (2).$$

Similarly, given a bundle E, one defines the dual bundle E^* to be the bundle with fibres E^*_x. Given a connexion ∇ on E, E^* inherits a canonical connexion ∇^*, defined by requiring that for all $X \in \Gamma(TM)$, $\xi \in \Gamma(E)$ and $\nu \in \Gamma(E^*)$,

$$X[\nu(\xi)] = (\nabla^*_X \nu)(\xi) + \nu(\nabla_X \xi) \qquad (3).$$

We can think of this differently. Let $\kappa : E^* \otimes E \to \mathbb{R}$ be the contraction map $\kappa(\nu \otimes \xi) = \nu(\xi)$. Combining (1),(2) and (3) one sees that

$$\nabla_X(\kappa(\nu \otimes \xi)) = \kappa(\nabla^{E^*\otimes E}_X \nu \otimes \xi).$$

That is, the connexion commutes with contraction, by definition.

Once again, we can apply all of these definitions in the case of principal interest, that is, $E = TM$, the tangent bundle of M. Using the above constructions, once we have specified a connexion on TM, it has a unque extension to every tensor bundle $T^{(p,q)}M$. We just identify $T^{(p,q)}M$ with $\otimes_p TM \otimes_q T^*M$.

Example 86 Given a connexion ∇ on TM, what is the associated connexion ∇^E on $E = T^{(0,2)}M$? Sections of E are $C^\infty(M)$ bilinear maps $\Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$. Given such a section g, say, and any two vector fields Y, Z, $g(Y, Z)$ is a smooth function on M. Hence

$$\nabla_X(g(Y, Z)) = X[g(Y, Z)].$$

But, by definition,

$$\nabla_X(g(Y, Z)) = (\nabla^E_X g)(Y, Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Hence

$$(\nabla^E_X g)(Y, Z) = X[g(Y, Z)] - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$$

for all $X, Y, Z \in \Gamma(TM)$ and all $g \in \Gamma(E)$. This uniquely defines ∇^E.

In practice, one usually uses the same symbol ∇ to denote a connexion on E and the associated connexions it defines on all bundles constructed from E.

Remark 87 Just as described in section 4.4, we can interpret maps $\Gamma(E) \to \Gamma(F)$ as sections of $E^* \otimes F$, provided they are $C^\infty(M)$ linear. For example, let ∇ be a connexion on E. Then, for a *fixed* $\xi \in \Gamma(E)$ the map $\Gamma(TM) \to \Gamma(E)$ given by

$$X \mapsto \nabla_X \xi$$

is $C^\infty(M)$ linear. Hence, we can think of this map as a section of $T^*M \otimes E$. This section is usually denoted $\nabla\xi$.

Another way to build new bundles, and connexions, from old ones is by *pullback*.

Definition 88 Let $\phi : M \to N$ be a smooth map and E be a vector bundle over N. Then $\phi^{-1}E$ is the vector bundle over M whose fibre at $x \in M$ is the vector space $E_{\phi(x)}$. The bundle $\phi^{-1}E$ is called the **pullback** of E to M.

Example 89 Let $c : I \to M$ be a smooth curve. Then $c^{-1}TM$ is a rank m vector bundle over the interval I. A section of $c^{-1}TM$ is often called a **vector field along the curve** c.

In fact we can transfer any connexion on E to $\phi^{-1}E$, as follows:

Definition 90 Let $\phi : M \to N$ be a smooth map, E be a vector bundle over N and ∇ be a connexion on E. Then the **pullback connexion** ∇^ϕ is the connexion on $\phi^{-1}E$ satisfying the property

$$\nabla^\phi_X(\xi \circ \phi) = (\nabla_{\mathrm{d}\phi X}\xi) \circ \phi.$$

Exercise 91 Show that ∇^ϕ exists and is unique. [Hint: any section of $\phi^{-1}E$ can be written locally uniquely as a superposition of sections of the form $e_a \circ \phi$, where $e_1, \ldots, e_k$ are a local frame for E. The above definition determines what ∇^ϕ does to such sections, and the axioms of a connexion determine everything else.]

We will use ∇^c in the case of a smooth curve $c : I \to M$ to define parallel transport.

5.3 Parallel transport

Definition 92 Let (π, E, M) be a vector bundle and ∇ be a connexion on M. Let $c : I \to M$ be a curve in M. A section $\xi \in \Gamma(c^{-1}E)$ is **parallel** if

$$\nabla^c_{d/dt}\xi = 0.$$

Locally, we can define $c : I \to M$ by giving its local coordinate expression $c^i(t)$, $i = 1, \ldots, m$, and define ∇ by its connexion coefficients Γ^a_{ib} with respect to some choice of frame $e_1, \ldots, e_k$ for E and with respect to the obvious local frame for TM (the coordinate basis $\partial/\partial x^i$, $i = 1, \ldots, m$). Then a section of $c^{-1}E$ is a map $I \to E$ of the form

$$\xi(t) = \xi^a(t) e_a(c(t)).$$

Hence, the condition that ξ is parallel be can be written

$$\begin{aligned}
\nabla^c_{d/dt}(\xi^a(t)e_a(c(t)) &= \dot{\xi}^a(t)e_a(c(t)) + \xi^a(t)\nabla^c_{d/dt}(e_a(c(t)) \\
&= \dot{\xi}^a(t)e_a(c(t)) + \xi^a(t)(\nabla_{dc_t d/dt}(e_a))(c(t)) \\
&= \dot{\xi}^a(t)e_a(c(t)) + \xi^a(t)\dot{c}^i(t)(\nabla_{\partial/\partial x^i}e_a)(c(t)) \\
&= \dot{\xi}^a(t)e_a(c(t)) + \xi^a(t)\dot{c}^i(t)\Gamma^b_{ia}(c(t))e_b(c(t)) \\
&= \left[\dot{\xi}^b(t) + \Gamma^b_{ia}(c(t))\dot{c}^i(t)\xi^a(t)\right] e_b(c(t)) = 0.
\end{aligned}$$

Since $e_b(c(t))$ are linearly independent, we see that ξ is (locally) parallel if and only if its coefficients $\xi^a(t)$, $a = 1, \ldots, k$, satisfy the coupled **linear** system of ODEs

$$\dot{\xi}^b(t) + \Gamma^b_{ia}(c(t))\dot{c}^i(t)\xi^a(t) = 0, \qquad b = 1, \ldots, k.$$

Note that the coefficients in this system are all smooth by assumption. Hence, by standard ODE theory, given any $t_0 \in I$ and $\xi_0 \in E_{c(t_0)}$, there exists a unique parallel section ξ of $c^{-1}E$ with $\xi(t_0) = \xi_0$. The parallel section ξ is called the **parallel transport** of ξ_0 along c. Furthermore, the map which takes $\xi_0 \in E_{c(t_0)}$ to its parallel transport $\xi \in \Gamma(c^{-1}E)$ is **linear** (by linearity of the above ODE system).

Hence, associated to any curve $c : [0,1] \to M$ connection points $x = c(0)$ and $y = c(1)$ we have a linear map $Tr_c : E_x \to E_y$ defined by

$$Tr_c(\xi_0) = \xi(1)$$

where $\xi(t)$ is the parallel transport of $\xi \in E_x$ along c.

Exercise 93 Given curve $c : [0,1] \to M$, let $\bar{c}(t) = c(1-t)$ be the reverse curve. Show that if ξ is a parallel section of $c^{-1}E$ then $\bar{\xi}(t) = \xi(1-t)$ is a parallel section of $\bar{c}^{-1}E$. Deduce that $Tr_{\bar{c}} \circ Tr_c = \mathrm{Id}_{E_{c(0)}}$ and $Tr_c \circ Tr_{\bar{c}} = \mathrm{Id}_{E_{c(1)}}$. It follows that Tr_c is an invertible linear map (with inverse $Tr_{\bar{c}}$).

Remark 94 We can extend the definition of parallel transport to the case of piecewise smooth curves in the obvious way.

Remark 95 Parallel transport is independent of the parametrization of the curve c. That is, let $c : I \to M$ be a curve and $\tau : J \to I$ be an increasing smooth function from interval J to interval I. Then $\bar{c} : J \to M$, $\bar{c} = c \circ \tau$, is a reparametrization of c (geometrically, they have the same image in M and are traversed in the same direction). Given any section ξ of $c^{-1}E$, then $\bar{\xi} = \xi \circ \tau$ is a section of $\bar{c}^{-1}E = \tau^{-1}(c^{-1}E)$, and

$$\nabla^{\bar{c}}_{d/dt}\bar{\xi} = (\nabla^{c}_{\mathrm{d}\tau d/dt}\xi) \circ \tau = \tau'(\nabla^{c}_{\mathrm{d}/dt}\xi) \circ \tau.$$

Now $\tau'(t) > 0$, so $\bar{\xi}$ is parallel if and only if ξ is parallel.

It's important to realize that parallel transport depends on both the connexion ∇ and the choice of path $c : I \to M$ between x and y.

Example 96 Let's take $M = \mathbb{R}^2$, $E = TM$, and ∇ to be the connexion defined in Example 85, that is $\Gamma^1_{22} = x^1$, all other coefficients zero. Let $x = (0,0)$ and $y = (1,0)$. An obvious path from x to y is $c(t) = (t,0)$. Given a general tangent vector $\xi = a\partial/\partial x^1 + b\partial/\partial x^2 \in E_x$, its parallel transport along c is

$$\xi(t) = \xi^1(t)\frac{\partial}{\partial x^1} + \xi^2(t)\frac{\partial}{\partial x^2}$$

where the coefficients $\xi_1(t), \xi_2(t)$ solve the initial value problem

$$\dot{\xi}^k(t) + \Gamma^k_{ij}(c(t))\dot{c}^i(t)\xi^j(t), \qquad \xi^1(0) = a, \xi^2(0) = b.$$

In this case $\dot{c}^1(t) = 1$, $\dot{c}^2(t) = 0$ and almost all the $\Gamma^k_{ij} = 0$, so the system reduces to

$$\dot{\xi}^1 = 0, \qquad \dot{\xi}^2 = 0.$$

Hence $\xi^1(t) = a$, $\xi_2(t) = b$, and, with respect to the obvious bases for E_x, E_y, $Tr_c = \mathbb{I}_2$, the identity matrix.

