A survey of homogeneous structures

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Abstract.

A relational first order structure is *homogeneous* if it is countable (possibly finite) and every isomorphism between finite substructures extends to an automorphism. This article is a survey of several aspects of homogeneity, with emphasis on countably infinite homogeneous structures. These arise as Fraissé limits of amalgamation classes of finite structures. The subject has connections to model theory, to permutation group theory, to combinatorics (for example through combinatorial enumeration, and through Ramsey theory), and to descriptive set theory. Recently there has been a focus on connections to topological dynamics, and to constraint satisfaction. The article discusses connections between these topics, with an emphasis on examples, and on special properties of an amalgamation class which yield important consequences for the automorphism group.

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1 Introduction

For the purposes of this article, a *homogeneous structure* is a countable first order structure M over a relational language (here usually assumed finite) such that any isomorphism between finite substructures of M extends to an automorphism of M. This definition is couched in first order logic, in model-theoretic language, but the notion is essentially combinatorial. One fixes at the outset a first order relational language L, or signature, by specifying a collection $(\bar{R}_i : i \in I)$ of relation symbols, each with a specified *arity* $a_i \in \mathbb{N}^{>0}$. An L-structure $\mathcal{M} = (M, (R_i)_{i \in I})$ is then just a set M (the *domain*), equipped, for each $i \in I$, with a subset R_i of M^{a_i} (the *interpretation* of $\bar{R}_i$). If we are dealing with graphs (with no multiple edges), or digraphs, or partial orders, the language L will consist just of a single binary relation symbol $\bar{R}$. In the case of (loopless) graphs, the interpretation of $\bar{R}$ will always be symmetric and irreflexive. If we wish to deal with k-hypergraphs, that is with sets equipped with a family of k-subsets (the *hyperedges* or *k-edges*), the relation symbol $\bar{R}$ will have arity k; its interpretations will be irreflexive and symmetric in the natural sense. If working in the context of totally ordered graphs, we would fix a language with two binary relation symbols, one interpreted by the set of edges, the other by the ordering. Mostly, we will be considering infinite homogeneous structures M, where $|M| = \aleph_0$.

The context of homogeneous structures provides a meeting-point of ideas from combinatorics, model theory, permutation group theory, and descriptive set theory, and connections to theoretical computer science and to universal algebra are beginning to emerge. I aim in this article to give an introduction and overview to the subject, emphasising recently emerging themes and the connections between areas. In particular, I focus on combinatorial aspects, and on certain families of examples which often arise. The article contains virtually nothing new. Proofs are in general omitted, or at most sketched.

The subject arose first in work of R. Fraissé in the early 1950s. A basic example of a homogeneous structure is the countable dense linear order without endpoints, $(\mathbb{Q}, <)$; for this structure, homogeneity can be seen either by a 'back-and-forth' argument to build automorphisms, or just by extending finite order-preserving maps to piecewise linear automorphisms. The ordered set of rationals is built as a direct limit of finite (so discrete, and hence rigid) total orders. The high symmetry of $(\mathbb{Q}, <)$ is explained by the fact that any two finite total orders can be *amalgamated* over any common subordering. This led to Fraissé's Amalgamation Theorem ([61], see Theorem 2.1.3 below). This theorem asserts that any homogeneous structure arises (by a limiting process) from an *amalgamation class* of finite structures in the same language. Thus, examples are found by constructing amalgamation classes, and classification problems reduce to the classification of amalgamation classes.

The focus of Fraissé and associates was on relational structures in general, viewed combinatorially, with homogeneous structures as a very special case. However, infinite homogeneous structures are of immediate model-theoretic interest, since (assuming the language is finite) they are ω-categorical; that is, any infinite homogeneous structure is determined up to isomorphism, among countable structures, by its first order theory. This motivated Henson's construction of $2^{\aleph_0}$ homogeneous directed graphs [75], followed by a body of work by Lachlan, Cherlin and others (e.g. Harrington, Schmerl, Shelah, Woodrow) in the late 1970s and 1980s.

The initial focus of this work was on classification, with the Lachlan-Woodrow classification of infinite homogeneous graphs [99], and Cherlin's description of the infinite homogeneous digraphs [38]. In another direction, Gardiner [62] and independently Golfand and Klin [68] classified finite homogeneous graphs, and Lachlan classified finite homogeneous digraphs, and observed that, aside from very small structures, the examples fall into finitely many infinite families, such that in each family the isomorphism type is determined by the values of finitely many independent and free-ranging dimensions. This led to a connection to model-theoretic stability theory, a subject developed in the 1970s and still central to model theory. Lachlan showed that any stable countably infinite homogeneous structure over a finite relational language is the union of a chain of finite homogeneous structures with additional properties. For homogeneous structures, stability is extremely restrictive, eliminating many of the interesting phenomena. However, Lachlan's work led to the study of ω-categorical ω-stable structures in [40], to work of Ahlbrandt and Ziegler [5] on axiomatisation and covering constructions, and to the monograph [42] by Cherlin and Hrushovski on smoothly approximable structures. Many ideas in modern geometric stability and simplicity theory first emerged in this work.

Another major theme has been the connection to infinite permutation groups. Permutation group theory (including the classification of finite simple groups) is used in the above work on stable homogeneous structures, but the focus is on the structures, with group theory used as a tool to study them. In the other direction, homogeneous structures, and in particular Fraissé's Theorem, provide a wonderful source of examples of infinite permutation groups. This emerges particularly in work of Cameron, and [29] provides an excellent general reference. One topic of interest is the abstract structure of the automorphism group (e.g. normal subgroup structure, embeddability properties, subgroups of small index, and other phenomena more recently considered). There are connections to combinatorial enumeration: many interesting integer sequences familiar from enumeration problems arise by counting orbits of the automorphism groups on finite sets or tuples. Also, it is possible to translate between the language of permutation group theory and the language of model theory, which suggests that one may be able to *recover* a homogeneous structure, at least up to 'bi-interpretability', from its automorphism group; this is not always possible, but is achievable in many cases. The automorphism group of any countably infinite first order structure is a Polish group with respect to a natural topology, the 'topology of pointwise convergence'. Thus, techniques from descriptive set theory are available for some of the above questions, and also, automorphism groups of homogeneous structures provide important examples of Polish groups.

More recently, the subject has moved in other directions. There is a notion of 'Ramsey class' of finite structures (see Definition 6.5.1), developed by Nešetřil, Rödl, and others. Any Ramsey class of finite ordered structures is an amalgamation class (Proposition 6.5.2), and there are partial characterisations of those amalgamation classes which, augmented by an ordering, give Ramsey classes. In [92], Kechris, Pestov and Todorcevic found an application of this subject to topological dynamics, that is, to questions about continuous actions of Polish groups on compact spaces. In another direction, Bodirsky has recently investigated versions of *constraint satisfaction* with infinite (usually homogeneous) M. For a fixed structure M, one asks whether a finite structure (given as input) has a homomorphism to M, or equivalently whether a certain positive primitive sentence is true of M. For a number of interesting constraint satisfaction problems, it is appropriate to choose M to be homogeneous.

In this survey, we do not consider just homogeneous structures, but work in the broader framework of ω-categorical structures. However, I have emphasised topics for which the *combinatorics* of homogeneous structures plays a prominent role, and have stressed the connection to infinite permutation groups. I have also tried to emphasise the role of certain examples, such as the various ω-categorical structures associated with trees. Another theme of the article is that specific properties of amalgamation (free amalgamation, monotone free amalgamation) have very strong consequences for the structure of the automorphism group of the Fraissé limit, such as simplicity and extreme amenability. These consequences are summarised in Theorems 6.5.6 and 6.5.7. A number of open questions are included.

Many topics are omitted, including the following: the rich theory of ω-categorical ω-stable structures, developed in [40] and in later papers of Ahlbrandt and Ziegler, Hrushovski and others; smoothly approximable structures (the subject of the monograph [42]); the theory of finite covers of ω-categorical structures, initiated by Ahlbrandt and Ziegler, and developed particularly by Evans and coauthors; results on ω-categorical groups and rings; 'Hrushovski constructions', which are variants of Fraissé amalgamation based on a predimension; homogeneous metric spaces – a subject of considerable activity recently, particularly concerning Urysohn's universal metric space.

Several other articles and books survey parts of this subject, from different points of view. Cameron's monograph [29] focusses mainly on the automorphism group, in the slightly wider setting of ω-categorical structures, and there are other helpful surveys by Cameron. For a range of constructions of ω-categorical structures, see [59]. The paper [93] has many new results, but also collates material around automorphism groups of Polish groups. The introduction of [38] discusses classification, and both it and [39] have many open problems. For material around Jordan groups, [14] is a possible source.

Permutation groups play a major role in this subject. We fix some standard permutation group-theoretic notation and terminology. The automorphism group of a first order structure M is denoted by $\mathrm{Aut}(M)$. We often write (G, X) for a permutation group G on a set X, and we write permutations to the left of their arguments. We say (G, X) is *transitive* if, for all $u, v \in X$, there is $g \in G$ with $g(u) = v$; if $k \geq 1$, then (G, X) is k*-transitive* if, for any distinct $u_1, \ldots, u_k \in X$ and distinct $v_1, \ldots, v_k \in X$, there is $g \in G$ with $g(u_i) = v_i$ for $i = 1, \ldots, k$. The permutation group (G, X) is k*-homogeneous* if G is transitive on the collection of *unordered* k-subsets of X. It is *highly transitive* (respectively, *highly homogeneous*) if it is k-transitive (respectively, k-homogeneous) for all $k \in \mathbb{N}$. We say (G, X) is *primitive* if there is no G-invariant equivalence relation on X other than the trivial one and the universal one; it is *imprimitive* otherwise. Likewise, a first order structure M is *transitive* (respectively, *primitive*) if $\mathrm{Aut}(M)$ acts transitively (respectively, primitively) on M. If G is a permutation group on X and $x \in X$, then G_x denotes the *stabiliser* $\{g \in G : g(x) = x\}$. If $A \subset X$, then $G_{\{A\}} := \{g \in G : g(A) = A\}$ (the *setwise stabiliser* of A) and $G_{(A)} := \{g \in G : g|_A = \mathrm{id}|_A\}$ (the *pointwise stabiliser* of A).

In the context of graphs, we use $\sim$ for adjacency: if x, y are vertices, $x \sim y$ means that they are adjacent. Model-theoretic notation is fairly standard. We write $M \models \sigma$ if the first order sentence σ is true in the structure M. If A, B are first order structures, we write $A \leq B$ if A is a substructure of B. (This is the model theorist's notion of 'substructure', corresponding to the graph theorist's notion of 'induced substructure'.) If R is a relation (of arity n) on a structure M, we say R is *irreflexive* if, whenever $\bar{a} = (a_1, \ldots, a_n) \in M^n$ lies in R, the entries in $\bar{a}$ are all distinct. We say R is *symmetric* if it is invariant under permutations of the arguments. We then talk of an *irreflexive* relational structure, if all its relations are irreflexive, or of an irreflexive class of structures; likewise for 'symmetric'. A *homomorphism* between relational structures A, B in a fixed language is a function h from A to B such that for any $n > 0$, any relation symbol R of arity n, and any $a_1, \ldots, a_n \in A$, if $R(a_1, \ldots, a_n)$ holds in A then $R(h(a_1), \ldots, h(a_n))$ holds in B.

The article does not require much background from model theory. I assume familiarity with the notions of first order language and structure, formula, and interpretation of a formula in a language. Beyond this, I have tried to give definitions if they are needed. A good background source in model theory is [80].

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2 Background to homogeneous structures

2.1 Amalgamation classes and Fraissé limits

In this article, I will adopt the following definition of homogeneity. By a *relational structure*, I mean a structure $(M,(R_i)_{i\in I})$. Such a structure has *domain* or *universe* M. Each R_i has a prescribed *arity*, and a relation R_i of arity a_i is just a subset of M^{a_i}. The corresponding *language* L has relation symbols corresponding to the R_i, of appropriate arity a_i, and I do not distinguish notationally between a relation symbol (in the language) and the corresponding relation in the structure. I tend to use the same symbol for a structure and its domain, but where there is ambiguity (e.g. if several structures have the same domain) I may write $\mathcal{M}$ for $(M,(R_i)_{i\in I})$. Usually the language L is finite – this means that $|I|$ is finite. Much of the theory of homogeneous structures can be developed for languages which also have function symbols and constant symbols, replacing 'finite' by 'finitely generated'. To keep with the combinatorial emphasis of the volume, I mostly avoid this, but it would be fairly harmless at least to allow finitely many constant symbols. Often, we will not be very specific about the language. For example, if we are talking about graphs, or digraphs, or partial orders, it is *assumed* that the language has a single binary relation symbol. The language for 3-hypergraphs would consist of a ternary relation symbol.

Definition 2.1.1 A *homogeneous structure* is a countable, possibly finite, relational structure such that, for every isomorphism $f: U \to V$ between finite substructures U and V of M, there is an automorphism f' of M extending f.

In some sources, the word 'ultrahomogeneous' is used for this notion (possibly with the requirement that $|M| = \aleph_0$), due to overload for the word 'homogeneous'. Also, we have built into the definition of homogeneity the requirement that $|M|$ is countable. This is to save words, since in this paper we only consider countable structures. Some sources do not do this.

Example 2.1.2 The structure $(\mathbb{Q},<)$, where $<$ is the usual order on the rationals, is homogeneous. For if $f: U \to V$ is a finite partial isomorphism and $a \in \mathbb{Q} \setminus U$, then there is $b \in \mathbb{Q} \setminus V$ such that f extends to a partial isomorphism taking a to b; just choose any b in the appropriate interval. Using Cantor's back-and-forth procedure, one iterates this step, alternately putting new elements into the domain and range of f, until, at the limit, an automorphism, i.e. order-preserving permutation of $\mathbb{Q}$, is constructed. This method for constructing automorphisms or isomorphisms is ubiquitous in this subject. In the particular case of $(\mathbb{Q}, <)$, the back-and-forth method is not needed: a piecewise linear automorphism extending f could be constructed directly.

A few homogeneous structures, such as $(\mathbb{Q}, <)$, and disjoint unions of complete graphs all of the same size, require no special construction technique. However the standard method of construction of homogeneous structures, described next, is by Fraissé's Theorem.

Following Fraissé, we shall say that the *age* of a countable relational structure M with language L, denoted $\mathrm{Age}(M)$, is the collection of finite structures which are isomorphic to a substructure of M. It is easily seen that $\mathrm{Age}(M)$ has the following properties:

(i) $\mathrm{Age}(M)$ is closed under isomorphism and substructure,

(ii) $\mathrm{Age}(M)$ has countably many members up to isomorphism.