Alternatively, we could take $\bar{c}(t) = (t, \sin \pi t)$ as the curve from x to y. Then the system for $\xi^1(t), \xi^2(t)$ is

$$\begin{aligned} \dot{\xi}^1 + t\pi \cos \pi t \xi^2 &= 0 \\ \dot{\xi}^2 &= 0. \end{aligned}$$

The solution with initial data $\xi^1(0) = a, \xi^2(0) = b$ is

$$\begin{aligned} \xi^1(t) &= a - bt \sin \pi t + \frac{b}{\pi}(1 - \cos \pi t) \\ \xi^2(t) &= b. \end{aligned}$$

Hence, with respect to the obvious bases, $Tr_{\bar{c}} : E_x \to E_y$ is represented by the matrix

$$Tr_{\bar{c}} = \begin{bmatrix} 1 & \frac{2}{\pi} \\ 0 & 1 \end{bmatrix}.$$

Clearly $Tr_{\bar{c}} \neq Tr_c$.

5.4 Holonomy

Let ∇ be a connexion on a vector bundle E over M. Let $c : [0, 1] \to M$ be a **closed** curve based at $x \in M$, meaning that $c(0) = c(1) = x$. Then $Tr_c : E_x \to E_x$, is an invertible linear map from E_x to itself, that is, an automorphism of the fibre E_x. Consider the subset

$$H_x = \{Tr_c \,:\, c \text{ a closed path based at } x\} \subset \mathrm{Aut}(E_x).$$

Given two closed paths based at x, c_1, c_2, say, we can construct the path obtained by concatenating them, $c_2 \cdot c_1$ say:

$$c_2 \cdot c_1 : [0, 1] \to M, \qquad c_2 \cdot c_1(t) = \left\{ \begin{array}{cc} c_1(2t) & 0 \leq t \leq \frac{1}{2} \\ c_2(2t - 1) & \frac{1}{2} \leq t \leq 1. \end{array} \right.$$

This is also a closed path based at x, and

$$Tr_{c_2 \cdot c_1} = Tr_{c_2} \circ Tr_{c_1}.$$

Hence H_x is closed under composition. As remarked in the previous section, given any path c, its reverse path $\bar{c}(t) = c(1-t)$ has $Tr_{\bar{c}} = Tr_c^{-1}$. If c is a closed path based at x, so is $\bar{c}$. Hence every element of H_x has an inverse in H_x. It follows that H_x is a group, called the **based holonomy group** of the connexion ∇.

Example 97 Let's compute the holonomy group $H_{(0,0)}$ for the connexion ∇ on $E = T\mathbb{R}^2$ defined in Example 85, with $\Gamma^1_{22} = x^1$, all other coefficients 0.

Let $c(t) = (c^1(t), c^2(t))$, $c(0) = c(1) = (0,0)$. We calculate Tr_c by solving the parallel transport equation for a section $V \in c^{-1}T\mathbb{R}^2$ with general initial data $V(0) = a\partial/\partial x^1 + b\partial/\partial x^2$:

$$\begin{aligned} \dot{V}^k + \Gamma^k{}_{ij}(c(t))\dot{c}^i V^j &= 0 \\ \Rightarrow \dot{V}^1 + c^1\dot{c}^2 V^2 &= 0 \\ \dot{V}^2 &= 0. \end{aligned}$$

Clearly $V^2(t) = b$ for all t, whence

$$\begin{aligned} \dot{V}^1 &= -bc^1(t)\dot{c}^2(t) \\ \Rightarrow V^1(t) &= a + b\int_0^1 c^1(t)\dot{c}^2(t)\,dt \\ &= a + \frac{b}{2}\int_0^1 (c^1(t)\dot{c}^2(t) - c^2(t)\dot{c}^1(t))dt \\ &= a + b \times [\text{signed area enclosed by curve } c]. \end{aligned}$$

To see the last equality, just think of the integral as a line integral and invoke Stokes's theorem. (N.B. this is a nice property of *this example*, studied using *this coordinate system*! In general, we don't yet have enough structure on M to make sense of the idea of "area enclosed by c.") In any case, by a suitable choice of closed path, we can make this integral any real number we want, call it α, say. Then

$$Tr_c = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix},$$

and the based holonomy group of this connexion is

$$H_{(0,0)} = \left\{ \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\} \cong (\mathbb{R}, +).$$

Is there a relationship between holonomy groups based at different points x, y say? On a connected manifold, **yes**, in fact, they're isomorphic. Recall that a connected manifold is path connected (Theorem 38), so there exists a path γ, say, from x to y. But then, given any closed path c based at y, the concatenation $\bar{\gamma} \cdot c \cdot \gamma$ (where $\bar{\gamma}$ denoted the reverse of γ) is a closed path based at x, and

$$Tr_{\bar{\gamma}\cdot c\cdot\gamma} = (Tr_\gamma)^{-1} \circ Tr_c \circ Tr_\gamma.$$

Hence H_x and H_y are isomorphic, and the right conjugation map above gives an isomorphism $H_y \to H_x$. Note that this isomorphism *depends on the choice of path* γ. There is no canonical isomorphism between H_x and H_y in general.

Remark 98 If M is connected, it is customary to consider each H_x as a representation of an abstract group called **the holonomy group** of (E, ∇). For example, one would say that $(T\mathbb{R}^2, \nabla)$ defined as in Example 85 has holonomy group $(\mathbb{R}, +)$.

5.5 Curvature

Locally, information about the holonomy of (E, ∇) can be encoded in a "tensor" field.

Definition 99 Let (π, E, M) be a vector bundle and ∇ be a connexion on E. The **curvature** of ∇ is the map

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

defined by

$$R(X, Y, \xi) = \nabla_X(\nabla_Y \xi) - \nabla_Y(\nabla_X \xi) - \nabla_{[X,Y]}\xi.$$

A connexion is **flat** if $R \equiv 0$.

Notes:

- *WARNING:* sometimes this is defined to be minus the above.
- It's customary to write the image of (X, Y, ξ) under R as $R(X, Y)\xi$ rather than $R(X, Y, \xi)$. We will use this convention henceforth. This is natural since $\xi \mapsto R(X, Y, \xi)$ is a second order linear differential operator acting on ξ.

Lemma 100 *R is a section of $T^*M \otimes T^*M \otimes \mathrm{End}(E)$.*

Proof: We need to show that $(X, Y, \xi) \mapsto R(X, Y)\xi$ is a $C^\infty(M)$ multilinear map. First note that this map is manifestly $\mathbb{R}$ multilinear (by the definition of a connexion). Second, note that $R(X, Y)\xi \equiv -R(Y, X)\xi$. So it suffices to demonstrate that, for all $f \in C^\infty(M)$,

(a) $R(fX, Y)\xi = fR(X, Y)\xi$
(Exercise for reader.)

(b) $R(X, Y)(f\xi) \equiv fR(X, Y)\xi$

Now, recalling that, by Definition 83, $\nabla_X(f\xi) = X[f]\xi + f\nabla_X\xi$, we see that

$$\begin{aligned} R(X,Y)(f\xi) &= \nabla_X(Y[f]\xi + f\nabla_y\xi) - \nabla_Y(X[f]\xi + f\nabla_X\xi) - ([X,Y](f)\xi + f\nabla_{[X,Y]}\xi) \\ &= (X[Y[f]]\xi + Y[f]\nabla_X\xi + X[f]\nabla_Y\xi + f\nabla_X\nabla_Y\xi) \\ &\quad -(Y[X[f]]\xi + X[f]\nabla_Y\xi + Y[f]\nabla_X\xi + f\nabla_Y\nabla_X\xi) \\ &\qquad -([X,Y](f)\xi + f\nabla_{[X,Y]}\xi) \\ &= fR(X,Y)\xi \end{aligned}$$

as required. □

If we choose a coordinate chart around a point x, and a local frame of sections $e_1, \ldots, e_k$ for E, we can think of R as m^2 linear maps $E_x \to E_x$, one for each pair $\partial/\partial x^i$, $\partial/\partial x^j$, represented by matrices $(R_{ij})^{\cdot}_{\cdot}$ with matrix elements $(R_{ij})^a_b$ where

$$R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})e_b) = (R_{ij})^a_b.$$

Exercise 101 Let Γ^a_{ib} be the connexion coefficients of ∇. Show that

$$(R_{ij})^a_b = \partial_i\Gamma^a_{jb} - \partial_j\Gamma^a_{ib} + \Gamma^c_{jb}\Gamma^a_{ic} - \Gamma^c_{ib}\Gamma^a_{jc}.$$

Hence show that the curvature operator of the connexion on $T\mathbb{R}^2$ defined in Example 85 is

$$(R_{12})^{\cdot}_{\cdot} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(Note that on a two-dimensional manifold $(R_{21})^{\cdot}_{\cdot} = -(R_{12})^{\cdot}_{\cdot}$ and $(R_{11})^{\cdot}_{\cdot} = (R_{22})^{\cdot}_{\cdot} = 0$, so this completely determines the curvature).

This allows us to give a geometric interpretation to R. In local coordinates, consider a closed loop c which is a square of side length ε in the $x^i - x^j$ plane. Construct the parallel transport of any vector $\xi_0 = \xi^a_0 e_a$ about this

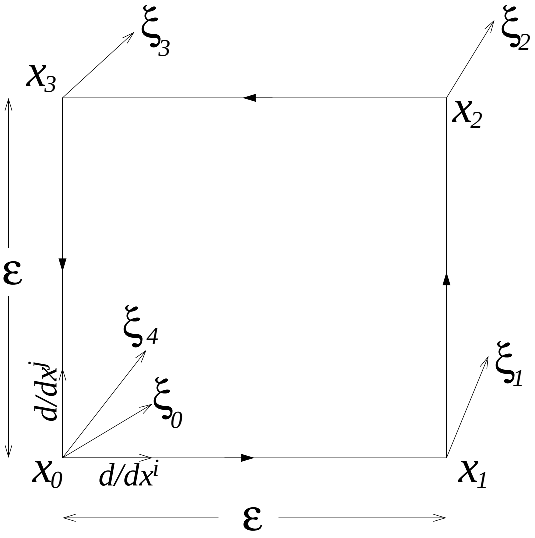


Figure 15: Parallel transport around a coordinate plaquette.

square: solve the parallel transport equation by Taylor expansion in ε. We then obtain the following

$$\delta\xi^a = \xi^a_4 - \xi^a_1 = \varepsilon^2 (R_{ij})^a_b \xi^b_0 + o(\varepsilon^2)$$

So, expressed in the basis $e_1, \ldots, e_k$, the holonomy of ∇ around c is

$$Tr_c = \mathbb{I}_k + \varepsilon^2 (R_{ij})^{\cdot}_{\cdot} + o(\varepsilon^2).$$

That is, we can think of curvature as infinitesimal holonomy. It follows that if a connexion has trivial holonomy (that is $Tr_c = \mathrm{Id}$ for all closed loops c), then it is flat ($R \equiv 0$).
WARNING: $R \equiv 0 \nRightarrow$ trivial holonomy.

Counterexample 102 Let E be a vector bundle over $M = S^1$. Since M is one-dimensional, any connexion on E is trivially flat. But there is no reason why the holonomy of ∇ around a noncontractible loop in S^1 should be trivial.

5.6 Geodesics

In this section, we specialize to the case where $E = TM$. In this case it's common to refer to a connexion ∇ on E as a connexion on M (rather than on TM).

Definition 103 Given a connexion ∇ on M, a **geodesic** $c : I \to M$ is a curve whose tangent vector field is parallel, i.e. $\nabla^c_{d/dt}\dot{c} = 0$.

Notes:

(a) In local coordinates the geodesic equation $\nabla^c_{d/dt}\dot{c} = 0$ becomes

$$\ddot{c}^k + \Gamma^k{}_{ij}(c(t))\dot{c}^i\dot{c}^j = 0. \tag{4}$$

This is a set of coupled, 2^{nd} order, nonlinear ODEs for m real-valued functions $(c^1(t), \ldots, c^m(t))$.

(b) The initial value problem requires initial data $c(0) \in M$, $\dot{c}(0) \in T_{c(0)}M$ (2^{nd} order). Since all coefficients in system (4) are smooth, we are guaranteed uniqueness and *local* existence of a solution to any initial value problem, but *not* global existence (nonlinear), i.e. a geodesic may become singular in finite time.

(c) The definition of geodesic means the curve c is "as straight as possible." In $\mathbb{R}^n$, with the usual ∇ ($\Gamma^k{}_{ij} = 0$ in Cartesian coordinates), the geodesics are straight lines traversed at constant speed – geodesics are analogues in (M, ∇) of straight lines in Euclidean space.

(d) The condition of being a geodesic is *not* invariant under time reparametrization. Let $c : I \to M$ be a geodesic and $\tau : J \to I$ be a smooth increasing function, so that $\bar{c} = c \circ \tau : J \to M$ is a reparametrization of c. Then $\nabla^c_{d/dt}\dot{c} = 0$, but

$$\begin{aligned}
\nabla^{\bar{c}}_{d/dt}\dot{\bar{c}}(t) &= \nabla^{\bar{c}}_{d/dt}(\tau'(t)\dot{c}(\tau(t))) \\
&= \tau''(t)\dot{c}(\tau(t)) + \tau'(t)\nabla^{\bar{c}}_{d/dt}((\dot{c} \circ \tau)(t)) \\
&= \tau''(t)\dot{c}(\tau(t)) + \tau'(t)((\nabla^c_{\mathrm{d}\tau d/dt}\dot{c}) \circ \tau)(t) \\
&= \tau''(t)\dot{c}(\tau(t)) \neq 0
\end{aligned}$$

unless $\dot{c}(t) = 0$ (that is, the geodesic is constant) or τ is linear. So the definition of geodesics concerns the parametrization of the curve c, not just its image set.

Example 104 Solve the general geodesic problem for $(\mathbb{R}^2, \nabla)$ as in Example 85 with $c(0) = (x_0, y_0)$, $\dot{c}(0) = (u, v)$ say, $c(t) = (x(t), y(t))$.
Geodesic equations:

$$\begin{aligned}
\ddot{x} + x\dot{y}^2 &= 0 \qquad (5) \\
\ddot{y} &= 0 \qquad (6)
\end{aligned}$$

$$\begin{array}{llll}
(6) \Rightarrow & \dot{y}(t) = \dot{y}(0) = v & \Rightarrow & y(t) = y_0 + vt \\
(5) \Rightarrow & \ddot{x} + v^2 x = 0 & \Rightarrow & x(t) = A\cos vt + B\sin vt \\
 & & & x_0 = A \\
 & & & u = \dot{x}(0) = Bv \ \Rightarrow B = {}^{u}\!/_{v}, \ \ v \neq 0
\end{array}$$

So $c(t) = \begin{cases} (x_0 \cos vt + {}^u\!/_v \sin vt, y_0 + vt), & v \neq 0 \\ (x_0 + ut, y_0), & v = 0 \end{cases}$ $\square$

5.7 Torsion

In the final section of this chapter on connexions, we again restrict attention to connexions on TM.

Definition 105 The **torsion** of a connexion ∇ on M is the map

$$T : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

∇ is **symmetric** or **torsion free** if $T \equiv 0$. $\square$

Exercise 106 Show that T is a $(1, 2)$-tensor field and that $T(X, Y) \equiv -T(Y, X)$. $\square$

Example 107 ∇ from Example 85 is torsion free. This is clear from the fact that

$$T(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = ({\Gamma^k}_{ij} - {\Gamma^k}_{ji})\frac{\partial}{\partial x^k}. \qquad \square$$

Torsion controls the failure of the parallelogram law. In $\mathbb{R}^n$ with its usual connexion, we can construct a parallelogram as follows

(i) Choose a pair of vectors V, W at the same point x_0, then transport them along the geodesic (straight line) with initial data $(c(0), \dot{c}(0)) = (x_0, V)$ for time $\epsilon > 0$. Denote the endpoint of this geodesic by x_1, and the transported vectors V_1, W_1.

(ii) Now transport V_1, W_1 along the geodesic with initial data $(c(0), \dot{c}(0)) = (x_1, W_1)$ for time ϵ, to reach x_2, where their transports are denoted V_2, W_2.

(iii) Now transport V_2, W_2 along the geodesic with initial data $(c(0), \dot{c}(0)) = (x_2, -V_2)$ for time ϵ, to reach x_3, where their transports are denoted V_3, W_3.

(iv) Travel along the geodesic with initial data $(c(0), \dot{c}(0)) = (x_3, -W_3)$ for time ϵ, to reach x_4, the final point.

The point is that in standard $\mathbb{R}^n$ the union of all these geodesics is a closed parallelogram, that is, $x_4 = x_0$. In a general manifold with connexion (M, ∇) we can construct the same sequence of geodesics, but now we find that the geodesic "parallelogram" no longer (necessarily) closes, i.e. $x_0 \neq x_4$ in general. By solving the geodesic and parallel transport equations by Taylor expansion (for small edge duration ϵ), one can show that

$$x_4^i(\varepsilon) = x_0^i - \varepsilon^2 T^i{}_{jk}(x_0) v^j w^k + o(\varepsilon^2)$$

– closed to first order in ε
– not closed to 2^{nd} order
– non-closure controlled (up to $o(\varepsilon^2)$) by torsion.

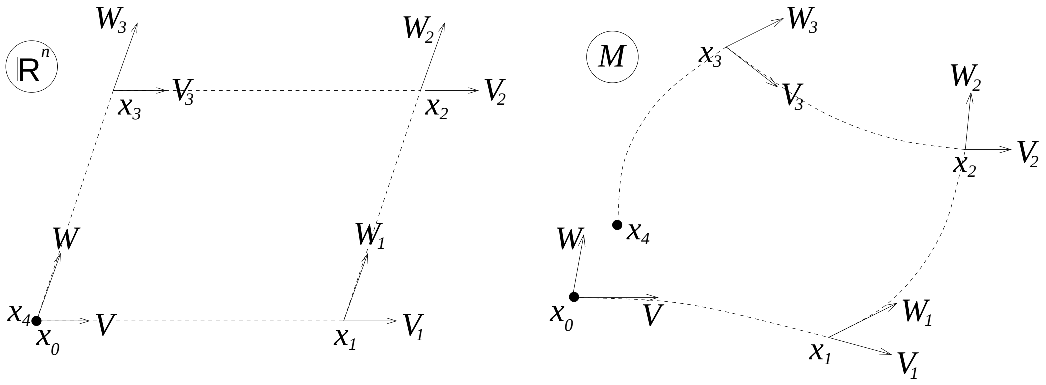
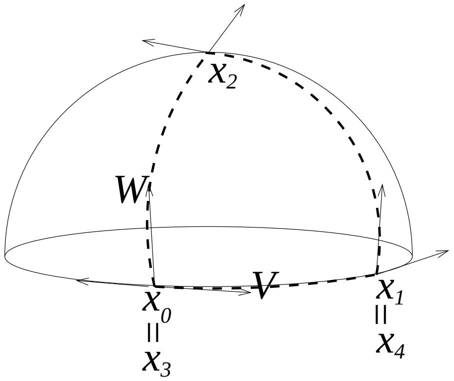


Figure 16: Geodesic parallelograms are closed in standard $\mathbb{R}^n$, but not on a general manifold M with connexion.

WARNING: Exact closure of geodesic parallelograms $\Rightarrow T \equiv 0$, but $T \equiv 0$ $\not\Rightarrow$ exact closure of geodesic parallelograms (only up to $o(\varepsilon^2)$).

Counterexample 108 The standard connexion on S^2 (to be defined shortly) is torsion free, but geodesic parallelograms on S^2 are certainly not closed.