(iii) $\mathrm{Age}(M)$ has the *joint embedding property* (JEP): if $U, V \in \mathrm{Age}(M)$ then there is $W \in \mathrm{Age}(M)$ such that both U and V embed in W.

Conversely, if $\mathcal{C}$ is a class of finite L-structures satisfying (i), (ii), and (iii), then there is a countable L-structure M such that $\mathcal{C} = \mathrm{Age}(M)$: just build M as a union of a chain of finite structures in $\mathcal{C}$, repeatedly using (iii).

We are interested in the following additional condition:

(iv) A class $\mathcal{C}$ of finite L-structures has the *amalgamation property* (AP) if the following holds: whenever $A, B_1, B_2 \in \mathcal{C}$ and $f_i : A \to B_i$ (for $i = 1, 2$) are embeddings, there is $C \in \mathcal{C}$ and embeddings $g_i : B_i \to C$ (for $i = 1, 2$) such that $g_1 \circ f_1 = g_2 \circ f_2$.

Theorem 2.1.3 (Fraissé's Theorem [61])

(a) Let M be a homogeneous relational structure. Then $\mathrm{Age}(M)$ *has the amalgamation property.*

(b) Let $\mathcal{C}$ be a non-empty class of finite L-structures satisfying (i)–(iv) above. Then there is a homogeneous L-structure M with $\mathrm{Age}(M) = \mathcal{C}$. *If N is another homogeneous L-structure whose age is $\mathcal{C}$, then $M \cong N$.*

Sketch Proof. (a) Consider f_1, f_2, A, B_1, B_2 as in the statement of (AP). We may suppose B_1 and B_2 are substructures of M, and that $f_1 = \mathrm{id}_A$. The map f_2 extends to an automorphism g of M. Put $C := g(B_1) \cup B_2$, $g_2 := \mathrm{id}_{B_2}$, and $g_1 := g|_{B_1}$.

(b) It suffices to build M so that for every $A, B \in \mathcal{C}$ with $A \leq B$, and every embedding $f : A \to M$, there is an embedding $g : B \to M$ extending f. There are countably many such configurations (f, A, B) to consider. We shall build M as a union of a chain of finite substructures $(M_i)_{i \in \mathbb{N}}$, so countably many steps occur in the construction. We use some of these steps to ensure $\mathrm{Age}(M) = \mathcal{C}$, using (JEP). At other steps, we consider some $A \le B$ as above, with some embedding $f : A \to M_n$, and build M_{n+1} as an amalgam of the embeddings $f : A \to M_n$, $\mathrm{id} : A \to B$.

The uniqueness of M follows by a back-and-forth argument. □

The homogeneous structure M with age $\mathcal{C}$ is called the *Fraissé limit* of $\mathcal{C}$.

We shall say that $\mathcal{C}$ has the *disjoint amalgamation property* (DAP), also called the *strong amalgamation property*, if in (iv) above, C and the g_i can be chosen so that $g_1(B_1) \cap g_2(B_2) = f_1(A)$. When actually considering whether a class $\mathcal{C}$ has the amalgamation property, we usually view f_1, f_2 as the identity maps, so B_1, B_2 are L-structures with a common substructure A. The amalgamatiom problem is to build a structure $C \in \mathcal{C}$ with union $B_1 \cup B_2$, so that B_1 and B_2 are substructures of C. One must specify whether any elements of $B_1 \setminus A$ are identified with elements of $B_2 \setminus A$, and which tuples of C which meet both $B_1 \setminus A$ and $B_2 \setminus A$ satisfy relations of L. Viewed this way, the amalgamation is disjoint if and only if C can be chosen so that $B_1 \cap B_2 = A$, that is, if no additional identifications are forced.

Lemma 2.1.4 *Let M be a homogeneous L-structure. Then* $\mathrm{Age}(M)$ *has (DAP), if and only if, for any finite $A \subset M$,* $\mathrm{Aut}(M)_{(A)}$ *has no finite orbits on $M \setminus A$.*

Proof. See [29, (2.15), p. 37]. □

There is a further refinement of amalgamation: we say that an amalgamation class $\mathcal{C}$ with (DAP) has *free amalgamation* if, in the definition above of (AP), the structure C can be chosen so that no tuple of C which satisfies a relation intersects both $g_1(B_1) \setminus g_1 f_1(A)$ and $g_2(B_2) \setminus g_2 f_2(A)$ non-trivially. With the informal view above (where A is identified with substructures of B_1 and B_2) this means that the structure C with union $B_1 \cup B_2$ can be built so that $B_1 \cap B_2 = A$, and, in addition, no tuple which meets both $B_1 \setminus A$ and $B_2 \setminus A$ satisfies any relation of L. We denote such an amalgam as $B_1 \oplus_A B_2$. If $\mathcal{C}$ is a free amalgamation class, we call its Fraissé limit M a *free homogeneous structure*.

We shall say that a class $\mathcal{C}$ of finite L-structures is *monotone* if the following holds, where L has, for each $i \in I$, a relation symbol R_i of arity m_i: for each structure $\mathcal{A} \in \mathcal{C}$ with domain A, if $\mathcal{B}$ is any L-structure with domain $B \subseteq A$ and for each $i \in I$,

$$\{\bar{x} \in B^{m_i} : B \models R_i \bar{x}\} \subseteq \{\bar{x} \in B^{m_i} : A \models R_i \bar{x}\},$$

then $\mathcal{B} \in \mathcal{C}$. More informally, there is a distinction, in graph theory, between the notion of *subgraph* and *induced subgraph.* The standard model theoretic notion of 'substructure' (as used in this paper) corresponds to 'induced subgraph'. The class $\mathcal{C}$ is monotone if it is closed under the 'graph-theoretic' weaker notion of substructure. Clearly, any monotone amalgamation class with disjoint amalgamation is a free amalgamation class. We shall call the Fraissé limit of a monotone amalgamation class with disjoint amalgamation a ‘monotone homogeneous structure’.

We shall say that an amalgamation class $\mathcal{C}$ over a language L is *finitely bounded* (or that its Fraissé limit is finitely bounded) if there is a finite set $\mathcal{F}$ of finite L-structures, such that $\mathcal{C}$ is the set of all finite L-structures which have no substructure (in the model theorist’s sense) which is isomorphic to any member of $\mathcal{F}$.

Remark 2.1.5 1. A homogeneous structure M with a transitive binary relation R (e.g. a partial ordering or equivalence relation) satisfied by a pair of distinct elements cannot be free.

2. The notions ‘free’ and ‘monotone’ involve a distinction between ‘positive’ and ‘negative’ properties. For example, the complement of a monotone homogeneous graph is not in general monotone.

3. It is easy to give examples which are free but not monotone. For example, let L be a language with two binary relation symbols R and G, and let $\mathcal{C}$ be the class of all finite L-structures such that (i) R and G are symmetric and irreflexive, (ii) there is no triple of distinct elements x, y, z such that R holds of all pairs from $\{x, y, z\}$ but G holds from no pairs from this set.

Free and monotone amalgamation classes will be a major theme of this paper – see for example the results at the end of Section 6.5.

2.2 Some classification results

We are now ready to give some examples and describe some classification results. First, to get used to homogeneity, note that any connected homogeneous graph has diameter at most two. For otherwise, a pair of vertices at distance two would be isomorphic to a pair at distance three, but these two pairs would lie in distinct orbits on pairs.

Example 2.2.1 Let $\mathcal{C}$ be the class of all finite graphs. Then $\mathcal{C}$ is a free amalgamation class. Its Fraissé limit R is known as Rado’s graph, or more commonly as *the random graph*, for reasons discussed after Theorem 3.2.3. A back-and-forth argument shows that R is the unique countably infinite graph which satisfies, for all $n \in \mathbb{N}^{>0}$, the following condition (sometimes called the ‘Alice’s Restaurant Axiom’, or just an ‘extension axiom’):

$(*)_n$ let U, V be finite disjoint subsets of (the vertex set of) R with $|U \cup V| = n$. Then there is a vertex $x \in R$ joined to all vertices in U and to no vertices in V.

Example 2.2.2 Let $n \geq 3$, and $\mathcal{C}_n$ be the class of all finite K_n-free graphs, that is, graphs which do not have an n-vertex complete graph as an induced subgraph. Then $\mathcal{C}_n$ is a free (in fact, monotone) amalgamation class. The Fraissé limit R_n is known as the *generic* K_n-free graph (or universal homogeneous K_n-free graph).

It can be checked directly that any countable graph which is a disjoint union of complete graphs, all of the same size, is homogeneous. Its automorphism group is just a wreath product of symmetric groups. In this case, amalgamation will not be free unless the complete graphs are singletons. As a particular case, a countable independent set is homogeneous, and will often be referred to as a *pure set*. Observe too that if Γ is a homogeneous graph, so is its *complement* $\bar{\Gamma}$, which has the same vertex set, but with two vertices joined in $\bar{\Gamma}$ if and only if they are not joined in Γ; for $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\bar{\Gamma})$, and likewise, a partial map $U \to V$ is an isomorphism between substructures of Γ if and only if the same holds for $\bar{\Gamma}$. The complement of R_n (for $n \geq 3$) is not monotone or free.

We can now state a difficult classification theorem of Lachlan and Woodrow. It extends an earlier classification by Woodrow of countably infinite triangle-free homogeneous graphs. In the theorem below, the graphs R and R_n, and an infinite independent set, are all monotone, but the others are not. All are finitely bounded.

Theorem 2.2.3 ([99]) *Let Γ be a countably infinite homogeneous graph. Then Γ or Γ^c is isomorphic to one of: R, R_n (for $n \geq 3$), or a disjoint union of complete graphs, all of the same size.*

It is convenient to mention now a graph which just fails to be homogeneous, namely the *random bipartite graph*. This is a countably infinite bipartite graph such that each part of the bipartition is infinite, and if U, V are finite disjoint sets of vertices from one part, there is a vertex in the other part adjacent to all vertices of U and to none of V. This graph is not homogeneous, since there are two orbits on pairs of non-adjacent vertices, but it becomes homogeneous when one adds a binary relation symbol interpreted as the bipartition. Homogeneous bipartite graphs (in this language) are classified in [67] (a paper dealing mainly with a notion of homogeneity for uncountable graphs). The examples are: the complete bipartite graph, an independent set, the random bipartite graph, a perfect matching, and the complement (in the bipartite sense) of a perfect matching, namely, a complete bipartite graph with the edges of a perfect matching between the parts removed.

Earlier, Gardiner [62], and independently Golfand and Klin [68], classified finite homogeneous graphs, first investigated by Sheehan [136].

Theorem 2.2.4 *[62] Let Γ be a finite homogeneous graph. Then either Γ or Γ^c is isomorphic to a disjoint union of complete graphs all of the same size, or to the pentagon, or to the line graph $L(K_{3,3})$.*

Observe the basic format of this theorem. Every finite homogeneous graph is either sporadic (the pentagon or $L(K_{3,3})$, both of which are isomorphic to their complements) or belongs to one of two families, each parameterised by two natural numbers, which range freely. By very deep work of Lachlan and coauthors, discussed in Section 3.3 below, this phenomenon is completely general for finite homogeneous structures over a finite relational language.

Prior to the classification of homogeneous graphs, there was a more elementary classification of homogeneous partial orders, due to Schmerl [133]. The examples are easily described. First, as noted above, $(\mathbb{Q}, <)$ is homogeneous. Next, the collection $\mathcal{C}$ of all finite partial orders has the disjoint amalgamation property: given two finite partial orders B_1, B_2 with common substructure A, first form the free amalgam $B_1 \oplus_A B_2$; it may not be a partial order, but it is easily checked that its transitive closure is a partial order which induces no extra structure on the B_i. The Fraissé limit of $\mathcal{C}$ is the universal homogeneous partial order, denoted $\mathcal{P}$. There are also some rather trivial examples of homogeneous partial orders, namely (for $1 \leq n \leq \aleph_0$): A_n (an antichain on n vertices); $B_n = A_n \times (\mathbb{Q}, <)$, with $(a,p) < (b,q)$ if and only if $a = b$ and $p < q$; and C_n, which has the same domain as B_n, but with $(a,p) \leq (b,q)$ if and only if $p < q$.

Theorem 2.2.5 ([133]) *Any homogeneous partial order is one of:* $(\mathbb{Q}, <), \mathcal{P}$, A_n, B_n, C_n *(for* $1 \leq n \leq \aleph_0$*).*

To prove this, the key step is to verify that if $\mathcal{C}$ is an amalgamation class of finite partial orders and $\mathcal{C}$ contains the structure $\{u, v, w\}$ (where $u < v, w$ and v, w are incomparable) and $\mathcal{C}$ also contains the structure on u, v, w where $u < v$ and w is incomparable to u and v, then $\mathcal{C}$ consists of all finite partial orders. This is done by bare-hands arguments with transitivity. If $\mathcal{C}$ does not embed one of the above structures, this gives highly restrictive structural information from which the other examples are recovered.

The Lachlan-Woodrow classification of homogeneous graphs is much more intricate, and requires many amalgamation arguments. This programme was taken much further by Cherlin [38], who classified the infinite homogeneous digraphs. (Here, we view a digraph as a structure with a single binary irreflexive relation R such that given distinct vertices x, y, either they are unrelated by R, or exactly one of Rxy, Ryx holds.)

I describe briefly the infinite homogeneous digraphs. I have organised the list in a different way to [38, Ch. 5], allowing some overlaps between classes, and given less detail.

(i) The homogeneous partial orders classified by Schmerl (Theorem 2.2.5 above) may be viewed as digraphs.

(ii) A *tournament* is a digraph $(D, \to)$ such that for any two distinct vertices x, y, either $x \to y$ or $y \to x$. Lachlan, [98], introducing a Ramsey-theoretic method greatly developed in [38], classified the countably infinite homogeneous tournaments. There are three of them: we may view $(\mathbb{Q}, <)$ as a homogeneous tournament, and also the class of all finite tournaments has the amalgamation property, so there is a universal homogeneous tournament, sometimes called the *random* tournament. In addition, there is a homogeneous tournament constructed by distributing countably many points densely on the unit circle, no two antipodal, and putting $x \to y$ if and only if $0 < \arg(x, y) < \pi$: this last example is known as the *local order*, or *circular tournament* – see also [29].

Following some notation adopted later (Example 2.3.1), we denote it by $S(2)$.