6 Riemannian metrics

6.1 Definitions

Definition 109 A **Riemannian metric** on a manifold is a $(0,2)$ tensor field $g : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ which is

(a) symmetric: $g(X,Y) \equiv g(Y,X)$ for all $X, Y \in \Gamma(TM)$,

(b) positive definite: $g(X,X) \geq 0 \quad \forall\, X$, $g(X,X) = 0 \Leftrightarrow X = 0$

A **Riemannian manifold** is a pair (M, g). □

For each $p \in M$, g equips T_pM with an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$, often denoted $\langle\ ,\ \rangle_p$:

$$\langle u, v\rangle_p = g_p(u,v)$$

In particular, we can define the **length** of $u \in T_pM$ to be

$$\|u\| = \sqrt{g_p(u,u)},$$

and the **angle** between $u, v \in T_pM$ to be θ where

$$\cos\theta = \frac{g_p(u,v)}{\|u\|\|v\|}.$$

Given a local coordinate system on M it is conventional to define the metric coefficients

$$g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}), \qquad 1 \leq i \leq m, 1 \leq j \leq m$$

so that

$$g = g_{ij} dx^i \otimes dx^j.$$

Note that the metric coefficients form a positive definite, symmetric matrix.

Example 110 *(a)* The Euclidean metric on $\mathbb{R}^m$ is

$$g = \sum_{i=1}^m \mathrm{d}x^i \otimes \mathrm{d}x^i = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \cdots + \mathrm{d}x^m \otimes \mathrm{d}x^m$$

in Cartesian coordinates. So if $u = u^i \frac{\partial}{\partial x^i}$ and $v = v^i \frac{\partial}{\partial x^i}$ then

$$g(u,v) = \sum_{i=1}^m \mathrm{d}x^i \otimes \mathrm{d}x^i \left(u^j \frac{\partial}{\partial x^j}, v^k \frac{\partial}{\partial x^k} \right) = \sum_{i=1}^m u^j v^k \delta^i{}_j \delta^i{}_k = \sum_{i=1}^m u^i v^i$$

(b) The hyperbolic plane (H^2, g): $H^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$

$$g = \frac{\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y}{y^2}$$ □

Recall (Problem Set 2, question 2) that a manifold M which is a subset of another manifold N is a **submanifold** if the inclusion map $\iota : M \to N$ is an **embedding** (an immersion which is a homeomorphism onto its image). Every submanifold of a Riemannian manifold inherits a natural Riemannian metric:

Definition 111 Let (N, h) be a Riemannian manifold and M be a submanifold of N. Then the induced metric on M is $g = \iota^* h$, where $\iota : M \to N$ is the inclusion map. More explicitly, let $u, v \in T_pM$. Let c_1, c_2 be representative curves for u and v. Then c_1, c_2 are also curves in N, and their classes $[c_1], [c_2] \in T_pN$ are independent of the curves $c_1 \in u, c_2 \in v$ chosen. Hence, we may canonically identify u, v with vectors in T_pN. Then $g(u, v) = h(u, v)$.

Example 112 The usual metric g on S^2, is just the induced metric from the inclusion $\iota : S^2 \to \mathbb{R}^3$ where $\mathbb{R}^3$ is given the Euclidean metric. Exercise: Show that in terms of stereographic coordinates (ρ^1, ρ^2) this is

$$g = \frac{4[\mathrm{d}\rho^1 \otimes \mathrm{d}\rho^1 + \mathrm{d}\rho^2 \otimes \mathrm{d}\rho^2]}{(1 + (\rho^1)^2 + (\rho^2)^2)^2}$$

and that in polar coordinates,

$$\phi^{-1}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \in S^2,$$

this becomes

$$g = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta\, \mathrm{d}\varphi \otimes \mathrm{d}\varphi.$$

Definition 113 (Musical isomorphisms) Given a Riemannian metric g, there is a canonical isomorphism $\flat : T_pM \to T_p^*M$ at each $p \in M$, namely

$$v \mapsto \flat v, \qquad \flat v(u) = g_p(v, u) \quad \forall\, u \in T_pM$$

The inverse isomorphism $T_p^*M \to T_pM$ is denoted by $\sharp$.

Example 114 On (H^2, g) let $X = x^2 \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \in \Gamma(TM)$. What is $\flat X$?

$$\begin{aligned}
\flat X &= a(x,y)\mathrm{d}x + b(x,y)\mathrm{d}y \\
\text{where} \quad a(x,y) &= \flat X\left(\frac{\partial}{\partial x}\right) = g\left(X, \frac{\partial}{\partial x}\right) = \left\langle x^2\frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right\rangle = \frac{x^2}{y^2} \\
b(x,y) &= \flat X\left(\frac{\partial}{\partial y}\right) = g\left(x^2\frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = -\frac{1}{y^2} \\
\text{i.e.} \quad \flat X &= \frac{x^2\mathrm{d}x - \mathrm{d}y}{y^2}. \qquad \Box
\end{aligned}$$

6.2 Metric connexions

Definition 115 Let (M, g) be a Riemannian manifold and ∇ be a connexion on TM. Then ∇ is **metric compatible** (or, is a **metric connexion**) if

$$\nabla g = 0.$$

The above condition means that $\nabla_X g = 0$ for all $x \in \Gamma(TM)$, where ∇ is the induced connexion on $T^*M \otimes T^*M$. Recall that this connexion is defined by

$$(\nabla_X g)(Y, Z) = X[g(Y, Z)] - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$$

for all vector fields X, Y, Z. So ∇ is metric compatible if (and only if)

$$X[g(Y, Z)] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \qquad \text{for all } X, Y, Z \in \Gamma(TM) \quad (\clubsuit).$$

Parallel transport using a metric connexion preserves inner products:

Proposition 116 *Let $c : I \to M$ be a smooth curve, and u, v be parallel sections of $c^{-1}TM$. Then $g(u(t), v(t))$ is constant.*

Proof: Consider $g \circ c \in \Gamma(c^{-1}T^{(0,2)}M)$. By the definition of pullback connexion, and metric compatibility of ∇, $\nabla^c_{d/dt}(g \circ c) = 0$. But for any sections $\omega \in \Gamma(c^{-1}T^{(0,2)}M)$, $u, v \in \Gamma(c^{-1}TM)$,

$$(\nabla^c_{d/dt}\omega)(u, v) = \frac{d}{dt}\omega(u, v) - \omega(\nabla^c_{d/dt}u, v) - \omega(u, \nabla^c_{d/dt}v).$$

Applying this in the case where $\omega = g$ and u and v are both parallel sections, one sees that

$$0 = \frac{d}{dt}g(u(t), v(t)) - g(0, v(t)) - g(u(t), 0)$$

whence the result immediately follows. $\square$

Corollary 117 *Given curve $c : [0, 1] \to M$ the parallel transport map $Tr_c : T_{c(0)}M \to T_{c(1)}M$ for a metric connexion is an isometry (with respect to the inner products $g_{c(0)}$, $g_{c(1)}$).*

Corollary 118 *The based holonomy group H_x for a metric connexion on TM is a subgroup of* $\mathrm{Orth}(T_xM, g_x) \cong O(m)$. *(Here* Orth *denotes the group of isometries of an inner product space.)*

In fact, on an orientable Riemannian manifold, we can say more:

Proposition 119 *Let (M, g) be an orientable Riemannian manifold, ∇ be a metric connexion on TM and $x \in M$. Then the based holonomy group H_x is a subgroup of $\mathrm{Orth}_+(T_xM, g_x) \cong SO(m)$, the group of orientation preserving isometries of (T_xM, g_x).*

Proof: Let $c : [0, 1] \to M$ be a closed path based at x. That Tr_c is an isometry follows from Corollary 118. It remains to show that Tr_c is orientation preserving. Let $\mathcal{A}$ be an orientation atlas on M. Since c is compact, it is covered by a finite collection of charts from this atlas. Hence, there exists a dissection

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

of $[0, 1]$ so that $c_{(i)} = c|_{[t_{i-1},t_i]}$ lies in U_i for each $i = 1, \ldots, n$. Denote by $Tr_i(t)$ the matrix representing parallel transport along $c_{(i)}$ from $c(t_{i-1})$ to $c(t)$ with respect to the coordinate basis vectors for chart (U_i, ϕ_i). Note that $Tr_i(t_{i-1}) = \mathbb{I}_m$ and $Tr_i(t)$ is a smooth curve in $GL(m, \mathbb{R})$, so $\det Tr_i(t) > 0$ for all $t \in [t_{i-1}, t_i]$. In particular, $T_i = Tr_i(t_i)$ has positive determinant.

Now solve the parallel transport equation around c by solving it piecewise on each segment $c_{(i)}$. Since we must change basis from chart (U_i, ϕ_i) to (U_{i+1}, ϕ_{i+1}) at each transition point, one finds that

$$Tr_c = d(\phi_1 \circ \phi_n^{-1})T_n \cdots d(\phi_3 \circ \phi_2^{-1})T_2 d(\phi_2 \circ \phi_1^{-1})T_1.$$

Since $\mathcal{A}$ is an orientation atlas $\det d(\phi_i \circ \phi_j^{-1}) > 0$ for all i, j. Hence $\det Tr_c > 0$. $\square$

Example 120 Consider $M = \mathbb{R}^2$ with the connexion defined in Example 85. We found (see Example 97) that $H_{(0,0)} \cong (\mathbb{R}, +)$ which is not isomorphic to any subroup of $SO(2)$. Hence, we conclude that ∇ is *not* compatible with *any* Riemannian metric on M.

One of the miraculous facts that makes Riemannian geometry so natural and rich is that, given a choice of metric, there is a unique "best" connexion on TM.