(iii) Henson [75] observed that there are $2^{\aleph_0}$ homogeneous digraphs constructed as follows. Let X be any collection of finite tournaments such that no member of X is a subtournament of any other member of X, and let $\mathcal{C}_X$ be the collection of all finite digraphs which do not embed any member of X. Then $\mathcal{C}_X$ is a free amalgamation class, so the Fraïssé limit M_X is a homogeneous digraph. Furthermore, if X and Y are distinct such sets of tournaments, then $\mathcal{C}_X \neq \mathcal{C}_Y$, so $M_X \not\cong M_Y$. To obtain $2^{\aleph_0}$ pairwise non-isomorphic such M_X, it suffices to find an infinite collection $\mathcal{A}$ of pairwise non-embeddable finite tournaments, since then X can range through subsets of $\mathcal{A}$. Henson finds such $\mathcal{A}$ by encoding finite cycles into tournaments. We refer to homogeneous digraphs of the form M_X as 'Henson digraphs'. Since we may choose X to be empty, this class includes the universal homogeneous digraph. These examples are all monotone. If X is infinite then M_X is not finitely bounded.

(iv) Given any $n \geq 2$, let I_n denote the finite digraph with n vertices and no edges, and $\mathcal{C}_n$ the class of all finite digraphs not embedding I_n. Then $\mathcal{C}_n$ is an amalgamation class, and its Fraïssé limit Γ_n is a homogeneous digraph. These examples are not monotone (or free).

(v) There are countably many homogeneous digraphs with imprimitive automorphism groups, not listed here, but easily described. Schmerl's partial orders B_n and C_n (for $n > 1$) also belong here.

(vi) There are two further 'sporadic' primitive digraphs, denoted by Cherlin $S(3)$ and $\mathcal{P}(3)$. The digraph $S(3)$ (Example 2.3.1(2)) belongs to the family listed in Example 2.3.1 below, and $\mathcal{P}(3)$ is closely related to the universal homogeneous partial order.

Theorem 2.2.6 (Cherlin) *The infinite homogeneous digraphs are exactly the members of (i)–(vi) above.*

Cherlin's monograph [38] contains also a revised treatment of the classifications of infinite homogeneous graphs. It has an extremely helpful introduction, with an overview of the classification, and a number of open problems.

In my view, we know rather little about homogeneous structures in general. There are many questions which are answered for homogeneous graphs and digraphs using the classification (and other results) but seem out of range in general, and it would be valuable to take classification to an additional level of complexity. In addition to other questions scattered around this survey, we ask:

Question 2.2.7 *1. Does every structure which is homogeneous over a finite relational language have the small index property (Definition 5.2.1).*

2. Is there a binary homogeneous structure with primitive automorphism group and non-disjoint amalgamation [39, Problem D]?

3. Which are the homogeneous structures whose age has no infinite antichains under embeddability? Is this condition equivalent to the growth rate of $(f_k(\mathrm{Aut}(M))$ *(see Section 6.3) being bounded above by an exponential function?*

4. If M is homogeneous over a finite relational language, must $\mathrm{Aut}(M)$ *have just finitely many normal subgroups which are closed (in the topology described in Section 4.1)?*

There are many other interesting questions in [39].

Cherlin's classification of homogeneous digraphs is at present the strongest such result, but there are other more recent classifications. Torrezao de Sousa [140] extended Schmerl's result by classifying homogeneous partial orderings in a language expanded by unary predicates (coloured partial orderings). Also, there are results of Jenkinson, Rose, and Truss towards a classification of homogeneous *2-graphs* (as distinct from the *two-graphs* of Example 2.3.1!); these are graphs with the vertex set partitioned into two pieces by unary predicates, and with finitely many colours for those edges which meet each piece.

In another direction, Cameron [33] classified the countably infinite homogeneous *permutations*, where a permutation of a set X is viewed as a pair of total orders of the set. There are five of these: the identity permutation (where the two total orders agree everywhere), the universal homogeneous permutation (where the two total orders are independent), one in which $<_2$ is the reverse of $<_1$, and two others, each of which has automorphism group $\mathrm{Aut}(\mathbb{Q}) \mathrm{wr} \, \mathrm{Aut}(\mathbb{Q})$. For these last two there is an $\emptyset$-definable equivalence relation E on the domain, with classes convex with respect the order $<_1$ (which is dense without endpoints), such that the order induced by $<_1$ on the quotient by E is dense without endpoints. For one of these last two structures, $<_2$ agrees with $<_1$ on each class and disagrees for pairs in distinct classes, and the other structure is obtained by reversing $<_2$. Cameron also describes the *circular* permutations (homogeneous structures in a language with two ternary relations, each interpreted by a circular order), and poses several problems, for example the classification problem for homogeneous structures which consist of a set equipped with $n > 2$ total orders.

There are two obvious further directions for classification: one would be to classify homogeneous graphs with edges coloured with two (or more) colours; equivalently, to classify homogeneous structures in a language with finitely many symmetric irreflexive binary relations. Already, with two such relations, there are $2^{\aleph_0}$ non-isomorphic homogeneous structures, by a variant of Henson's argument. Indeed, we may view such structures as graphs with edges coloured red or green. Let $\mathcal{K}$ be the collection of all red/green colourings of a finite complete graph such that the red edges form a cycle; this is an infinite set of graphs which are pairwise non-embeddable. For any $X \subset \mathcal{K}$, let $\mathcal{C}_X$ be the collection of coloured graphs which do not embed any member of $\mathcal{A}$. It is easily checked that $\mathcal{C}_X$ is a free amalgamation class, and different subsets X yield different Fraïssé limits.

Another goal would be to classify homogeneous 3-hypergraphs, that is, homogeneous structures (M, R), where R is a ternary irreflexive symmetric relation. Some initial results here are obtained in [6] – for example, it is shown that there are $2^{\aleph_0}$ non-isomorphic examples. An important example is the universal homogeneous two-graph (Example 2.3.1(4) below). The *finite* homogeneous 3-hypergraphs are classified by Lachlan and Tripp [100]. Such structures have 2-transitive automorphism groups, and the latter are known, by the classification of finite simple groups.

2.3 Other examples of homogeneous structures

We draw attention here to some other interesting families of homogeneous structures, some described in more detail later.

Example 2.3.1 1. The example $(\mathbb{Q}, <)$ is one of a family of four. One may view the ordering up to reversal, and so obtain a (ternary) linear betweenness relation B on $\mathbb{Q}$, where $B(x; y, z)$ holds if and only if $y < x < z$ or $z < x < y$. Alternatively, by bending the rational line into a circle one obtains a natural (ternary) circular ordering K on $\mathbb{Q}$; here,

$$K(x, y, z) \Leftrightarrow (x < y < z) \vee (y < z < x) \vee (z < x < y).$$

The latter too may be viewed up to reversal, to obtain the (quaternary) separation relation S: $S(x, y; z, w)$ if the points x, y in the circular ordering separate the points z, w (so the group induced on 4 points is the dihedral group of order 8). All the structures $(\mathbb{Q}, <)$, $(\mathbb{Q}, B)$, $(\mathbb{Q}, K)$, $(\mathbb{Q}, S)$ are homogeneous. They are characterised by a theorem of Cameron [26] which asserts that any highly homogeneous but not highly transitive infinite permutation group preserves or reverses a linear or circular order; see also Theorem 6.2.1 below.

2. The homogeneous local order described above, in the list of homogeneous digraphs, is one of a family. Let n be a positive integer, and let $(S(n), \sigma_0, \ldots, \sigma_{n-1})$ be the structure whose domain is a countably infinite set of points, distributed densely around the unit circle with no two making an angle of $2k\pi/n$ at the centre (for any $k \in \mathbb{Z}$), and where each σ_j is a binary relation such that $\sigma_j(x, y)$ holds if and only if $2\pi j/n < \arg(x/y) < 2\pi j/(n + 1)$. Then each such structure $S(n)$ is homogeneous. The structure $S(1)$ is just $(\mathbb{Q}, K)$.

3. A *tree*, or *lower semilinear order* is a partial order $(T, \leq)$ such that

(i) $\forall x, y \exists z (z < x \wedge z < y)$, and

(ii) for all $x \in T$, the set $\{y \in T : y \leq x\}$ is totally ordered by $<$.

Droste [48] classified the countably infinite *2-homogeneous* trees, that is, those trees such that any isomorphism between 2-element subsets extends to an automorphism. These are not in general homogeneous (consider two possible kinds of antichains on 4 vertices) but it is easy to add finitely many automorphism-invariant relations to the language so that the structure becomes homogeneous. We refer the reader ahead to Section 6.1 for a description of Droste's classification, and of the related (treelike) C-relations, general betweenness relations, and D-relations.

4. A *two-graph* is a 3-hypergraph such that any 4-set contains an even number of 3-edges. It can be checked that the class of finite two-graphs has

the amalgamation property (but not free amalgamation), so there is a universal homogeneous two-graph H. Let $x \in H$, and define a graph on $K := H \setminus \{x\}$, putting $y \sim z$ if and only if $\{x, y, z\}$ is a 3-edge. Then $(K, \sim)$ is isomorphic to the random graph R, and in fact $\mathrm{Aut}(H)_x$ induces the full automorphism group of the random graph on K. Extending permutation group language, we say that the two-graph is a *transitive extension* of the random graph. For more on this see Section 6.2 below, and also Cameron [30, Section 7]. It would be interesting to classify the homogeneous two-graphs.

There is a curious analogy between H, the circular order $(\mathbb{Q}, K)$, and the universal homogeneous D-relation; we discuss this also in Section 4.2 (Proposition 4.2.8). They are useful examples to bear in mind. As noted in Section 6.2, the universal homogeneous two-graph H is both a transitive extension of the random graph, and a *reduct*; that is, there is a copy of the random graph R with the same domain as H, such that $\mathrm{Aut}(R) < \mathrm{Aut}(H)$. Likewise, the circular order on $\mathbb{Q}$ is both a transitive extension of a copy of $(\mathbb{Q}, <)$ and a reduct of $(\mathbb{Q}, <)$, and the universal homogeneous D-relation is both a transitive extension and a reduct of the universal homogeneous C-relation. Model-theoretically, they behave rather differently: the circular order and the D-relation have NIP but not simple theories, and the two-graph has a simple but not NIP theory (see Definition 6.4.1 below). These analogies are explored, in the context of finite covers, in [85].

2.4 Variants on homogeneity: extending some partial isomorphisms, set-homogeneity, homomorphism-homogeneity.

Extending some partial isomorphisms

In some situations, one considers a privileged class $\mathcal{C}$ of finite substructures of M, and only requires that isomorphisms between members of $\mathcal{C}$ extend to automorphisms of M. There is a version of Fraïssé amalgamation, where one works with a class $\mathcal{C}$ of finite structures, and a class $\mathcal{E}$ of maps between members of $\mathcal{C}$; see [59, Theorem 2.10] for a presentation. This is the basis of the 'Hrushovski constructions' mentioned in the introduction, but there are simpler versions as well.

Let M be a countable graph. We say that M is *connected-homogeneous* if every isomorphism between finite connected subgraphs extends to an automorphism of M. This is a strengthening of distance-transitivity: a graph M is *distance-transitive* if, for every positive integer k, $\mathrm{Aut}(M)$ is transitive on the set of ordered pairs of vertices at distance k. Extending results of Enomoto and Gardiner for finite and locally finite graphs, the countably infinite connected-homogeneous graphs are classified in [70]. In addition to the homogeneous graphs, and some imprimitive examples (the random bipartite graph, the line graph of the complete bipartite graph $K_{\omega,\omega}$, and the complement of a perfect matching), the list includes a family of graphs of infinite diameter, in which complete graphs of a fixed cardinality are glued together in a treelike way. In the proof, it is observed that if M is connected-homogeneous, then all vertex-neighbourhoods are homogeneous graphs. The possible vertex-neighbourhoods are then considered case-by-case. This result has possible generalisations, as there is a notion of connectedness for any relational structure.

Set-homogeneity

Following terminology of Fraissé, a countable relational structure M is said to be *set-homogeneous* if, whenever two finite substructures U, V of M are isomorphic, there is $g \in \mathrm{Aut}(M)$ with $g(U) = V$ (so we do not require that *all* isomorphisms extend). Clearly, if all finite substructures of M are rigid, then set-homogeneity of M implies homogeneity. Also, in a very short combinatorial argument, Enomoto [55] showed that any *finite* set-homogeneous graph is homogeneous.

In recent work of Gray, Macpherson, Praeger, and Royle [71], *finite* set-homogeneous digraphs are classified (even allowing some undirected edges, as well as directed edges). The list of examples is quite long, including a 27-vertex 'sporadic' digraph. The work also includes a classification of set-homogeneous not 2-homogeneous *infinite* digraphs (not allowing undirected edges). Here, a structure is *k-homogeneous* if every isomorphism between substructures of size k extends to an automorphism.

For graphs, there is a nice example in [51] of an infinite set-homogeneous graph which is not 3-homogeneous. It is related to $S(3)$, and is obtained by distributing countably many points densely on the unit circle, no two making an angle $\pi/3$ at the centre, with two vertices adjacent if they make an angle at the centre less than $\pi/3$. This graph is characterised in [51] as the unique graph (up to complementation) which is set-homogeneous but not 3-homogeneous.

A full classification of set-homogeneous graphs or digraphs seems rather difficult. There is a (twisted) version of Fraissé amalgamation in [51], but it seems difficult to use.

Homomorphism-homogeneity

There are the beginnings of a theory of homogeneous structures but with 'homomorphism' replacing 'isomorphism'. The subject was initiated in [34], and has been developed further in the PhD thesis of Lockett (see also [35]), and in a series of papers by Mašulović and coauthors. Given the connections of homogeneity to constraint satisfaction (see Section 6.6), this subject seems promising.

The definition of *homomorphism* is as in the introduction, and a *monomorphism* is just an injective homomorphism. Let I, M, H stand for 'isomorphism', 'monomorphism' and 'homomorphism' respectively, and for $X, Y \in \{\mathrm{I}, \mathrm{M}, \mathrm{H}\}$ say that a structure P is *XY-homogeneous* if every map of type X between finite substructures of P extends to a map $P \to P$ of type Y. Thus, II-homogeneity is just homogeneity. Since one should not expect to be able to extend maps to ones satisfying stronger assumptions, the interesting conditions are IH, MH, HH, IM, MM, and II.

A version of Fraissé's Theorem for MM-homogeneity is given in [34]. In Lockett's PhD thesis, full classification results for homomorphism-homogeneity for partial orders are obtained. The notion of homomorphism is sensitive to whether or not the partial orderings are strict. In [35], it is shown that for strict partial orders, the classes defined by IH, MH, HH, IM and MM all coincide, but properly contain II. For not strict partial orders, IH, MH and HH coincide, properly contain IM and MM (which coincide), and these in turn properly contain II. Similar results are obtained by Mašulović in [118].

3 Model theory of homogeneous structures

We develop here the basic model-theoretic framework of ω-categoricity, discuss the finite model property, and give an overview of Lachlan's theory of stable homogeneous structures. This is extended further in Section 6.4.

3.1 Omega-categoricity

From a model-theoretic viewpoint, homogeneity fits in a broader framework, since infinite homogeneous structures over a finite relational language are ω-categorical. There are other notions too, such as saturation and recursive saturation, which give similar symmetry information.

Definition 3.1.1 (i) A *first order theory* over a language L is a consistent set of first order sentences, that is, formulas without free variables. It is a *complete* theory if it is maximal subject to being consistent. Equivalently, a complete theory is the set of all sentences true of some L-structure M (and is then called the theory of M, denoted $\mathrm{Th}(M)$).

(ii) A complete theory T in a countable language is *ω-categorical* if it has an infinite model and any two models of size $\aleph_0$ are isomorphic. An *ω-categorical structure* is a structure M of size $\aleph_0$ whose theory is ω-categorical.

If G is a permutation group on a set X, then G has an induced action, coordinatewise, on X^n for each n. Following Cameron [29], we say that (G, X) is *oligomorphic* if $|X| = \aleph_0$ and G has finitely many orbits on X^n for all positive integers n. The central result on ω-categoricity is the following, usually called the Ryll-Nardzewski Theorem, though parts are due to Svenonius and Engeler. It gives an equivalence between model-theoretic and group-theoretic conditions. We omit some of the model-theoretic equivalent conditions. At the heart of the theorem, though hidden below, is Vaught's Omitting Types Theorem.

Theorem 3.1.2 ([130, 54, 141]) *Let M be a countably infinite first order structure in a countable language. Then the following are equivalent.*

(i) M is ω-categorical;

(ii) Aut(M) *acts oligomorphically on M;*

(iii) for each $n > 0$, there are finitely many formulas $\phi(x_1, \ldots, x_n)$ up to Th(M)*-provable equivalence.*

Corollary 3.1.3 *Let M be a countably infinite structure which is homogeneous over a finite relational language. Then M is ω-categorical.*

Proof. Since the language is finite, there are finitely many isomorphism types of n-element substructures of M. Any two isomorphic (labelled) structures of size n lie in the same $\mathrm{Aut}(M)$-orbit on M^n. $\square$

Remark 3.1.4 The assumption in 3.1.3 that the language is finite is unnecessarily strong. Provided that all the relations are irreflexive, it suffices that the language has finitely many relation symbols of each arity.

For an ω-categorical structure M, by Theorem 3.1.2 and its proof, one can translate between model-theoretic and group-theoretic concepts. Recall that if M is a first order structure then $X \subset M^n$ is *definable* if X is the set of solutions in M^n of some formula $\phi(x_1, \ldots, x_n)$, possibly with parameters. If the parameters come from $A \subset M$ (that is, $X = \{\bar{x} \in M^n : \phi(\bar{x}, a_1, \ldots, a_m)\}$ for some m and $a_1, \ldots, a_m \in A$) then we say that X is A-definable. If M is ω-categorical and $A \subset M$ is finite, then $X \subset M^n$ is A-definable if and only if X is a union of $\mathrm{Aut}(M)_{(A)}$-orbits on M^n. Also, recall that an *n-type* of M is a maximal set of formulas in $x_1, \ldots, x_n$ consistent with $\mathrm{Th}(M)$, and, for ω-categorical M, can be identified with the intersection of the solution sets of these formulas. For finite $A \subset M$, there is a corresponding notion of *n-type over A*, where the formulas are allowed to have parameters from A. The corresponding subset of M^n is just an $\mathrm{Aut}(M)_{(A)}$-orbit on M^n. Back-and-forth arguments underpin all these observations.

There are many ω-categorical structures which are not homogeneous, indeed are not naturally viewed as relational structures.

Example 3.1.5 An $\aleph_0$-dimensional vector space V over the finite field $\mathbb{F}_q$ is naturally parsed in the language $(+, -, 0, (f_a)_{a \in \mathbb{F}_q})$, where each f_a is a unary function symbol interpreted as scalar multiplication by a. The theory of V expresses that V is a vector space over $\mathbb{F}_q$. Also, if V is countably infinite then $\dim(V) = \aleph_0$. Since the dimension and the field determine the isomorphism type, ω-categoricity of V follows. Alternatively, observe that its automorphism group $\mathrm{GL}(\aleph_0, q)$ acts oligomorphically on V.

The first order theory T has *quantifier elimination* if, for every formula $\phi(x_1, \ldots, x_n)$ there is a *quantifier-free* formula $\psi(x_1, \ldots, x_n)$ such that $T \vdash \forall x_1 \ldots \forall x_n (\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$. If the language is not too complex, this condition makes it feasible to understand the definable sets in models of T.

Proposition 3.1.6 *Let M be an ω-categorical structure over a relational language L. Then M is homogeneous if and only if* $\mathrm{Th}(M)$ *has quantifier elimination.*

Sketch proof. $\Rightarrow$ Suppose that M is ω-categorical and homogeneous. By Theorem 3.1.2, for any n there are finitely many formulas in n variables (modulo $\mathrm{Th}(M)$). If two tuples satisfy the same quantifier-free formulas then they lie in the same orbit so satisfy all the same formulas. It follows easily that every formula is equivalent to a quantifier-free formula.

$\Leftarrow$ Suppose that M is ω-categorical and $\mathrm{Th}(M)$ has quantifier-elimination, and $f : U \to V$ is an isomorphism of finite substructures of M. If $\bar{u}$ enumerates U and $\bar{v} = f(\bar{u})$, then $\bar{u}$ and $\bar{v}$ satisfy the same quantifier-free formulas, and hence by quantifier-elimination satisfy the same formulas. It follows from the Ryll-Nardzewski Theorem (though this is not quite immediate from the conditions in Theorem 3.1.2) that there is $g \in \mathrm{Aut}(M)$ with $g(\bar{u}) = \bar{v}$. □

If M is any ω-categorical structure, then there is an ω-categorical homogeneous relational structure with the same domain and automorphism group as M: for each n, introduce a relation symbol for each $\mathrm{Aut}(M)$-orbit on M^n, interpreted by the orbit (and omit any function of constant symbols in the original language). Model-theoretically, this is known as *Morleyisation*, and there is no restriction to ω-categoricity. The process can also be done starting with an arbitrary oligomorphic permutation group. Given an oligomorphic permutation group (G, X) or ω-categorical structure M, the relational language thus obtained is referred to as the *canonical language*. For example, it is clear that $\mathrm{PGL}(\aleph_0, q)$ acts oligomorphically on the projective space $\mathrm{PG}(\aleph_0, q)$, and the latter can be viewed as a homogeneous relational structure in the canonical language.

Within the class of all ω-categorical structures, there is a subclass consisting of those ω-categorical structures M such that there is a homogeneous structure M' over a *finite* relational language such that M and M' have the same domain and $\mathrm{Aut}(M) = \mathrm{Aut}(M')$. Following Covington [45], who systematically investigated the notion, we call such structures *homogenizable*. It is easily seen that V in Example 3.1.5 is not homogenizable. Indeed, if $n \in \mathbb{N}$ and V_n is a structure which has the same domain as V and, for each orbit of $\mathrm{Aut}(V)$ on V^k for each $k \leq n$, has a relation symbol interpreted by the orbit, then V_n is not homogeneous; for if $v_1, \ldots, v_{n+1}$ are linearly independent vectors, then the map $(v_1, \ldots, v_n, v_{n+1}) \mapsto (v_1, \ldots, v_n, v_1 + \ldots + v_n)$ is an isomorphism in the language of V_n, but there is no automorphism of V_n taking the first tuple to the second. This argument is generalised in [112], where the following is proved using the affine Ramsey theorem. A structure M is *interpretable* in another structure N if an isomorphic copy of M lives on a quotient (by an $\emptyset$-definable equivalence relation) of an $\emptyset$-definable subset of a power of N, with the relations, functions and constants of the copy of M also $\emptyset$-definable in N.

Theorem 3.1.7 *Let M be a homogenizable structure. Then no infinite group* *is interpretable in M, even after naming constants from M.*

3.2 The finite model property

Homogeneous structures arise from amalgamation classes, but in general the model theory of a homogeneous structure is very different from that of its finite substructures. For example $(\mathbb{Q}, <)$ is dense, but finite total orders are discrete, and have endpoints.

Definition 3.2.1 A first order theory T has the *finite model property* (FMP) if every sentence in T is true of some finite structure. A structure M has the finite model property if $\mathrm{Th}(M)$ has the finite model property.

There is a strengthening of this – M has the *finite submodel property* (FSP) if every sentence of $\mathrm{Th}(M)$ is true of a finite substructure of M.

Question 3.2.2 *Is there an ω-categorical structure which has (FMP) but not (FSP)?*

We recall the following result (Theorem 3.2.3), a 'zero-one law', usually attributed to Fagin [60] but proved also in [66]. Since all finite graphs embed in the random graph, it shows the random graph has FSP in a very strong form. A very similar argument justifies the term 'random graph'. Indeed there is a probability measure on the collection of all graphs with vertex set $\mathbb{N}$ obtained, informally, by tossing a coin for each pair of vertices, distinct pairs having independent tosses, and making them adjacent if the coin shows 'heads'. For each n, the axiom $(*)_n$ of Example 2.2.1 then holds with probability 1, and hence, with probability 1, the graph obtained is isomorphic to the random graph, since a countable union of measure zero sets has measure zero.

Theorem 3.2.3 *Let R be the random graph, let $T := \mathrm{Th}(R)$, and let $\sigma \in T$. Then almost all finite graphs satisfy σ, in the sense that if a_n is the number of labelled graphs on $\{1, \ldots, n\}$ satisfying σ, and b_n is the number of labelled graphs on $\{1, \ldots, n\}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.*

Sketch proof. Recall that T is axiomatised by sentences σ_n (for $n \in \mathbb{N}$), where σ_n is:

$$\forall x_1 \ldots x_n \forall y_1 \ldots y_n [\bigwedge_{1 \leq i,j \leq n} x_i \neq y_j \rightarrow \exists z (\bigwedge_{i=1}^{n} (z \sim x_i \wedge z \not\sim y_i))].$$

Thus, it suffices to show that σ_n holds for almost all finite graphs. Observe that if S is a finite set of size m, and U, V are disjoint subsets of S with $|U \cup V| = n$, then the proportion of graphs on S such that no vertex outside $U \cup V$ is correctly joined to $U \cup V$ (i.e. joined to all vertices of U and to none of V) is $(1 - \frac{1}{2^n})^{m-n}$.

This has limit zero as $m \to \infty$. Of course, in view of the universal quantifiers, some additional argument is needed. $\square$

It is not immediately obvious how to construct *specific* finite graphs satisfying σ_n, but there is one construction which gives much stronger information. If q is a prime power with $q \equiv 1 \pmod 4$, then the *Paley graph* P_q has vertex set the finite field $\mathbb{F}_q$, with $x \sim y$ if and only if $x - y$ is a square; this is symmetric, as -1 is a square in $\mathbb{F}_q$. It is shown in [24] that if U, V are disjoint subsets of P_q with $|U \cup V| = n$, and $t = |\{z \in \mathbb{F}_q : \forall u \in U(z \sim u) \wedge \forall v \in V(z \not\sim v)\}|$, then $|t - \frac{q}{2^n}| \leq \frac{1}{2}(n-2+2^{-n+1})q^{\frac{1}{2}} + \frac{n}{2}$. The proof uses character sums, and is linked to the Lang-Weil estimates for finite fields. This result says in particular that σ_n holds in any sufficiently large Paley graph, and hence so does any sentence of T. So the set of sentences which are true in all but finitely many Paley graphs P_q (where $q \equiv 1 \pmod 4$) is a complete theory, namely that of the random graph. The Paley graphs form a *one-dimensional asymptotic class* in the sense of [115]; this means that the cardinalities of definable sets have asymptotically a rather uniform behaviour across the class of all Paley graphs (where the sets are defined by a fixed formula, but with parameters varying).

All the above considerations apply also to the universal homogeneous tournament, but with Paley tournaments P_q, where $q \equiv 3 \pmod 4$, replacing Paley graphs. The proof is given in [69].

Paley graphs have particular interest because they can be viewed as *pseudo-random graphs*. That is, they form a specific class of graphs which have many properties which should hold of a random graph. This viewpoint is developed in [23].

There are certain other homogeneous structures which clearly have the finite model property for probabilistic reasons: for example, the universal homogeneous n-coloured graph (where every edge is coloured with exactly one of n colours), the universal homogeneous digraph, the random bipartite graph, and the universal homogeneous k-hypergraph. It is not so clear in which cases there are corresponding analogues to the Paley graphs and tournaments. Beyarslan [13], with a beautiful construction, has shown that for any $k \geq 2$ the universal homogeneous k-hypergraph is definable in a *pseudofinite field* (that is, an infinite field satisfying every first order sentence which holds for all finite fields). By general model theory, this will give a class $\mathcal{C}_k$ of finite k-hypergraphs, definable in finite fields, which approximates the universal homogeneous k-hypergraph, in the sense that every sentence true in the latter holds in all but finitely many members of $\mathcal{C}_k$. I do not know if $\mathcal{C}_k$ has the same kinds of asymptotic regularity and 'pseudo-randomness' that Paley graphs exhibit.

We shall see in the next subsection that *stable* homogeneous structures (in a finite relational language) also have the FSP. In all those homogeneous structures which I know to have the finite model property, FSP arises either from probabilistic arguments as above or from stability, or conceivably from a mixture of these, e.g. for the disjoint union of the random graph and an infinite complete graph. It would be valuable to have a new method for proving the finite model property for homogeneous structures. A hard open problem, raised by Cherlin, is whether the universal homogeneous triangle-free graph has the FSP. The probabilistic arguments for the random graph do not work, since with probability tending to 1 with $|\Gamma|$, any finite triangle-free graph Γ is bipartite. It is also not known whether there are $2^{\aleph_0}$ homogenizable structures with the FSP.