Theorem 121 (The Fundamental Theorem of Riemannian Geometry) *On any Riemannian manifold (M, g) there exists a unique metric compatible, torsion free connexion, ∇ (usually called the* **Levi-Civita connexion***).*

Proof: (a) Assume that such a ∇ exists. Then $\forall\, X, Y, Z$

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0$$

and

$$\begin{aligned}
X[g(Y,Z)] \;&+\; Y[g(Z,X)] \;-\; Z[g(X,Y)] \\
&= g(\nabla_X Y, Z) + g(\nabla_X Z, Y) + g(\nabla_Y Z, X) \\
&\quad +g(\nabla_Y X, Z) - g(\nabla_Z X, Y) - g(\nabla_Z Y, X) \\
&= g(\nabla_X Y + \nabla_Y X, Z) + g(\nabla_X Z - \nabla_Z X, Y) \\
&\quad +g(\nabla_Y Z - \nabla_Z Y, X) \\
&= g(2\nabla_X Y - [X,Y], Z) + g([X,Z], Y) + g([Y,Z], X) \\
\Rightarrow\; 2g(\nabla_X Y, Z) &= X[g(Y,Z)] + Y[g(Z,X)] - Z[g(X,Y)] \\
&\quad +g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) \qquad (7)
\end{aligned}$$

Now, (7) holds for all Z, and hence uniquely defines a covector field $\flat(\nabla_X Y)$. But $\flat$ is an isomorphism, so (7) uniquely defines $\nabla_X Y$. So if ∇ exists, it is unique.
(b) Now, given any $X, Y \in \Gamma(TM)$, let $\nabla_X Y$ be the vector field defined by (7). We must verify that the map given by $(X,Y) \mapsto \nabla_X Y$ satisfies the axioms of definition 83, and that $\nabla_X Y - \nabla_Y X - [X,Y] = 0$. We leave this as an exercise for the reader.

□

One may use the formula (7) to compute connexion coefficients ${\Gamma^k}_{ij}$ relative to a particular coordinate basis $\{\frac{\partial}{\partial x^i} : i = 1, 2, \ldots, m\}$.

Choose $X = \dfrac{\partial}{\partial x^i}$, $Y = \dfrac{\partial}{\partial x^k}$, $Z = \dfrac{\partial}{\partial x^k}$, then

$$\begin{aligned}
2g\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) &= \frac{\partial}{\partial x^i}\left[g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right)\right] + \frac{\partial}{\partial x^j}\left[g\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i}\right)\right] \\
&\quad -\frac{\partial}{\partial x^k}\left[g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right] \\
\Rightarrow 2\,{\Gamma^l}_{ij} g_{lk} &= \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) \\
\Rightarrow \quad {\Gamma^m}_{ij} &= \frac{1}{2} g^{km} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right), \qquad (8)
\end{aligned}$$

where g^{ij} are the components of the inverse metric (i.e. the components of the inverse of the matrix whose components are g_{ij}). The coefficients Γ^i_{jk} are called the **Christoffel symbols**.

Note that (7) is more useful than (8) since one can use it to compute $\nabla_X Y$ relative to a non-coordinate basis for $\Gamma(TM)$. For example, let E_i, $i = 1, \ldots, m$ be a basis of local **orthonormal** vector fields, i.e.

$$g(E_i, E_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} .$$

Then $\nabla_{E_i} E_j = \Gamma^k{}_{ij} E_k$ where

$$\Gamma^k{}_{ij} = g(\nabla_{E_i} E_j, E_k) = \frac{1}{2}\left\{g([E_i, E_j], E_k) - g([E_j, E_k], E_i) + g([E_k, E_i], E_j)\right\}$$

This shows that ∇ is uniquely determined by the Lie bracket structure of any local orthonormal frame.

6.3 Geodesics in Riemannian Manifolds

A geodesic in (M, g) is simply a geodesic with respect to the Levi-Civita connexion, as in definition 103, i.e. a curve $\boldsymbol{\gamma} : I \to M$ such that

$$\nabla^{\gamma}_{d/dt} \dot{\gamma} = 0$$

Note that since ∇ is metric compatible we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\dot{\gamma}\|^2 = \frac{\mathrm{d}}{\mathrm{d}t} g(\dot{\gamma}, \dot{\gamma}) = 2g(\dot{\gamma}, \nabla^{\gamma}_{d/dt} \dot{\gamma}) = 0$$

So in a Riemannian manifold (M, g) geodesics are traversed at constant speed (N.B. this statement is meaningless for (M, ∇) in general).

A fundamental property of Riemannian manifolds is that we can give a variational formulation of the geodesic problem.

Definition 122 The **length**, $L[\gamma]$ of a smooth curve $\gamma : [a, b] \to (M, g)$ is

$$L[\gamma] := \int_a^b \sqrt{g(\dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t))}\,\mathrm{d}t = \int_a^b \|\dot{\gamma}(t)\|\mathrm{d}t. \qquad \Box$$

Exercise 123 Show that $L[\gamma]$ is invariant under reparametrization of the curve γ. That is, let $\tau : [c, d] \to [a, b]$ be a smooth increasing function and $\widetilde{\gamma} = \gamma \circ \tau$. Then $L[\widetilde{\gamma}] = L[\gamma]$.

Remark 124 Any regular curve $\gamma : I \to M$ (i.e. smooth curve with $\dot{\gamma}(t) \neq 0$ for all $t \in I$) can be reparametrized so that $\|\dot{\widetilde{\gamma}}(t)\| = 1$ for all $t \in I$. We just fix $t_0 \in I$ and choose $\tau : J \to I$ to be the inverse of the mapping

$$t \mapsto \int_{t_0}^t \|\dot{\gamma}(s)\| ds.$$

The reparametrized curve is said to be **arclength parametrized** or a **unit speed curve**. Every smooth curve has a piecewise smooth unit speed reparametrization.

Definition 125 Let $\gamma : [a,b] \to M$ be a smooth curve. A **variation** of γ is a smooth map $F : (-\varepsilon, \varepsilon) \times [a,b] \to M$ such that $F(0,t) = \gamma(t)$ for all $t \in [a,b]$. The curve $t \mapsto F(s,t)$ will often be denoted γ_s (so, in particular, $\gamma_0 = \gamma$). We say the variation has **fixed endpoints** if $F(s,a) = \gamma(a)$ and $F(s,b) = \gamma(b)$ for all $s \in (-\varepsilon, \varepsilon)$.

Definition 126 A smooth curve $\gamma : [a,b] \to M$ is a **critical point** of the length functional L if, given any variation F of γ with fixed endpoints,

$$\frac{d}{ds} L[\gamma_s]\bigg|_{s=0} = 0,$$

where γ_s is the curve $t \mapsto F(s,t)$, as above.

We will prove that the critical points of L are, when parametrized to have constant speed, precisely the geodesics in M with respect to the Levi-Civita connexion. First, we need a technical lemma.

Lemma 127 *Let $\gamma : [a,b] \to M$ be a curve in M and F be a variation of γ. Define $X, Y \in \Gamma(F^{-1}TM)$ by*

$$Y(s,t) = dF\frac{\partial}{\partial s}, \qquad X(s,t) = dF\frac{\partial}{\partial t}.$$

Let ∇ be a torsion free connexion on TM. Then

$$\nabla^F_{\partial/\partial s} X - \nabla^F_{\partial/\partial t} Y = 0.$$

Note: We can think of $X(s,t)$, for fixed s, as the velocity vector of the curve $\gamma_s(t)$. Similarly, we can think of $Y(s,t)$ for fixed t as the velocity vector of the curve $s \mapsto F(s,t)$. Hence, $Y(0,\cdot) \in \Gamma(\gamma^{-1}TM)$ is the vector field along γ generated by the variation F. In particular, if the variation has fixed endpoints, so $F(s,a)$ and $F(s,b)$ are constant, then $Y(s,a) = Y(s,b) = 0$.

Proof of Lemma 127: Choose a local coordinate system for M in a neighbourhood of a particular point $F(s,t) \in M$. Then F has local coordinate expression $(F^1(s,t), \ldots, F^m(s,t))$ for some smooth real valued functions $F^i(s,t)$. Further, $Y(s,t) = Y^i(s,t)E_i$, $X(s,t) = X^i(s,t)\partial/E_i$ where $E_i = \partial/\partial x^i$ and

$$Y^i = \frac{\partial F^i}{\partial s}, \qquad X^i = \frac{\partial F^i}{\partial t}.$$

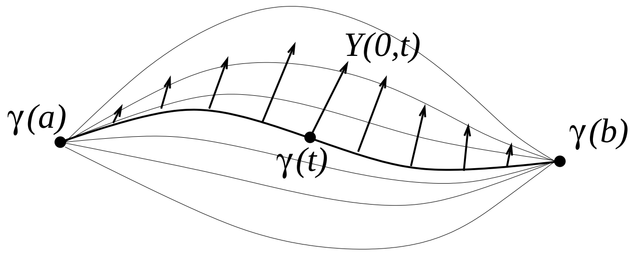


Figure 17: The vector field Y generated by a variation $(s,t) \mapsto \gamma_s(t)$ of a curve γ

Now

$$\begin{aligned}\nabla^F_{\partial/\partial s} X &= \nabla^F_{\partial/\partial s}(X^i E_i) = \frac{\partial X^i}{\partial s} E_i + X^i \nabla^F_{\partial/\partial s} E_i \\ &= \frac{\partial^2 F^i}{\partial s \partial t} E_i + X^i \nabla_{dF\partial/\partial s} E_i = \frac{\partial^2 F^i}{\partial s \partial t} E_i + X^i Y^j \nabla_{E_j} E_i.\end{aligned}$$

Similarly

$$\nabla^F_{\partial/\partial t} Y = \frac{\partial^2 F^i}{\partial t \partial s} E_i + Y^i X^j \nabla_{E_j} E_i.$$

Hence

$$\nabla^F_{\partial/\partial s} X - \nabla^F_{\partial/\partial t} Y = X^i Y^j (\nabla_{E_j} E_i - \nabla_{E_i} E_j) = X^i Y^j [E_j, E_i] = 0.$$

□

Theorem 128 *Let (M,g) be a Riemannian manifold. A curve $\gamma : [a,b] \to M$, parametrized to have constant speed, is a critical point of length if and only if it is a geodesic (with respect to the Levi-Civita connexion).*