Finally, we mention one further refinement of the finite model property. The following result is due to Thomas ([143], see also Section 6.2 below). In [143] Thomas describes the condition on H_k as the *strong finite submodel property*, and gives a short proof, due to Kahn, using the Borel-Cantelli Lemma. The result holds also for the random tournament [11].

Theorem 3.2.4 *For any $k \geq 2$, the generic k-hypergraph H_k can be written as the union of a chain $(\Delta_n)_{n\in\mathbb{N}}$ of finite k-hypergraphs such that*

(i) $|\Delta_n| = n$ for all n; and

(ii) for each sentence σ such that $H_k \models \sigma$, there is an integer N_σ such that $\Delta_n \models \sigma$ for all $n > N_\sigma$.

3.3 Stable homogeneous structures

In a serious of major papers in the 1980s, Lachlan and coauthors developed the model theory of *stable* homogeneous L-structures, where L is a finite relational language. Curiously, this runs in parallel with a structure theory of *finite* homogeneous L-structures.

Definition 3.3.1 A complete theory T is *unstable* if there is a formula $\phi(\bar{x}, \bar{y})$ (where $l(\bar{x}) = r$ and $l(\bar{y}) = s$), some model M of T, and $\bar{a}_i \in M^r$ and $\bar{b}_i \in M^s$ (for $i \in \mathbb{N}$) such that for all $i, j \in \mathbb{N}$,

$$M \models \phi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \leq j.$$

The theory T is *stable* otherwise. A structure M is *stable* if its theory is stable.

The theory T (over a countable language) is ω*-stable* if, for any countable model M, there are just countably many complete types over M.

See [117] for more on ω-stability. This is a stronger condition than stability.

Stability is equivalent to many other conditions on T. For example, stability of T is equivalent to no infinite subset of a power of any model of T being totally ordered by any formula; to there being an infinite cardinal λ such that for every model M of T of size λ, there are just λ types over M (this is λ*-stability*); to the condition that all types over models of T are definable; and to the finiteness of certain local ranks on formulas. Stability theory is a major branch of model theory. Notions related to stability provide dividing lines between those theories for which models of all cardinals can be classified by cardinal invariants, and those for which there are too many non-isomorphic models for this to hold (2^λ models of each sufficiently large cardinality λ.) Stability theory has yielded abstract notions of independence with a geometric flavour, with many ramifications and applications.

The random graph is unstable: we may simply take $\phi(x, y)$ to be the formula $x \sim y$. (Note here that the random graph embeds every countable graph, so in particular has vertices a_i, b_i – for $i \in \mathbb{N}$ – with $a_i \sim b_j \leftrightarrow i \leq j$.) It is also easily shown that the universal homogeneous K_n-free graph, and the 'Henson digraphs' are unstable. On the other hand, a disjoint union of complete graphs, all of the same size, is stable.

It is easily seen that any stable finitely homogeneous structure M is ω-stable. For this, it suffices to show that there are at most $\aleph_0$ 1-types over M. By standard stability theory arguments, for any formula $\phi(x, \bar{y})$, there are at most $\aleph_0$ ϕ-types over M, that is, partial types using only the formula ϕ; here the $\bar{y}$ are viewed a 'parameter-variables' taking values in M. Since $\mathrm{Th}(M)$ has quantifier elimination by Proposition 3.1.6, this is enough. Indeed, any type is determined by its subtypes given by the finitely many the atomic formulas (with all but one of the variables as parameter-variables).

Thus, stable finitely homogeneous structures belong to the broader class of ω-categorical ω-stable structures investigated in [40]. Note that for an ω-categorical structure, stability (or ω-stability) is a property determined by the automorphism group; it is not changed by moving to the canonical language.

Example 3.3.2 For an illuminating example of a stable ω-categorical structure, let Γ be the graph whose vertices are the 2-element subsets of $\mathbb{N}$, with two 2-sets adjacent if they intersect in a singleton. Then $\mathrm{Aut}(\Gamma) = \mathrm{Sym}(\mathbb{N})$ (in its action on 2-sets); the containment $\geq$ is clear, but a little work is needed for the other direction. The graph Γ is not homogeneous, since there are two orbits on triangles, the first containing $\{\{0,1\},\{0,2\},\{0,3\}\}$ and the second containing $\{\{0,1\},\{0,2\},\{1,2\}\}$. If a ternary relation is added to the language interpreted by triples of the first kind, namely three distinct 2-sets all sharing a single element, then the graph becomes homogeneous. When viewed as a homogeneous structure in this richer language, its age does not have (DAP). This does not yield a positive answer to Question 2.2.7(2), since the language is ternary. It is stable (in fact, ω-stable), for example because it is first-order interpretable in an ω-stable structure, namely a set in the language just with equality. In fact, these two structures, Γ and the pure set, are *bi-interpretable*; this is related to the fact that they have the same automorphism group (as a topological group). For more on this example, see Section 5.1. See also Thomas's result after Theorem 6.2.2.

The above graph is a well-known example of a distance-transitive graph. It is one of the *Johnson graphs*. The number 2 above can be replaced, without affecting the model theory much, by any integer $k > 2$: a Johnson graph then has as vertices the k-subsets of $\mathbb{N}$, with two such adjacent if they intersect in a $(k-1)$-set. There is also an analogous construction of the *Hamming* distance transitive graphs: if $k > 1$ is an integer, form a graph whose vertex set is

$\mathbb{N}^k$, with two k-tuples adjacent if they agree in $k-1$ coordinates. All of these examples are not homogeneous as graphs, but are homogenizable, and have stable theory.

Both the constructions in Example 3.3.2, and the disjoint union of complete graphs, arise as the union of a chain of finite homogeneous structures. For example, Γ is the disjoint union of the corresponding graphs Γ_n, which has vertex set the collection of 2-element subsets of $\{1, \ldots, n\}$. Lachlan proved that if M is any stable countably infinite homogeneous structure over a finite relational language L, then M is a union of a chain $(M_n : n \in \mathbb{N})$ of finite homogeneous L-structures, and each sentence $\sigma \in \mathrm{Th}(M)$ holds in all but finitely many of the M_n. The M_n are embedded into M very nicely, in that any two tuples of M_n lie in the same $\mathrm{Aut}(M)$-orbit if and only if they lie in the same $\mathrm{Aut}(M_n)$-orbit (this is the idea of *smooth approximation*). Furthermore, the finite homogeneous L-structures fall, apart from finitely many sporadics, into finitely many infinite families, with the isomorphism type in each family determined by finitely many $\mathbb{N}$-valued dimensions, which range freely above a certain minimum. If some of these dimensions are given value $\aleph_0$, one obtains a stable homogeneous structure. The main model-theoretic arguments are contained in [95]. A key ingredient is the finiteness of a certain rank function proved in [41] via the classification of finite simple groups, entering via the O'Nan-Scott Theorem in finite permutation group theory. In the case of a binary language there is a somewhat more direct argument, not using much group theory, given in [96]. This work on stable finitely homogeneous structures was extended to the much richer class of *smoothly approximable* structures (or *Lie coordinatizable* structures) in [42].

In some sense, the example above (the graph whose vertices are 2-subsets of $\mathbb{N}$) is typical of *primitive* stable homogeneous structures over a finite relational language L. For part of the structure theory in [41] and [95] yields the following description of sufficiently large finite homogeneous L-structures.

Theorem 3.3.3 *Let L be a finite relational language, and M a finite transitive homogeneous L-structure. Then if $\equiv$ is a maximal proper $\emptyset$-definable equivalence relation on M (so $\mathrm{Aut}(M)$ is primitive on $M/\equiv$), and $|M/\equiv|$ is large enough relative to L, there is a set X, an equivalence relation E on X with a finite number m (depending just on L) of E-classes, a number k (depending just on L) and a transitive permutation group H on X such that every permutation of X which fixes each E-class is induced by a member of H, so that the following holds: if N is the collection of subsets Y of X which meet each E-class in a k-set, then the permutation group $(\mathrm{Aut}(M), M)$ is isomorphic to the permutation group H acting on N.*

The Lachlan theory of homogeneous structures is surveyed in the article [97]. In the same volume there is another survey [114] with a slightly different treatment of part of [41], replacing part of the group theory by use of Theorem 3.1.7.

4 Automorphism groups: induced actions on subsets, free subgroups, normal subgroups, generic automorphisms.

We turn now to group-theoretic aspects of the subject. From the point of view of automorphism groups, it is more natural to work with ω-categorical structures, rather than just homogeneous structures over a finite relational language.

4.1 The Polish group topology

Recall that a *Polish space* is a topological space where the topology arises from some complete separable metric space. A *Polish group* is a topological group, often assumed to be Hausdorff, such that the underlying topology is that of a Polish space.

If $M = \{m_i : i \in \mathbb{N}\}$ is a countably infinite set, then $\mathrm{Sym}(M)$ is a Polish group. There are many possible metrics, and the exact form plays no role for us. We could define $d(f, g)$ (for $f, g \in \mathrm{Sym}(M)$) to be $\frac{1}{2^n}$ where n is the least r such that either $f(m_r) \neq g(m_r)$ or $f^{-1}(m_r) \neq g^{-1}(m_r)$. Of course, this metric depends on the enumeration of M, but the underlying topology does not. The topology is Hausdorff. Pointwise stabilisers of finite sets (subgroups $\mathrm{Sym}(M)_{(A)}$, where $A \subset M$ is finite) form a system of neighbourhoods of the identity. So a typical basic open set has the form $g\,\mathrm{Sym}(M)_{(A)} := \{h \in \mathrm{Sym}(M) : h|_A = g|_A\}$. The topology is often referred to as the 'topology of pointwise convergence'. If we endow M with the discrete topology, and view $\mathrm{Sym}(M)$ as a subset of the function space M^M, then the topology is that induced from the product topology.

Lemma 4.1.1 *Let $G \leq \mathrm{Sym}(M)$, where M is countably infinite. Then G is closed in $\mathrm{Sym}(M)$ if and only if $G = \mathrm{Aut}(\mathcal{M})$ for some first order structure $\mathcal{M}$ with domain M.*

Proof. Given a first order structure $\mathcal{M}$ with domain $M = \{m_i : i \in \mathbb{N}\}$, the group $\mathrm{Aut}(\mathcal{M})$ is a closed subgroup of $\mathrm{Sym}(M)$. Indeed, if $(f_i)_{i\in\mathbb{N}}$ is a Cauchy sequence of elements of $\mathrm{Aut}(\mathcal{M})$ then the f_i, as well as their inverses, agree on longer and longer initial subsequences of $(m_i)_{i\in\mathbb{N}}$ and converge to a permutation, which will also be an automorphism. Conversely, if G is a closed subgroup of $\mathrm{Sym}(M)$, then let $\mathcal{M}$ be the structure on M which has the *canonical language* for the action of G, namely a relation for each orbit on tuples. We find $G = \mathrm{Aut}(\mathcal{M})$. $\square$

It follows that if M is a countably infinite first order structure, then $G := \mathrm{Aut}(M)$ is itself a Polish group. The left cosets of point stabilisers $\mathrm{Aut}(M)_{(A)}$ form a basis of open sets. A subgroup H of G is dense in G if and only if H meets each coset $gG_{(A)}$ for all finite $A \subset M$, and this holds if and only if H

has the same orbits as G on M^n for all $n \in \mathbb{N}$. A permutation group G on a countably infinite set M is the automorphism group of some ω-categorical structure on M if and only if it is an oligomorphic closed subgroup of $\mathrm{Sym}(M)$.

Observe that the group $\mathrm{Aut}(M)$ is compact if and only if all its orbits are finite (in which case it is profinite). The group is locally compact if and only if the pointwise stabiliser of some finite set has only finite orbits on M. Thus, the automorphism group of a countably infinite locally finite connected graph is locally compact, but the automorphism group of an ω-categorical structure is not, since it is oligomorphic.

4.2 Abstract group structure of automorphism groups.

We aim here to give a picture of the kind of groups which arise as automorphism groups of ω-categorical structures. Throughout this section, M denotes an ω-categorical structure with automorphism group G. Clearly, $|G| = 2^{\aleph_0}$. This can be seen directly by building a tree of height ω with nodes labelled by partial maps, so that branches correspond to automorphisms. Alternatively, observe that G, as a topological space, has no isolated points.

Induced group actions on subsets

In general, if M is ω-categorical and A is an infinite coinfinite subset of M, we have very little control over the permutation group induced by $\mathrm{Aut}(M)_{\{A\}}$ on A. It may not be closed as a subgroup of $\mathrm{Sym}(A)$, even if A is definable. For example, if R_3 is the universal homogeneous triangle-free graph, v is a vertex of R_3, and A is the set of neighbours of v, then $H := \mathrm{Aut}(R_3)_{\{A\}}$ does not have a closed action on A. For by homogeneity, as A has no edges, the action induced by H on A is highly transitive, but not every permutation of A is induced: indeed, some (countably many) infinite coinfinite subsets of A have the form $\{x \in A : x \sim a\}$ for some vertex a of R_3, and others do not, and two such sets must lie in different orbits of H on the power set of A, so H does not induce $\mathrm{Sym}(A)$. However, we do have the following.

Proposition 4.2.1 *Let M be an ω-categorical structure.*

(i) There is a subset $X = \{x_i : i \in \mathbb{Q}\}$ such that, if $H := \mathrm{Aut}(\mathbb{Q}, <)$, there is an embedding of groups $\phi : H \to G$, where, for $h \in H$, we have $(\phi(h))(x_i) = x_{h(i)}$ for all $i \in \mathbb{Q}$.

(ii) If M has stable theory, then there is a subset $X = \{x_i : i \in \mathbb{N}\}$ such that, if $H = \mathrm{Sym}(\mathbb{N})$, there is an embedding $\phi : H \to G$, such that, for $h \in H$, $(\phi(h))(x_i) = x_{h(i)}$ for all $i \in \mathbb{N}$.

(iii) If R is the random graph, then the conclusion of (ii) holds with $M = R$, and X can be chosen so that in addition $\mathrm{Aut}(R)_{(X)} = \{1\}$.

(iv) Let R denote the random graph. Then $\mathrm{Aut}(R)$ is a universal closed permutation group*: that is, for every closed permutation group (H, Y) of countably infinite degree, there is an infinite co-infinite subset X of R such that*

$\mathrm{Aut}(R)_{(X)} = \{1\}$ *and* $\mathrm{Aut}(R)_{\{X\}}$ *induces on* X *a permutation group isomorphic to* (H, Y).

Proof. Part (i) is a classical model-theoretic result. Add Skolem functions to the language, and in the richer language, build a structure M^+ which has reduct M in the original language, as an Ehrenfeucht-Mostowski model; so M^+ is the Skolem hull of a sequence of indiscernibles indexed by $\mathbb{Q}$. Part (ii) is due to Lachlan. Proofs of both results can be found in [108].