Proof: Let F be any variation of γ with fixed endpoints and the curves γ_s, $-\varepsilon < s < \varepsilon$ and the sections $X, Y \in \Gamma(F^{-1}TM)$ be defined as above. Let $\hat{X}(s,t) = X(s,t)/\|X(s,t)\|$, which is well defined for $|s|$ sufficiently small, since $\|X(0,t)\| = u \neq 0$ constant and $[a,b]$ is compact. Then

$$L[\gamma_s] = \int_a^b \sqrt{g(X(s,t), X(s,t))} dt.$$

Let ∇ be the Levi-Civita connexion. Then, since ∇ is metric compatible,

$$\begin{aligned}\frac{d}{ds} L[\gamma_s] = \int_a^b \frac{\partial}{\partial s} \sqrt{g(X(s,t), X(s,t))} dt &= \int_a^b \frac{g(X(s,t), \nabla^F_{\partial/\partial s} X(s,t))}{\sqrt{g(X(s,t), X(s,t))}} dt \\ &= \int_a^b g(\hat{X}(s,t), \nabla^F_{\partial/\partial s} X(s,t)) dt.\end{aligned}$$

But ∇ is torsion free, so, by Lemma 127,

$$\begin{aligned}
\frac{d}{ds}L[\gamma_s] &= \int_a^b g(\hat{X}(s,t), \nabla^F_{\partial/\partial t}Y(s,t))dt \\
&= \int_a^b \left\{\frac{\partial}{\partial t}g(\hat{X}(s,t), Y(s,t)) - g(\nabla^F_{\partial/\partial t}\hat{X}(s,t), Y(s,t))\right\} dt \\
&= [g(\hat{X}(s,t), Y(s,t)]_{t=a}^{t=b} - \int_a^b g(\nabla^F_{\partial/\partial t}\hat{X}(s,t), Y(s,t))dt.
\end{aligned}$$

The variation has fixed endpoints, so $Y(s,a) = Y(s,b) = 0$, and the boundary term above vanishes. Recalling that $X(s,t) = \dot{\gamma}_s(t)$, and that $\|\dot{\gamma}(t)\| = u$, constant, one sees that

$$\left.\frac{d}{ds}L[\gamma_s]\right|_{s=0} = -\frac{1}{u}\int_a^b g(\nabla^\gamma_{d/dt}\dot{\gamma}(t), Y(0,t))dt.$$

Clearly, if γ is a geodesic w.r.t. ∇ then the right hand side vanishes for all variations and we conclude that γ is a critical point of L. Conversely, if γ is a critical point of L, then the integral above must vanish for all variations, and hence, for all sections $Y(0,\cdot) \in \Gamma(\gamma^{-1}TM)$. But g is nondegenerate so, by the Fundamental Lemma of the Calculus of Variations, $\nabla^\gamma_{d/dt}\dot{\gamma} = 0$ for all $t \in [a,b]$. Hence, γ is a geodesic. □

So *if* a length *minimizing* curve between points $x, y \in M$ exists then it must be, up to reparametrization, a geodesic. We should note, however, that

(a) There may be no length minimizing curve between x, y, even if M is connected. For example, consider the punctured plane $M = \mathbb{C}\backslash\{0\}$ with the Euclidean metric and $x = 1$, $y = -1$. Then there are curves between x and y of any length greater than 2, but there is no curve of length 2, so no length minimizing curve exists.

(b) In general a geodesic $\gamma : [a,b] \to M$ is length *extremizing* but not necessarily length minimizing. For example, let $M = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ with the obvious metric. Then $\gamma : [0, 3\pi/2] \to M$, $\gamma(t) = e^{it}$ is a geodesic connecting $x = 1$ and $y = -i$, but it has length $3\pi/2$, and is certainly not the shortest curve connecting these two points.

It turns out that if M is compact and connected then there exists a length minimizing geodesic between any pair of points in M, though this is very far from easy to prove. Another interesting result along these lines, which we shall state but not prove, concerns *periodic* geodesics:

Definition 129 A geodesic $\gamma : \mathbb{R} \to M$ is **periodic** if there exists $T > 0$ such that $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$. Alternatively, it is a closed geodesic $\gamma : [0, T] \to M$ with $\dot{\gamma}(0) = \dot{\gamma}(T)$.

Theorem 130 *Let (M, g) be a compact Riemannian manifold. Then every free homotopy class of closed curves in M contains a length-minimizing periodic geodesic.*

Clearly compactness is crucial for this result. For example, let M be the punctured plane $\mathbb{C}\backslash\{0\}$ with the Euclidean metric. Then every nontrivial free homotopy class of loops contains curves of arbitrarily small length

$$\gamma : [0, 2\pi] \to M, \qquad \gamma(t) = \varepsilon e^{it},$$

so no nontrivial class has a length minimizing representative.

6.4 Curvature and topology

Recall that for a general vector bundle E with connexion ∇, the curvature R is a section of $T^{(0,2)}M \otimes \mathrm{End}E$ which determines the holonomy of the connexion around infinitesimally small closed loops. Since a Riemannian manifold has a canonical connexion on its tangent bundle (the Levi-Civita connexion), one can talk of the curvature of a Riemannian manifold. This turns out to have very direct geometric significance. In particular, there is a large body of work establishing links between the topology of a manifold and the curvature properties of any metric it can support. The aim of this section is to state and prove a representative example of such results.

Definition 131 The **scalar curvature** of a Riemannian manifold (M, g) is the real-valued function $S : M \to \mathbb{R}$ defined by

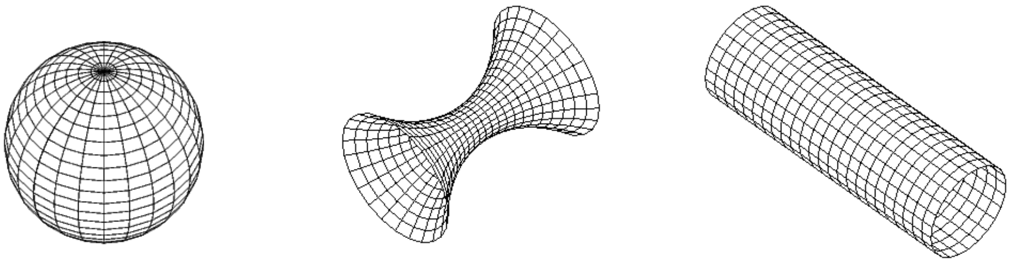
$$S(p) = \sum_{i,j=1}^{m} g(E_i, R(E_i, E_j)E_j)$$

where $E_1, \dots, E_m$ is an orthonormal basis for T_pM, and R is the curvature tensor of the Levi-Civita connexion on TM.

Fact: Let (M, g) be a two-dimensional submanifold of Euclidean $\mathbb{R}^3$, with g the induced metric. Then S coincides with the classical Gauss curvature of M. So a sphere has $S > 0$, a saddle has $S < 0$ and a cylinder has $S = 0$.

Definition 132 Let P be a (two dimensional) plane in T_pM and X, Y be an orthonormal basis for P. The **sectional curvature** of the plane P is

$$\sigma(P) = g(X, R(X, Y)Y).$$



$S > 0$ $S < 0$ $S = 0$

Figure 18: The scalar curvature of surfaces embedded in $\mathbb{R}^3$

Exercise 133 Show that $\sigma(P)$ is independent of the orthonormal basis chosen to span P.

The proof uses the first two identities asserted in the following proposition, whose proof we also leave as an exercise for the reader:

Proposition 134 *Let R be the curvature of the Levi-Civita connexion on (M, g). Then, for all vector fields X, Y, Z,*

- *(a)* $R(X,Y)Z + R(Y,X)Z = 0$
- *(b)* $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$
- *(c)* $g(X, R(Y,X)X) = 0.$

[In fact (a) is true for any connexion, (b) is true for any torsion free connexion and (c) is true for any metric connexion.]

Fact: Consider the (locally defined) two-dimensional submanifold of (M, g) formed by taking the union of all geodesics through $p \in M$ with initial velocities in P. This is called the **geodesic spray** of the plane P. If we give it the induced metric from g, its scalar curvature at p is precisely $\sigma(P)$.

We are now ready to state our representative "topology constrains curvature" result (recall that a space M is simply connected if every closed loop in M is contractible).

Theorem 135 (Synge) *Any compact, orientable, even-dimensional manifold with strictly positive sectional curvature is simply connected.*

Remark: All the hypotheses are necessary:

- (a) If we drop "orientable" then $\mathbb{R}P^2 = S^2/\mathbb{Z}_2$ with the projected metric (from the usual metric on S^2) is a counterexample.

(b) If we drop "even-dimensional" then $\mathbb{R}P^3 = S^3/\mathbb{Z}_2$ with the projected metric is a counterexample.

(c) If we weaken "strictly positive" to "non-negative" sectional curvature then $T^2 = S^1 \times S^1$ with the obvious product metric is a counterexample.

The proof of Synge's Theorem has several ingredients. The simplest of these is a linear algebra lemma, whose proof we leave as an exercise.

Lemma 136 *Let $A \in SO(2k+1)$. Then there exists $v \in \mathbb{R}^{2k+1}$ such that $v \neq 0$ and $Av = v$.*

The most subtle ingredient is the second variation formula for the length functional evaluated on a periodic geodesic.

Proposition 137 *Let $\gamma : [0,T] \to M$ be a unit speed periodic geodesic and $(s,t) \mapsto F(s,t) = \gamma_s(t)$ be a smooth variation of γ through periodic curves (so $\gamma_s : [0,T] \to M$ is periodic for each $s \in (-\varepsilon, \varepsilon)$). Let $Y(s,t) = dF\frac{\partial}{\partial s} \in \Gamma(F^{-1}TM)$ and*

$$\widetilde{Y}(t) = Y(0,t) - g(Y(0,t), \dot{\gamma}(t))\dot{\gamma}(t),$$

the component of $Y(0,t)$ orthogonal to the curve γ. Then

$$\frac{d^2}{ds^2}L[\gamma_s]\bigg|_{s=0} = \int_0^T \left\{ \|\nabla^\gamma_{d/dt}\widetilde{Y}\|^2 - g(\dot{\gamma}, R(\dot{\gamma}, \widetilde{Y})\widetilde{Y}) \right\} dt$$

Proof: As before, let $X(s,t) = dF\frac{\partial}{\partial t} = \dot{\gamma}_s(t)$ and $\hat{X} = X/\|X\|$. Then, arguing as in the proof of Theorem 128 we have, by Lemma 127

$$\begin{aligned} \frac{d}{ds}L[\gamma_s] &= \int_0^T \frac{g(X, \nabla^F_{\partial/\partial t}Y)}{\|X\|} dt \\ \text{so} \qquad \frac{d^2}{ds^2}L[\gamma_s] &= \int_0^T \left\{ \frac{1}{\|X\|}\left(g(\nabla^F_{\partial/\partial s}X, \nabla^F_{\partial/\partial t}Y) + g(X, \nabla^F_{\partial/\partial s}\nabla^F_{\partial/\partial t}Y)\right) \right. \\ &\qquad \left. - \frac{1}{\|X\|^3} g(X, \nabla^F_{\partial/\partial s}X) g(X, \nabla^F_{\partial/\partial t}Y) \right\} dt \\ &= \int_0^T \left\{ \frac{1}{\|X\|}\|\nabla^F_{\partial/\partial t}Y\|^2 + g(\hat{X}, \nabla^F_{\partial/\partial t}\nabla^F_{\partial/\partial s}Y + R^F(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})Y) \right. \\ &\qquad \left. - \frac{1}{\|X\|^3} g(X, \nabla^F_{\partial/\partial t}Y)^2 \right\} dt \end{aligned}$$

where R^F is the curvature of ∇^F and we have used Lemma 127 again. Now

$$\begin{aligned}\int_0^T g(\hat{X}, \nabla^F_{\partial/\partial t}\nabla^F_{\partial/\partial s} Y) dt &= \Big[g(\hat{X}(s,t), \nabla^F_{\partial/\partial t} Y(s,t))\Big]_{t=0}^{t=T} - \int_0^T g(\nabla^F_{\partial/\partial t}\hat{X}, \nabla^F_{\partial/\partial s} Y) dt \\ &= -\int_0^T g(\nabla^F_{\partial/\partial t}\hat{X}, \nabla^F_{\partial/\partial s} Y) dt\end{aligned}$$

since $X(s,t)$ and $Y(s,t)$ are periodic in t for all t (F is a variation through periodic curves). This integral vanishes at $s = 0$ since $\hat{X}(0,t) = \dot{\gamma}(t)$ and γ is a geodesic (so $\nabla^\gamma_{d/dt}\dot{\gamma} = 0$). Furthermore,

$$R^F(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})Y = R(dF\frac{\partial}{\partial s}, dF\frac{\partial}{\partial t})Y = R(Y, X)Y.$$

Hence,

$$\frac{d^2}{ds^2} L[\gamma_s]\bigg|_{s=0} = \int_0^T \Big\{\|\nabla^\gamma_{d/dt} Y\|^2 + g(\dot{\gamma}, R(Y, \dot{\gamma})Y) - g(\dot{\gamma}, \nabla^\gamma_{d/dt} Y)^2\Big\} dt.$$

Now, since γ is a geodesic,

$$\begin{aligned}\nabla^\gamma_{d/dt}\widetilde{Y} &= \nabla^\gamma_{d/dt} Y - \frac{d}{dt} g(\dot{\gamma}, Y)\dot{\gamma} - g(\dot{\gamma}, Y)\nabla^\gamma_{d/dt}\dot{\gamma} \\ &= \nabla^\gamma_{d/dt} Y - g(\dot{\gamma}, \nabla^\gamma_{d/dt} Y)\dot{\gamma} \\ \text{so} \qquad \|\nabla^\gamma_{d/dt}\widetilde{Y}\|^2 &= \|\nabla^\gamma_{d/dt} Y\|^2 - g(\dot{\gamma}, \nabla^\gamma_{d/dt} Y)^2\end{aligned}$$

where we have used the fact that $\|\dot{\gamma}\| = 1$. Finally, by Proposition 134,

$$\begin{aligned}g(\dot{\gamma}, R(Y, \dot{\gamma})Y) &= g(\dot{\gamma}, R(\widetilde{Y} + f\dot{\gamma}, \dot{\gamma})(\widetilde{Y} + f\dot{\gamma})) \qquad \text{where } f(t) = g(\dot{\gamma}(t), Y(0,t)) \\ &= g(\dot{\gamma}, R(\widetilde{Y}, \dot{\gamma})\widetilde{Y}) + g(\dot{\gamma}, R(\widetilde{Y}, \dot{\gamma})\dot{\gamma}) \qquad \text{identity (a)} \\ &= g(\dot{\gamma}, R(\widetilde{Y}, \dot{\gamma})\widetilde{Y}) \qquad \text{identity (c)} \\ &= -g(\dot{\gamma}, R(\dot{\gamma}, \widetilde{Y})\widetilde{Y}) \qquad \text{identity (a)}\end{aligned}$$

which completes the proof. □

Remark: $g(\dot{\gamma}, R(\dot{\gamma}, \widetilde{Y})\widetilde{Y}) = \|\widetilde{Y}\|^2\sigma(P)$ where P is the plane spanned by $\dot{\gamma}(t), Y(t)$. So the above proposition gives a link between the curvature in a neighbourhood of a periodic geodesic and the *stability* of that geodesic. That is, if the manifold is positively curved, one expects there to exist deformations of the geodesic which shrink its length to second order.

We are now ready to prove Theorem 135:

Proof of Synge's Theorem: Assume (M, g) is even dimensional, orientable, with strictly positive sectional curvature, but *not* simply connected.

Then, in any nontrivial free homotopy class of loops there exists a length minimizing periodic geodesic $\gamma : [0,T] \to M$ (Theorem 130) which, without loss of generality, we can assume is unit speed. Consider Tr_γ, the parallel transport map around γ. Since γ is a geodesic, $\dot{\gamma}$ is parallel so $Tr_\gamma \dot{\gamma}(0) = \dot{\gamma}(0)$. But Tr_γ is an isometry, so maps the hyperplane orthogonal to $\dot{\gamma}(0)$, E say, to itself. Choose an orthonormal basis $e_1, \ldots, e_m$ for $T_{\gamma(0)}M$ such that $e_1 = \dot{\gamma}(0)$, so $e_2, \ldots, e_m$ span E. With respect to this basis, Tr_γ is block diagonal with 1 in the top 1×1 block. By Proposition 119, Tr_γ is in $SO(m)$, so the bottom $(m-1)\times(m-1)$ block lies in $SO(m-1)$. Hence, by Lemma 136, there exists $v \in E$, $v \neq 0$ such that $Tr_\gamma v = v$ (since m is even). Consider the section of $\gamma^{-1}TM$ defined by parallel transport of v, call it $Y(t)$. By construction, Y is a periodic section and $\nabla^\gamma_{d/dt} Y \equiv 0$. Let $F : (s,t) \mapsto \gamma_s(t)$ be any variation of γ through periodic curves with $dF\frac{\partial}{\partial s}|_{s=0} = Y$ (such a variation certainly exists: we can take $F(s,t)$ to be the point $\alpha(s)$ on the geodesic with initial data $\alpha(0) = \gamma(t)$, $\dot{\alpha}(0) = Y(t)$, for example). Then, by Theorem 128,

$$\frac{d}{ds} L[\gamma_s]\bigg|_{s=0} = 0$$

and by Proposition 137,

$$\frac{d^2}{ds^2} L[\gamma_s]\bigg|_{s=0} = \int_0^T \left\{ \|0\|^2 - g(\dot{\gamma}, R(\dot{\gamma}, Y)Y) \right\} < 0$$

since the sectional curvature of M is strictly positive. Hence $L[\gamma_s] < L[\gamma]$ for all $s > 0$ sufficiently small. But γ_s is clearly homotopic to γ, a contradiction. □

Problem Set 1

1 Let M be a complex manifold of real dimension 2. (Recall that "complex" means it has an atlas whose transition functions are holomorphic.) Show that M is orientable.

2 Write down holomorphic atlases on

   (a) S^2 (hint: stereographic projection)

   (b) $\mathbb{R} \times S^1$ (hint: you can cover this with a single chart, if you're clever)

   (c) $\mathbb{C}P^n$ (hint: copy Example 42).

   So all these are naturally complex manifolds. Does the atlas of part 2a generalize to S^{2m} for $m > 1$?

3 Let M, N be complex manifolds.

   (a) What does it mean to say that a map $f : M \to N$ is holomorphic?

   (b) Use your answers to parts 2a and 2c to show that S^2 and $\mathbb{C}P^1$ are biholomorphic, that is, there is a bijection $f : S^2 \to \mathbb{C}P^1$ such that f and f^{-1} are holomorphic.

4 Let M be a complex manifold of real dimension 2. Such manifolds are often called *Riemann surfaces*. Given a pair of tangent vectors $X, Y \in T_pM$, we can define the (cosine of the) angle θ between them as follows. Pick a complex chart (U, ϕ) around p, and let $x, y : (-\varepsilon, \varepsilon) \to M$ be generating curves for X, Y respectively. Then $\phi \circ x$, $\phi \circ y$ are curves in $\mathbb{C}$. Let $u = (\phi \circ x)'(0) \in \mathbb{C}$ and $v = (\phi \circ y)'(0) \in \mathbb{C}$ and define

$$\cos\theta = \frac{\mathrm{Re}(u\bar{v})}{|u||v|}.$$

   so that θ is the angle between the complex numbers u and v. Show that $\cos\theta$ is canonical, that is, independent of the choice of the complex chart about p. So Riemann surfaces possess a canonical *conformal structure*, that is, the notion of *angle* is well defined without equipping them with any extra structure.

5 Let $p = (0, 1, 0) \in S^2$ and $X = (2, 0, -1) \in T_pS^2$.

   (a) Write down a generating curve for X.

(b) Express X relative to the coordinate bases for stereographic coordinates and spherical polar coordinates.