(iii) This result was first proved by Henson [74, Theorem 3.1]. Build a copy of R in layers from the countably infinite set X. At a typical step, suppose we have constructed layers $X = X_0, \ldots, X_{n-1}$. For each set A of size n^2 consisting of n points from each X_i, introduce a vertex x_A adjacent to the vertices of $X_0 \cup \ldots \cup X_{n-1}$ which lie in A, and to no other vertices of $X_0 \cup \ldots \cup X_{n-1}$. The layers X_i are themselves all independent sets. The resulting graph with domain $\bigcup_{i \in \mathbb{N}} X_i$ satisfies the extension axioms for graphs, so is isomorphic to the random graph. It is easily seen that each permutation of X_0 extends uniquely to an automorphism of R, to give an embedding of groups.

(iv) This is a small extension of (iii), from [113]. As a first step (also done by Henson in [74, Theorem 3.1]), note that we could have put an arbitrary graph structure on X_0; then every automorphism of this graph extends uniquely to an automorphism of R (which will still be the random graph). To arrange that a more general closed group is induced on X_0, some coding of relations on X_0 is needed. □

I take this point to note that [113] mistakenly does not cite [74] – we were unaware that essentially the argument in (iii) was given there – and thank Eric Jaligot for pointing this out.

Problem 4.2.2 *Generalise Proposition 4.2.1(iv), by finding universal closed permutation groups for restricted classes of structures.*

For example, Jaligot [88] has shown that if M is the random tournament, and N is any countably infinite tournament, then there is a copy N' of N embedded in M such that every automorphism of N' extends uniquely to an automorphism of M.

Free subgroups

If M is ω-categorical and $G = \mathrm{Aut}(M)$, then G has many free subgroups. As a first step it was noted in [108] that G has a dense free subgroup of rank $\aleph_0$. One builds countably many automorphisms of M, which will be a free basis of the group they generate, as unions of finite partial maps. First list the words in the generators; there are countably many. Then build the generating automorphisms in countably many steps, apportioning tasks so that for any non-identity reduced word $w(x_1, \ldots, x_n)$ and partial maps $g_1, \ldots, g_n$ approximating the generators, the g_i are extended to g_i' so that $w(g_1', \ldots, g_n')$ moves some element of M so is not the identity. We also have the following variant for uncountable structures, part (ii) of which is satisfied by any ω-categorical M. For a discussion of 'saturation', a generalisation of ω-categoricity, see [80, Ch. 10]; a structure is *saturated* if any type over any subset of size less than $|M|$ is realised in M. Below, G is *dense* in $\mathrm{Aut}(M)$ if every finite restriction of an automorphism extends to an element of G.

Proposition 4.2.3 *Let M be a saturated model of a complete theory T, with $|M| = \lambda$.*

(i) [119] If $\lambda > |T|$ then $\mathrm{Aut}(M)$ *has a dense free subgroup of cardinality* 2^λ.

(ii) (Hodges, unpublished) If $\lambda = |T| = \lambda^{<\lambda}$, then $\mathrm{Aut}(M)$ *has a dense free subgroup of rank* 2^λ.

It was shown in [108] that the automorphism group of the random graph has a dense free subgroup of rank 2. In another direction, Gartside and Knight [63], proved the following. It stems from a long line of results, starting with the theorem of Dixon [46] that, asymptotically, almost all pairs of permutations of $\{1, \ldots, n\}$ generate Alt_n or Sym_n. Recall that a subset K of a complete metric space X is *comeagre* in X if K contains the intersection of countably many dense open subsets of X. By the Baire Category Theorem, any comeagre set is dense and has size $2^{\aleph_0}$. Thus, the comeagre sets form a filter of subsets of X, and comeagreness provides a notion of largeness.

Theorem 4.2.4 *[63] Let M be ω-categorical and $G = \mathrm{Aut}(M)$. For every positive integer n, let K_n be the set of n-tuples $(g_1, \ldots, g_n) \in G^n$ such that $g_1, \ldots, g_n$ are free generators of a subgroup of G. Then K_n is a comeagre subset of the Polish space G^n.*

It would be interesting to obtain dense subgroups H of $\mathrm{Aut}(M)$ (for various ω-categorical M), where H belongs to a prescribed class of groups *not* including free groups. Little is known on this, but we mention the following, from [15].

Theorem 4.2.5 *Let R be the random graph. Then* $\mathrm{Aut}(R)$ *has a dense locally finite subgroup.*

This is proved via the extension property (EP) for graphs (see Section 5.3 below, in particular Lemma 5.3.2). It is also easily seen that for many ω-categorical structures which are *smoothly approximable* (see [42]) – for example any infinite-dimensional vector space over a finite field equipped with a symplectic, orthogonal or hermitian form, or the projective or affine versions – the automorphism group has a dense locally finite subgroup. We do not know whether this holds for *all* smoothly approximable structures, in particular whether it holds for every stable structure which is homogeneous over a finite relational language. At the other extreme, any locally finite subgroup of $\mathrm{Aut}(\mathbb{Q}, <)$ is trivial, and more generally an ω-categorical structure with the *strict order property* (see Definition 6.4.1 below) cannot admit a locally finite group dense in the full group of automorphisms. The point here is that if $(P,<)$ is a finite partially ordered set and g is an automorphism of P, then for every $x \in P$, $g(x)$ is incomparable to x. See also Remark 5.3.8 (4) below.

Normal subgroup structure

Experience from other contexts suggests that, if M is homogeneous (or more generally, ω-categorical) and $G = \mathrm{Aut}(M)$, then if G has proper non-trivial normal subgroups, it has some obvious ones. After all, it is hard to conceive a description of G except as an automorphism group, and anything fundamental about its group structure should be visible from the action. As a starting point, we have the following.

Theorem 4.2.6 *(i) Let M be a countably infinite set. Then the only proper non-trivial normal subgroups of* $\mathrm{Sym}(M)$ *are* $\mathrm{FSym}(M)$*, the group of* finitary permutations *(those which move just finitely many elements of M), and* $\mathrm{Alt}(M)$*, the group of finitary* even *permutations.*

(ii) Let $G = \mathrm{Aut}(\mathbb{Q},<)$. Define

$$L(\mathbb{Q}) := \{g \in G : \exists q \in \mathbb{Q} \forall q' > q(g(q') = q')\},$$

$$R(\mathbb{Q}) := \{g \in G : \exists q \in \mathbb{Q} \forall q' < q(g(q') = q')\},$$

and $B(\mathbb{Q}) := L(\mathbb{Q}) \cap R(\mathbb{Q})$. Then the only proper non-trivial normal subgroups of G are $L(\mathbb{Q})$, $R(\mathbb{Q})$ and $B(\mathbb{Q})$.

Proof. For (i), see [134]. A proof of (ii) can be found in [64, Theorem 2.3.2]. □

Both of the above proofs are based on a full combinatorial description of conjugacy classes. For more general homogeneous structures, this seems out of reach, and perhaps unrewarding. However, Truss [144] proved that the automorphism group of the random graph is a simple group. In fact, he proves simplicity of the automorphism group of the generic n-coloured graph, where each vertex is coloured (randomly) with one of n colours. Truss also gives a description of the possible cycle types of automorphisms of these structures.

Extending this, Rubin, in unpublished notes, proved simplicity of the automorphism group of any *binary* homogeneous structure for which the amalgamation satisfies a condition close to our *free amalgamation*; his restriction on amalgamation covers the generic K_n-free graphs, and, unlike that in Theorem 4.2.7 below, covers the random tournament. More recently, Lovell, extending the methods of Truss, proves simplicity of various automorphism groups, such as the automorphism group of the universal k-hypergraph, and the universal tournament (handled independently by Jaligot). In addition, it was shown in [65] that the automorphism group of the universal homogeneous partial order is simple.

In recent work of the author and Tent (manuscript in preparation), there is a more topological approach to proving simplicity. We prove

Theorem 4.2.7 *Let M be a transitive free homogeneous structure whose automorphism group is not the whole symmetric group. Then $G = \mathrm{Aut}(M)$ is a simple group.*

Sketch proof. This is based on an argument of Lascar [102]. First, it is easily shown that G has no proper *open* normal subgroup. Next, by replacing an automorphism g with a commutator $[g, h]$ for appropriate choice of h, it can be shown that any non-trivial normal subgroup N of G contains an element g with no fixed points and no 2-cycles. Now define $\alpha : G^6 \to G$ by putting $\alpha(u, v, w, x, y, z) := g^u g^v g^w g^x g^y g^z$. One easily finds that if H is the group generated by $\mathrm{Im}(\alpha)$, then H has the *Baire property*, that is, there is non-empty open U such that the symmetric difference $H\Delta U$ is meagre. The key combinatorial lemma now is that if $U_1, U_2, U_3, U_4, U_5, U_6$ are non-empty open subsets of G, then there is open $Y \subset G$ such that $\alpha(U_1, U_2, U_3, U_4, U_5, U_6)$ is dense in Y. From this it follows rapidly that H is not meagre. Since, in a Polish group, any subgroup with the Baire property is meagre or open [91], it follows that H is open. Hence, as noted above, $H = G$. $\Box$

The above method seems very flexible. We have in mind other structures (certain 'Hrushovski constructions', and free amalgamation constructions of projective planes) to which it might be applicable.

There are other examples where normal subgroup structure has been calculated. To emphasise an analogy made in Example 2.3.1, I note:

Proposition 4.2.8 *Let M be a circular order, a universal homogeneous D-relation, or the universal homogeneous two-graph. Then $\mathrm{Aut}(M)$ is simple.*

Proof. For the D-relation, see [50, Theorem 5.5]. In each case, if $x \in M$ then there is a homogeneous structure M' on $M \setminus \{x\}$ such that $(\mathrm{Aut}(M))_x$ induces $\mathrm{Aut}(M')$ on M'. The information available on normal subgroups of $\mathrm{Aut}(M')$ then suffices. For example, suppose M is the universal homogeneous two-graph, $G := \mathrm{Aut}(M)$, and K is a non-trivial normal subgroup of G. Define a graph structure on $M' = M \setminus \{x\}$, putting $y \sim z$ whenever $\{x, y, z\}$ is an edge of the two-graph M. Then M' is the random graph, which by Truss's result (or Theorem 4.2.7) has simple automorphism group; that is, G_x is simple. By an easy conjugation argument, $K \cap G_x \neq \{1\}$, so $G_x \subseteq K$. Since $G = \langle G_y : y \in M \rangle$, it follows that $K = G$. $\Box$

The above results suggest that automorphism groups of homogeneous structures are close to being simple. However, as a warning, we note the following.

Theorem 4.2.9 ([49]) *Let T be a countably infinite 2-homogeneous tree (see Example 2.3.1(3)). Then $\mathrm{Aut}(T)$ has $2^{2^{\aleph_0}}$ distinct normal subgroups (but no proper non-trivial closed normal subgroups).*

In another direction, Evans and Hewitt [58] showed that for any profinite group H, there is an ω-categorical structure M with automorphism group G, such that G has a closed normal subgroup K so that G/K is isomorphic, as a topological group, to H. If H is the Cartesian product of $\aleph_0$ copies of the cyclic group C_2, then H, and hence G, has $2^{2^{\aleph_0}}$ normal subgroups of index 2. For such constructions, the structure M in general will have infinitely many relation symbols.

The special case when $H = \Pi_{i \in \mathbb{N}} C_2$ was introduced by Hrushovski to give an example of an ω-categorical structure without the small index property - see Section 5.2 below. Cherlin gave the same construction earlier, but for different purposes. The idea is to work over a language with a relation symbol P_n of arity $2n$ for each $n > 0$, and consider the class $\mathcal{C}$ of finite structures in which P_n is interpreted by an equivalence relation on n-sets with at most 2 classes, with the different P_n independent. Then $\mathcal{C}$ is an amalgamation class, and if M is the Fraissé limit, then $\mathrm{Aut}(M)$ has a normal subgroup fixing setwise each P_n-class for all n, with quotient of the required form.

Generic automorphisms

Following Truss [146], we say that an automorphism g of an ω-categorical structure M is *generic* if the conjugacy class C of g is comeagre in $\mathrm{Aut}(M)$. By the Baire Category Theorem, any two comeagre sets intersect non-trivially, so $\mathrm{Aut}(M)$ has at most one comeagre conjugacy class.

This notion can be viewed game-theoretically (the Banach-Mazur game [91]). Consider the two-player game where each player successively builds a larger and larger finite elementary map on M, each player extending the previous player's map by defining it on finitely many additional points. The second wins if she is able to force that the resulting map is an automorphism lying in C. Then C is comeagre if and only if the second player has a winning strategy. Viewed this way, it is obvious that the comeagre conjugacy class of $\mathrm{Sym}(\mathbb{N})$ consists of permutations with infinitely many cycles of each finite length and no infinite cycles; for the second player can 'close up' any finite partial cycle to a finite cycle, and can add new finite cycles. There is an analogous description in [146] of generic automorphisms of $(\mathbb{Q}, <)$, and a more complicated description of generic automorphisms of the random graph.

Truss observed in [146] that a certain amalgamation condition for structures with partial isomorphisms guarantees existence of a generic automorphism. The following necessary and sufficient condition for existence was identified by Ivanov [86] and also by Kechris and Rosendal in [93, Theorem 2.4]. We say that a class $\mathcal{K}$ of finite structures satisfies the *weak amalgamation property* (WAP) if for every $A \in \mathcal{K}$ there is $B \in \mathcal{K}$ and an embedding $e : A \to B$ such that: for any $C_1, C_2 \in \mathcal{K}$ and embeddings $f_i : B \to C_i$ (for $i = 1, 2$) there is $D \in \mathcal{K}$ and embeddings $g_i : C_i \to D$ such that $g_1 \circ f_1 \circ e = g_2 \circ f_2 \circ e$. Below, if $\mathcal{C}$ is a class of finite structures, then $\mathcal{C}_p$ denotes the class of expansions of structures $W \in \mathcal{C}$ by a relation symbol interpreted by the graph of a partial isomorphism $U \to V$, where $U, V \leq W$.

Theorem 4.2.10 ([86, 93]) *Let $\mathcal{C}$ be an amalgamation class of finite relational* *structures with Fraissé limit M. Then the following are equivalent.*

(i) M has a generic automorphism.

(ii) The class $\mathcal{C}_p$ satisfies (WAP) and (JEP).