6 The purpose of this question is to show that the requirement that M be Hausdorff in the definition of a smooth manifold is not redundant, that is, it does not follow from the properties of an atlas. Let $X = \{(x, y) \in \mathbb{R}^2 : |y| = 1\}$, a disjoint union of two parallel straight lines, with the subspace topology. Define an equivalence relation on X as follows:

$$(x, y) \sim (x', y') \text{ iff } (x, y) = (x', y') \text{ or } (x, y) = (x', -y') \text{ and } x \neq 0.$$

Let $M = X/\sim$, with the quotient topology. (You can think of M as being the real line but with two distinct numbers "zero".) Show that M is not Hausdorff. Show that the sets $U_+ = M\backslash\{[(0, -1)]\}$ and $U_- = M\backslash\{[(0, 1)]\}$ are open amd cover M. Show that the maps

$$\begin{aligned} \phi_+ : U_+ \to \mathbb{R} \qquad & \phi_+([(x, 1)]) = x \\ \phi_- : U_- \to \mathbb{R} \qquad & \phi_-([(x, -1)]) = x \end{aligned}$$

are homeomorphisms. Show that $\phi_+ \circ \phi_-^{-1} : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ and $\phi_- \circ \phi_+^{-1} : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ are smooth. Hence $\{(U_+, \phi_+), (U_-, \phi_-)\}$ has all the properties of an atlas on M, despite M being non-Hausdorff.

Problem Set 2

1 A smooth map $f : M \to N$ is an *immersion* if $df_x : T_xM \to T_{f(x)}N$ is injective everywhere, and a *submersion* if $df_x : T_xM \to T_{f(x)}N$ is surjective everywhere. An immersion is an *embedding* if it is a homeomorphism onto its image. Determine whether the following maps are immersions, submersions or neither. Determine whether the immersions (if any) are embeddings.

   (a) $f : S^2 \to \mathbb{R}$, $f(x_1, x_2, x_3) = x_3$.

   (b) $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x/\cosh x, x^2/\cosh x)$.

   (c) $f : \mathbb{R}P^2 \to \mathbb{R}P^3$, $f([x_1, x_2, x_3]) = [x_1, x_2, x_3, x_1 + x_2 + x_3]$.

   (d) $\det : GL(2, \mathbb{R}) \to \mathbb{R}$.

   (e) $\pi : TM \to M$

   (f) $X : M \to TM$ where X is any smooth vector field.

2 Let M, N be manifolds and M be a subset of N. Then M is a *submanifold* of N if $\iota : M \to N$ is an embedding, where ι denotes the inclusion map.

   (a) Show that S^2 is a submanifold of $\mathbb{R}^3$.

   (b) Show that the unit square X, equipped with the differentiable structure defined in Example 37 is *not* a submanifold of $\mathbb{R}^2$.

3 Let $\phi : M \to N$ be a diffeomorphism and X be a vector field on M. The *push forward* of X by ϕ is the vector field ϕ_*X on N defined by

$$(\phi_*X)(p) = d\phi_{\phi^{-1}(p)}X(\phi^{-1}(p)).$$

   (a) This definition doesn't make sense if ϕ is not a diffeomorphism. Why not?

   (b) Denote the derivation associated with a vector field X by δ_X. Show that, for all $f \in C^\infty(N)$,

$$\delta_{\phi_*X}(f) = \delta_X(f \circ \phi) \circ \phi^{-1}.$$

   (Hint: note that for all $g \in C^\infty(M)$, $(\delta_X(g))(p) = dg_pX(p)$. Apply this to $\delta_X(f \circ \phi)$.)

   (c) Deduce that $[f_*X, f_*Y] = f_*[X, Y]$ for all vector fields X, Y on M.

4 Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(x, y) = (x \cos\alpha + y \sin\alpha, -x \sin\alpha + y \cos\alpha)$, where α is a constant. Compute $\phi_* \frac{\partial}{\partial x}$ and $\phi_* \frac{\partial}{\partial y}$.

5 Let M be a complex manifold, $p \in M$, and (U, ϕ) be a complex chart around p with local coordinates $(z^1, \ldots, z^m) \in \mathbb{C}^m$. Define a linear map $J_p : T_pM \to T_pM$ by

$$J_p : \frac{\partial}{\partial x^i} \mapsto \frac{\partial}{\partial y^i}, \qquad J_p : \frac{\partial}{\partial y^i} \mapsto -\frac{\partial}{\partial x^i}$$

where $z^i = x^i + iy^i$. Note that $J_p \circ J_p = -\mathrm{Id}$.

(a) This map is called the *almost complex structure* of M. Show that it is well-defined, that is, independent of the choice of complex coordinate chart.

(b) Let N be a second complex manifold, with almost complex structure J', and $f : M \to N$ be a holomorphic map. Show that $df \circ J = J' \circ df$.

(c) Using the embedding $\iota : S^2 \to \mathbb{R}^3$, we can identify elements of T_pS^2 with vectors in $\mathbb{R}^3$ orthogonal to p. We can then define a map

$$\hat{J}_p : T_pM \to T_pM, \qquad \hat{J}_p : V \mapsto -p \times V \qquad \text{(vector product).}$$

Show that $\hat{J}_p$ coincides with the almost complex structure on S^2 (as defined by stereographic projection from $(0, 0, 1)$).

Problem Set 3

1 Let E be a vector bundle over M and ∇, ∇' be two connexions on E. Show that $\nabla - \nabla'$ is a section of $T^*M \otimes \mathrm{End}(E)$. That is, show that the map

$$\Gamma(TM) \times \Gamma(E) \to \Gamma(E), \qquad (X, \xi) \mapsto \nabla_X \xi - \nabla'_X \xi$$

is $C^\infty(M)$ bilinear.

2 Let $M = \{(x^1, x^2) \in \mathbb{R}^2 \ : \ x^2 > 0\}$, the upper half plane, and ∇ be the connexion on TM whose connexion coefficients with respect to the coordinate basis $\partial/\partial x^1$, $\partial/\partial x^2$ are

$$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = \frac{-1}{x^2} = -\Gamma^2_{11}, \qquad \Gamma^i_{jk} = 0 \quad \text{otherwise.}$$

(a) Solve the parallel transport equation for ∇ along the curve $c : [0,1] \to M$, $c(t) = (at, b)$, with general initial data, and hence construct the linear map $Tr_c : T_{(0,b)}M \to T_{(a,b)}M$.

(b) Solve the parallel transport equation for ∇ along the curve $c : [0,1] \to M$, $c(t) = (a, b + vt)$, with general initial data, and hence construct the linear map $Tr_c : T_{(a,b)}M \to T_{(a,b+v)}M$.

(c) Deduce the holonomy of ∇ about a square with vertices at $(0,1)$, $(a,1)$, $(a,1{+}a)$, $(0,1{+}a)$ traversed once anticlockwise. Hence show that the based holonomy group $H_{(0,1)}$ has a subgroup isomorphic to $SO(2)$ (alternatively, the circle group $\mathbb{R}/\mathbb{Z}$).

3 Let M, ∇ be as in question 2.

(a) Solve the geodesic equation for ∇ with initial data $c(0) = (0,1)$, $\dot{c}(0) = \partial/\partial x^2$.

(b) Solve the geodesic equation for ∇ with initial data $c(0) = (0,1)$, $\dot{c}(0) = \partial/\partial x^1$.

4 Let $M = S^1$ and E be the trivial line bundle $E = M \times \mathbb{R}$. Then sections of E can be interpreted as smooth functions $S^1 \to \mathbb{R}$. Consider the following map

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E), \qquad (X, \xi) \mapsto X[\xi] + \alpha(X)\xi$$

where α is one-form on S^1.

(a) Verify that ∇ is a connexion on E.

(b) Compute the holonomy of ∇ around S^1 traversed once anticlockwise.

Note that ∇ is trivially flat but generically (i.e. for almost all choices of α) has nontrivial holonomy group.

Problem Set 4

1 Let (M, g) be a Riemannian manifold and ∇ be a connexion on TM. Show that for all vector fields X, Y, Z,

  (a) $R(X, Y)Z + R(Y, X)Z = 0$.

  (b) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ if ∇ is torsion free.

  (c) $g(X, R(X, Y)X) = 0$ if ∇ is metric compatible.

  Hence show that the sectional curvature of a plane $P \subset T_pM$

$$\sigma(P) = g(X, R(X, Y)Y),$$

  where R is the curvature of the Levi-Civita connexion, is independent of the orthonormal basis X, Y chosen for P.

2 Given a curve $\gamma : [a, b] \to M$, define its *energy* to be

$$E[\gamma] = \frac{1}{2} \int_a^b g(\dot{\gamma}(t), \dot{\gamma}(t))dt.$$

  Show that γ is a critical point of E (with respect to all smooth variations with fixed endpoints) if and only if it is a geodesic.

3 Let $M = \mathbb{R} \times S^1$, a cylinder, and $f : \mathbb{R} \to (0, \infty)$ be a smooth function. Then the map $\phi : M \to \mathbb{R}^3$, $\phi(x, \theta) = (x, f(x) \cos \theta, f(x) \sin \theta)$, where θ is a periodic coordinate on S^1, embeds M as a surface of revolution in $\mathbb{R}^3$. Let g be the metric induced on M by this embedding.

  (a) Show that

$$g = (1 + f'(x)^2)dx \otimes dx + f(x)^2 d\theta \otimes d\theta.$$

  (b) Compute the scalar curvature of (M, g).

  (c) Show that all the *meridians* (curves with θ constant) are geodesics (when appropriately parametrized).

  (d) Which, if any, of the *parallels* (curves with x constant) are geodesics?

  (e) Find a function f such that the corresponding surface of revolution is geodesically complete (that is, for all initial data, the geodesic equation has a global solution for all time) but has no periodic geodesics.

[Hint: use the fact that geodesics are critical points of E to give a "Lagrangian mechanical" formulation of the geodesic problem.]

4 Let $A \in SO(2k+1)$, meaning $A^T A = \mathbb{I}$ and $\det A = 1$. Show that there exists $v \in \mathbb{R}^{2k+1}$ such that $v \neq 0$ and $Av = v$. [Hint: think about the spectrum of A.]

5 Let (M, g) be a two-dimensional Riemannian manifold and $\gamma : [0, T] \to M$ be a periodic geodesic.

(a) Show that if M is orientable $Tr_\gamma = \mathrm{Id}$.

(b) Show that if γ is contractible $Tr_\gamma = \mathrm{Id}$.