The following result is proved by Ivanov in [87].

Theorem 4.2.11 *Let M be a free homogeneous structure over a finite relational language. Then M has a generic automorphism.*

Next, we collect some consequences of existence of generic automorphisms.

Proposition 4.2.12 *Let M be a structure such that $G = \mathrm{Aut}(M)$ has a comeagre conjugacy class C. Then*

(i) $G = \mathrm{Aut}(M)$ has no proper normal subgroup of countable index;

(ii) every element of G is a product of two elements of C;

(iii) each element of G is a commutator, so G equals its derived subgroup G',

(iv) if G acts as a group of automorphisms of a graph-theoretic tree, and has no inversions (i.e. no element of G swaps over any two adjacent vertices), then for every $g \in G$ there is $v \in T$ such that $g(v) = v$;

(v) if G is a free product with amalgamation $G = H_1 *_K H_2$, then $G = H_1$ or $G = H_2$.

(vi) if G acts by isometries on an $\mathbb{R}$-tree, then every element of G has a fixed point.

Proof. (i) Let K be a proper normal subgroup of G. If $K \cap C \neq \emptyset$ then $C \subset K$, so K is comeagre. Hence all cosets of K are comeagre, contradicting the fact that any two comeagre subsets of G intersect non-trivially. Thus, $K \cap C = \emptyset$, so K is meagre. However, it is shown in [81, Theorem 4.1] that any meagre subgroup of a Polish group has index $2^{\aleph_0}$. (This last point follows immediately from a theorem of Kuratowski and Mycielski – see [93, Lemma 5.6].)

(ii) Let $g \in G$. As the inversion map is continuous, C^{-1} is comeagre (so equals C), so also gC^{-1} is comeagre. Thus $gC^{-1} \cap C \neq \emptyset$, so there are $h_1, h_2 \in C$ with $gh_1^{-1} = h_2$. Then $g = h_2 h_1$.

(iii) Let $g \in G$. As in (ii), there are $h_1, h_2 \in C$ with $g = h_1 h_2$. Since $C = C^{-1}$, there is $k \in G$ with $h_2 = k^{-1} h_1^{-1} k$. Then $g = h_2 h_1 = [k, h_1] \in G'$.

(iv), (v) See [116].

(vi) This is a strengthening of (iv), due to Rosendal [129]. The result is further strengthened in [72], where the same conclusion is shown to hold if G is just assumed to have a non-nesting action on the $\mathbb{R}$-tree T (that is, G preserves the betweenness relation on T and there does not exist a segment I of T and $g \in G$ such that $g(I)$ is a proper subset of I). □

Remark 4.2.13 As noted in [146], Proposition 4.2.12(i) gives a way of proving *non-existence* of generic automorphisms. For example, many structures have automorphism group with a proper closed normal subgroup of finite index: examples include the disjoint union of two countably infinite complete graphs, the random bipartite graph, and the linear betweenness relation.

Another homogeneous structure not admitting a generic automorphism, also noted in [146], is the countable dense circular order (see Example 2.3.1(1). In the Banach-Mazur game, the first player could build a partial map fixing a point, or could ensure, by interchanging two elements of $\mathbb{Q}$, that no extension fixes a point. Thus, the second player does not have a winning strategy. A similar argument shows that the universal homogeneous D-relation (see Section 6.1) does not admit a generic automorphism. We omit the details.

We mention briefly another model-theoretic notion of generic automorphism. Let T be a complete theory over some countable language L, with quantifier elimination, and with an infinite model. Let $L_\sigma = L \cup \{\sigma\}$, where σ is a unary function symbol. Let T_σ be the (in general incomplete) extension of T given by sentences which express that, if $\mathcal{M}$ is a model, then σ is an automorphism of the reduct $\mathcal{M}|_L$. In certain cases, T_σ admits a model companion T_σ^*, that is, a $\forall\exists$-axiomatised theory extending T_σ such that every model of T_σ embeds in a model of T_σ^* and vice versa. The very important motivating example is the theory ACFA of algebraically closed fields with a generic automorphism, developed in [36]. There is a substantial literature on when T_σ^* exists, emanating from [37].

The connections between these two notions of generic automorphism are investigated in [10]. They work in a broader framework, considered earlier by Ivanov, of arbitrary expansions of a structure to a richer language, such that the expansions are generic, e.g. in a sense coming from Baire category.

5 Reconstruction from the automorphism group

5.1 Versions of reconstruction

As noted in Section 3.1, for ω-categorical structures it is convenient to translate between the language of model theory and the language of permutation group theory. This suggests that much information about an ω-categorical structure is encoded in the automorphism group, and that the latter can be viewed as a kind of invariant of the structure. So we ask: to what extent can an ω-categorical structure M be *recovered* from $\mathrm{Aut}(M)$. It should be noted that in general $\mathrm{Aut}(M)$ is a much more complicated object than M, so it is not too surprising if M is recoverable. For example, the familiar homogeneous structures have recursively axiomatised theories, so have decidable theory; that is, the set of Gödel numbers of sentences in their theory is recursive. On the other hand, we have the following.

Theorem 5.1.1 *[16] If M is ω-categorical then* $\mathrm{Aut}(M)$, *viewed just as a structure in the language of groups, has undecidable theory.*

In fact, if T is any complete theory in a countable language with infinite models, then it has a countable model whose automorphism group has undecidable *existential* theory. This is because there is a right orderable finitely presented group with insoluble word problem, every countable right orderable group embeds in $\mathrm{Aut}(\mathbb{Q}, \leq)$, and, by the Ehrenfeucht-Mostowski construction (cf. Proposition 4.2.1(i)), there is a countable model M of T such that $\mathrm{Aut}(\mathbb{Q}, \leq)$ embeds into $\mathrm{Aut}(M)$.

We could aim to recover the structure M from $G = \mathrm{Aut}(M)$, where G is given as a permutation group on M, or as a topological group, or as an abstract group. As a permutation group, there is nothing to do: the $\emptyset$-definable n-ary relations on M are the unions of orbits of G on M^n, so we can recover $\mathcal{M}$ up to interdefinability from G, without parameters. We could not hope for more – for example, a complete graph and a graph with no edges have the same automorphism group.

The case where G is viewed as a topological group is also well-understood. Two structures M and N are *bi-interpretable*, if each is interpretable in a copy of the other, and furthermore, there is an $\emptyset$-definable (in M) isomorphism between M and a copy M^{**} of M which lives in a copy N^* of N which lives in M (and likewise with M and N reversed). We now have the following, which can be expressed functorially. The authors of [5] describe (ii) as an unpublished result of T. Coquand.

Theorem 5.1.2 (Ahlbrandt and Ziegler [5])

(i) Let M be ω-categorical. Then a structure N is interpretable in M if and only if there is a continuous homomorphism $h : \mathrm{Aut}(M) \to \mathrm{Aut}(N)$ *such that* $h(\mathrm{Aut}(M))$ *has finitely many orbits on N.*

(ii) Let M and N be ω-categorical structures. Then M and N are bi-interpretable if and only if $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(N)$ *are isomorphic as topological groups.*

Thus, we focus on two reconstruction questions.

Question 5.1.3 *Let M be an ω-categorical structure and* $G = \mathrm{Aut}(M)$.

(1) If N is also ω-categorical and $\mathrm{Aut}(N)$ *is isomorphic to G as an abstract group, is* $\mathrm{Aut}(N)$ *isomorphic to G as a topological group?*

(2) Is the structure M interpretable (possibly with parameters) in the (abstract) group G?

There appear to be two known approaches to Question 5.1.3(1). One is via the *small index property*, discussed in the next two sections. The other stems from the paper [131] of Rubin, and is discussed in Section 5.4. Question 5.1.3(2) is subsidiary, but Rubin's method, when applicable, gives an interpretation.

5.2 Small index property

Definition 5.2.1 The ω-categorical structure M has the *small index property* if every subgroup of $G = \mathrm{Aut}(M)$ of index less than $2^{\aleph_0}$ is open. It has the *strong small index property* if for every $H \leq G$ with $|G : H| < 2^{\aleph_0}$, there is finite $A \subset M$ such that $G_{(A)} \leq H \leq G_{\{A\}}$.

Proposition 5.2.2 *Suppose that M is an ω-categorical structure with the small index property, and N is ω-categorical and* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(N)$ *are isomorphic as abstract groups. Then M and N are bi-interpretable.*

Proof. Let $\phi : \mathrm{Aut}(M) \to \mathrm{Aut}(N)$ be an isomorphism, and let H be an open subgroup of $\mathrm{Aut}(N)$. Then there is finite $A \subset N$ such that $H \geq \mathrm{Aut}(N)_{(A)}$, so $|\mathrm{Aut}(N) : H| \leq \aleph_0$, as any tuple enumerating A has countably many translates. Thus, $|\mathrm{Aut}(M) : \phi^{-1}(H)| \leq \aleph_0$, so as M has the small index property, $\phi^{-1}(H)$ is open in $\mathrm{Aut}(M)$. Thus, ϕ is continuous. It is well-known (see e.g. [91]) that any continuous isomorphism between Polish groups is a homeomorphism. The result now follows from Theorem 5.1.2. □

The above argument only requires that every subgroup of $\mathrm{Aut}(M)$ of *countable* index (rather than index less than $2^{\aleph_0}$) is open. We do not know of any example of an ω-categorical structure such that every countable index subgroup of the automorphism group is known to be open, but for which this has not been proved for subgroups of index less than the continuum.

The *strong* small index property has a further model-theoretic consequence. Namely, if the ω-categorical structure M has the strong small index property, then it has *weak elimination of imaginaries*. See [80, p. 161] for definitions and a proof. An example of a homogeneous structure with the small index property but not the strong small index property is the disjoint union of infinitely many complete graphs, all infinite: the setwise stabiliser of one of the maximal complete induced subgraphs has countable index, but is not of the required form.

As noted after Theorem 4.2.9 above, there is an ω-categorical structure $\mathcal{M}$, constructed by Cherlin and by Hrushovski, such that $G = \mathrm{Aut}(\mathcal{M})$ has a quotient (by a closed normal subgroup) isomorphic to the Cartesian product of $\aleph_0$ copies of C_2. Then G has $2^{2^{\aleph_0}}$ subgroups of index 2. The group G has at most $\aleph_0$ open subgroups, for there are just countably many basic open subgroups, and if $H < G$ is basic open then there are finitely many open K with $H < K < G$. (This is essentially because intermediate open groups K are closed, so correspond to $\emptyset$-definable equivalence relations in an ω-categorical structure arising from the action of G on cosets of H; there can only be finitely many such equivalence relations, by the Ryll-Nardzewski Theorem.) Thus some subgroup of G of index 2 is not open. Using a variant of the Cherlin-Hrushovski construction with $\Pi_{i \in \mathbb{N}} C_2$ replaced by a carefully chosen profinite group, Evans and Hewitt [58] constructed two ω-categorical structures which are not bi-interpretable but which have isomorphic automorphism group (as abstract groups).

Remark 5.2.3 The small index property has been proved, among others, for the following ω-categorical structures.

(i) A pure set (so the automorphism group is the symmetric group) [47].

(ii) The group $\mathrm{GL}(\aleph_0, q)$ for any prime power q, and the analogous symplectic, orthogonal, and unitary groups [56, 57].

(iii) 2-homogeneous trees (in the sense of Droste [48] – see Section 6.1 below) and the corresponding C and D relations [50].

(iv) $\mathrm{Aut}(\mathbb{Q}, <)$, and the countable atomless Boolean algebra [145].

(v) Those ω-categorical structures whose theory is ω-stable (see Definition 3.3.1) [81]. This class includes the stable homogeneous structures discussed in Section 3.3.

(vi) Many homogeneous structures whose age has been proved to have the *extension property* (EP) for finite partial automorphisms, defined in Section 5.3. By [77], such structures include the universal homogeneous K_n-free graph, the universal homogeneous k-hypergraph, and the 'Henson digraphs'.

There are essentially just two known methods for proving the small index property.

The proofs in (i)–(iv) of Remark 5.2.3 are based on a 'piecewise patching' property of the automorphism group – one can in certain situations obtain an automorphism as the union of infinitely many partial maps all defined on (disjoint) infinite sets; for example, we may partition $(\mathbb{Q}, <)$ into disjoint clopen convex sets X_i (for $i \in \mathbb{N}$), and then if $g_i \in \mathrm{Aut}(X_i, <)$ for each i and g is the union of the g_i, then $g \in \mathrm{Aut}(\mathbb{Q}, <)$. These proofs use in addition an argument with almost disjoint sets, seen first in [47]. This line of argument also works for certain other structures closely related to the above. We do not discuss this approach here.

For examples (v) and (vi) above, a completely different method is used, coming from [81]. The key definition in [81] is the following. For details of M^{eq}, see [80, p. 151]. For example (vi) in Remark 5.2.3, M^{eq} may be replaced below by M, and $\mathcal{B}(M)$ by the collection of all finite subsets of M.

Definition 5.2.4 Let M be an ω-categorical structure, and let $G := \mathrm{Aut}(M)$.
(i) A *base* for M is a countable set $\mathcal{B}(M)$ of subsets of M^{eq} such that

(a) $\mathcal{B}(M)$ is G-invariant: if $A \in \mathcal{B}(M)$ and $g \in G$, then $g(A) \in \mathcal{B}(M)$.

(b) for all $A \in \mathcal{B}(M)$, the set $\{G_{(B)} : B \in \mathcal{B}(M), B \supseteq A\}$ is a base of open neighbourhoods of 1 in G.

(ii) Let $\mathcal{B}(M)$ be a base for M, and $n \in \mathbb{N}^{>0}$. Then $(g_1, \ldots, g_n) \in G^n$ is $\mathcal{B}(M)$*-generic* if the following hold.

(a) $\{A \in \mathcal{B}(M) : g_i(A) = A \text{ for all } i \leq n\}$ is cofinal in $\mathcal{B}(M)$, where the latter is ordered by inclusion.

(b) Suppose $A \in \mathcal{B}(M)$ with $g_i(A) = A$ for each i, and that $B \supset A$ with $B \in \mathcal{B}(M)$, and let $h_i \in \mathrm{Aut}(M)$ extend g_i for each i, with $h_i(B) = B$. Then there is $f \in G_{(A)}$ such that $fg_i f^{-1}$ extends $h_i|_B$ for each i.

(iii) If $\mathcal{B}(M)$ is a base for M, then the structure M has *ample $\mathcal{B}(M)$-generic automorphisms* if, for all $n > 0$, the set of $\mathcal{B}(M)$-generic elements of G^n is comeagre in G^n in the product topology.

(iv) A Polish group H has *ample homogeneous generic automorphisms* if, for each $n > 0$, H has a comeagre orbit on H^n in its action by conjugation.

(v) The structure M has *ample homogeneous generic automorphisms* if G has ample homogeneous generic automorphisms.

The terminology in (v) differs slightly from [81], where $\mathrm{Aut}(M)$ is said to have ample homogeneous generic automorphisms if it has ample $\mathcal{B}(M)$-generic automorphisms for some base $\mathcal{B}(M)$. In (iv) we follow [93]. It seems possible that M might have ample homogeneous generic automorphisms without having ample $\mathcal{B}(M)$-generic automorphisms for any base $\mathcal{B}(M)$. In the other direction, by [81], every ω-categorical ω-stable structure has ample $\mathcal{B}(M)$-generic automorphisms, but some do not have ample homogeneous generic automorphisms. An example is a set equipped with an equivalence relation with two classes, both infinite; by Remark 4.2.13, this does not even admit a single generic automorphism (working over $\emptyset$).

However, if M has ample $\mathcal{B}(M)$-generic automorphisms, then some open subgroup of $\mathrm{Aut}(M)$ has ample homogeneous generic automorphisms. Indeed, let X be the (comeagre) set of $\mathcal{B}(M)$-generic elements of G^n, and let $A \in \mathcal{B}(M)$. Let

$$X_A := \{(g_1, \ldots, g_n) \in X : g_i|_A = \mathrm{id}|_A \text{ for } i = 1, \ldots, n\}.$$

By [81, Proposition 2.3], the elements of X_A are conjugate under $G_{(A)}$ (which is open in G by the definition of *base*). Also, $X_A = X \cap (G_{(A)})^n$, so is comeagre in $(G_{(A)})^n$. It follows that, in the theorem below, (ii) follows immediately from (i).

Theorem 5.2.5 *(i) [93] Let G be a Polish group with ample homogeneous generic automorphisms. Then every subgroup H of G with $|G : H| < 2^{\aleph_0}$ is open.*

(ii) [81, Theorem 5.3] Let M be an ω-categorical structure with ample $\mathcal{B}(M)$-generic automorphisms for some base $\mathcal{B}(M)$. Then M has the small index property.

The technique of [81] has been used to prove the small index property for some structures which are not ω-categorical, such as the free group of infinite rank, and certain other relatively free groups [25]. Similar methods have been used by Lascar to prove the small index property for any countable arithmetically saturated model of Peano Arithmetic [104], and also a version of the small index property for any uncountable saturated structure of cardinality $\kappa = \kappa^{<\kappa}$ [103].

To apply Theorem 5.2.5, we need tools for constructing ample $\mathcal{B}(M)$-generic automorphisms. The following is proved in [81, Theorem 2.9]

Theorem 5.2.6 *Let M be ω-categorical, let $\mathcal{B}(M)$ be a base for M, and suppose:*

(i) for any $A \in \mathcal{B}(M)$, and any finite partial elementary maps $e_1, \ldots, e_n$ between subsets of A, there is $B \in \mathcal{B}(M)$ containing A and automorphisms $f_1, \ldots, f_n \in \mathrm{Aut}(B)$ such that f_i extends e_i for each i;

(ii) if $A, B, C \in \mathcal{B}(M)$ with $A \subseteq B \cap C$, there is $g \in \mathrm{Aut}(M)_{(A)}$ such that if $f_1 \in \mathrm{Aut}(g(B))$ and $f_2 \in \mathrm{Aut}(C)$ fix A setwise and agree on A, then $f_1 \cup f_2$ is an elementary map on $g(B) \cup C$ (in the sense of M^{eq}).

Then M has ample $\mathcal{B}(M)$-generic automorphisms, so has the small index property.

In many situations, condition (ii) of Theorem 5.2.6 comes free: for example, if M is a free homogeneous structure, then its age satisfies (ii) – just choose g so that $g(B) \cup C = g(B) \oplus_A C$. Similarly, in stable theories, there is a well-behaved abstract theory of independence, and we may choose g so that $g(B)$ is independent from C over A (formally, we require here an extra condition, namely that $\mathrm{acl}^{\mathrm{eq}}(A) \subseteq \mathrm{dcl}(A)$). Usually, condition (i) is much more problematic. One situation where the reverse holds – that is, condition (i) comes almost free by definition but (ii) is open in general – is that of smoothly approximable structures; see [43].

Structures such as the random graph, the random K_n-free graph, the Henson digraphs, and the universal homogeneous k-hypergraph, are all known to have the small index property, via Theorems 5.2.5 and 5.2.6. In each case, as they have free amalgamation, by the last theorem the proof reduces to condition (i) in Theorem 5.2.6. This is the subject of the next section.

For ω-categorical structures, the small index property has so far only been proved by the two methods mentioned above: almost disjoint sets using 'piecewise patching' of partial automorphisms, or using Theorem 5.2.5. We mention four test cases for which these methods do not seem to work, and for which the question appears to be open.

Question 5.2.7 *Does the small index property hold for the following homogeneous structures?*

(i) The universal homogeneous tournament.

(ii) The universal homogeneous partial order.

(iii) The generic totally ordered graph. This is the unique countably infinite homogeneous structure in the language with two binary relations R and $<$, whose age consists of all finite structures such that R is a graph (i.e. is symmetric and irreflexive) and $<$ induces a total order.

(iv) The Fraissé limit (in a language with two binary relations $<_1$ and $<_2$) of the class of all finite sets totally ordered, independently, by $<_1$ and $<_2$ (so this is the universal homogeneous permutation in the sense of Cameron [33] – see Section 2.2 above).

It seems likely that the methods of Theorems 5.2.5 and 5.2.6 will not work for structures which non-trivially involve partial orders (e.g. those with the strict order property (Definition 6.4.1)). For example, Hodkinson [personal communication] showed that for $(\mathbb{Q}, <)$, the automorphism group G does not have a comeagre orbit in its diagonal action by conjugation on G^2.

We shall discuss in Section 5.5 other applications of ample homogeneous generic automorphisms.

5.3 Extension property for finite isomorphisms

The last section motivates the following general question, of independent combinatorial interest. If $\mathcal{C}$ is the age of a homogeneous structure M, a positive answer yields condition (i) of Theorem 5.2.6, where $\mathcal{B}(M) = \mathrm{Age}(M)$.

Question 5.3.1 *Let $\mathcal{C}$ be a class of finite structures in some fixed first order language (typically, an amalgamation class). Is it true that for every $A \in \mathcal{C}$, there is $B \in \mathcal{C}$ containing A such that every isomorphism between substructures of A extends to an automorphism of B?*

We say that the class $\mathcal{C}$ has the *extension property for partial automorphisms* (EP) if the answer to Question 5.3.1 is positive for $\mathcal{C}$.

There are quantitative versions of Question 5.3.1, which we shall not discuss here: for a given class $\mathcal{C}$, find a good bound $f : \mathbb{N} \to \mathbb{N}$ such that for any $A \in \mathcal{C}$ we can choose B so that $|B| \leq f(|A|)$. There are some remarks on this in [83], and also in [78].

For some amalgamation classes, Question 5.3.1 has an obviously negative answer. For example, every finite total order is rigid, so the answer is negative for the class of finite total orders, and Theorem 5.2.6 could not be used to prove the small index property for $(\mathbb{Q}, <)$. Likewise, if $\mathcal{C}$ is the age of the universal homogeneous partial order, then $\mathcal{C}$ does not have (EP). For if A is the 2-element total order $\{u, v\}$ with $u < v$ and e is the isomorphism with $\mathrm{dom}(e) = \{u\}$ and $e(u) = v$, there is no finite partial order B containing A such that e is the restriction of an automorphism of B. In fact, the small index property is open for the universal homogeneous partial order (Question 5.2.7). Note that it is conceivable that the universal homogeneous partial order has ample homogeneous generic automorphisms (and hence the small index property) even though its age does not have (EP) and Theorem 5.2.6 is not applicable. More generally, if $\mathcal{C}$ is the age of a homogeneous structure with the *strict order property* (see Definition 6.4.1 below), then $\mathcal{C}$ does not have (EP).

The first non-trivial proof of (EP) was by Hrushovski [83], for the class of all finite graphs. It is used in [81] to prove the small index property for the random graph. A rather different, extremely short proof of (EP) for graphs is given in [78, Section 4.1]. The latter proof yields in addition the following, used in the proof of Theorem 4.2.5

Lemma 5.3.2 *Let Δ be a finite graph, and $G := \mathrm{Aut}(\Delta)$. Then there is a finite graph Δ' such that Δ is an induced subgraph of Δ', every partial isomorphism between subgraphs of Δ extends to an automorphism of Δ', and there is a monomorphism $\phi : G \to \mathrm{Aut}(\Delta')$ such that $\phi(g)$ extends g for all $g \in G$.*

Hrushovski's proof of (EP) for graphs was greatly generalised by Herwig in [77, 76], and further (with Lascar) in [78]. I state below versions from [77] and [78].

Fix a finite relational language L, and a class $\mathcal{C}$ of finite L-structures. Recall from the introduction the definition of a *homomorphism* between relational structures. If $\mathcal{F}$ is a class of finite L-structures, and A is an L-structure, then we say A is *$\mathcal{F}$-free* if there do not exist $B \in \mathcal{F}$ and a homomorphism $f : B \to A$.

Theorem 5.3.3 *[78] Let L be a finite relational language, and $\mathcal{F}$ a finite set of finite L-structures, and let $\mathcal{C}$ be the class of (finite or infinite) $\mathcal{F}$-free L-structures. Then $\mathcal{C}$ has the property (EPPA), that is: for any finite $A \in \mathcal{C}$ and set P of partial isomorphisms between substructures of A, if there is $B \in \mathcal{C}$ with $A \leq B$ such that every $p \in P$ extends to an automorphism of B, then there is* finite *$B \in \mathcal{C}$ with $A \leq B$ such that every $p \in P$ extends to an automorphism of B.*

To avoid the assumption above that $\mathcal{F}$ is finite, we shall actually quote an earlier version of this theorem, from [77]. We first give some definitions.

Definition 5.3.4 Let L be a relational language.

(i) An L-structure A is a *link* structure if, for some n, $A = \{a_1, \ldots, a_n\}$ and $A \models Ra_1 \ldots a_n$ for some $R \in L$. If $\mathcal{P}$ is a collection of link structures, then an L-structure A has *link type* $\mathcal{P}$ if every substructure of A which is a link structure belongs to $\mathcal{P}$. An L-structure A is *packed* if any two distinct elements of A lie in a tuple satisfying a relation of L.

(ii) Let $\mathcal{P}$ be a class of link structures (for L) and $\mathcal{F}$ be a class of finite L-structures. Then $\mathcal{K}_{\mathcal{P}\mathcal{F}}$ is the collection of all finite L-structures which are $\mathcal{F}$-free and have link type $\mathcal{P}$.

Remark 5.3.5 1. If A is packed, B is irreflexive, and $f : A \to B$ is a homomorphism, then f is injective.

2. If $\mathcal{F}$ is a family of packed structures, and $\mathcal{P}$ is a family of link structures, then $\mathcal{K}_{\mathcal{P}\mathcal{F}}$ has the free amalgamation property.

Theorem 5.3.6 *[77] Let L be a finite relational language, $\mathcal{F}$ a set of finite irreflexive packed L-structures and $\mathcal{P}$ a set of irreflexive link structures. Then $\mathcal{K}_{\mathcal{P}\mathcal{F}}$ has (EP).*

Corollary 5.3.7 *[77] Let $\mathcal{C}$ be a monotone free amalgamation class over a finite relational language, with Fraïssé limit M. Then* *(i) $\mathcal{C}$ has (EP),*

(ii) M has ample homogeneous generic automorphisms, and

(iii) M has the small index property.

Proof. (i) First replace L by a finite relational language L', and each L-structure A by an interdefinable L'-structure A' on the same domain such that A' is irreflexive. The idea is that for each L-relation R of arity n (for all n) and each partition π of $\{1, \ldots, n\}$ into r parts, introduce a new relation R_π of arity r. We may then replace each R by the set of R_π, and require that all the R_π are irreflexive. We leave the details to the reader.

Thus, the links of the members of $\mathcal{C}$ are all irreflexive. Let $\mathcal{P}$ be the set of link structures in $\mathcal{C}$, and $\mathcal{F}$ the set of minimal irreflexive L-structures which do not embed in $\mathcal{C}$. Then, since $\mathcal{C}$ is free, the members of $\mathcal{F}$ are packed, and we have $\mathcal{C} = \mathcal{K}_{\mathcal{P}\mathcal{F}}$.

(ii), (iii) These follow immediately from Theorem 5.2.6 and the remarks following it. Note that Theorem 5.2.6(i) yields for M ample $\mathcal{B}(M)$-generic automorphisms for the base consisting of all finite subsets of M, which ensures that M has ample homogeneous generic automorphisms. □

Remark 5.3.8 1. This theorem yields the small index property for the homogeneous K_n-free graph, the Henson digraphs, and for the universal homogeneous k-hypergraph. For example, in the case of the Henson digraphs, $\mathcal{F}$ is just the (finite) set of minimal tournaments which do not embed in the digraph.

2. It is noted in [78] that results about (EPPA) have translations in combinatorial group theory. Recall that if F is a finitely generated free group, then the cosets of the finite index subgroups form the basis of the 'profinite topology' on F. Ribes and Zalesskii [128] proved that if $H_1, \ldots, H_n$ are finitely generated subgroups of F, then $H_1 \ldots H_n := \{h_1 \ldots h_n : h_i \in H_i\}$ is a closed set in the profinite topology on F. Now (EP) for the collection of all graphs (Hrushovski's Theorem) can be derived from the Ribes-Zalesskii Theorem, and the latter follows from (EPPA) for the class of 'n-partitioned cycle-free graphs'. In [78], the authors show that Theorem 5.3.3 is also 'equivalent' to a group-theoretic theorem of this kind.

3. An alternative treatment of Theorems 5.3.3 and 5.3.6 is given in [82]. This is used to give a proof of the finite model property of the clique-guarded fragment of first order logic. There is also an approach to Theorem 5.3.3 due to J. Almeida.

4. Solecki [138] has shown that if A is a finite metric space then there is a finite metric space B such that A is a substructure of B (as metric spaces) and every partial isometry on A extends to an isometry of B. The collection of all finite metric spaces is an amalgamation class with Fraïssé limit the *rational Urysohn space*, that is, the unique countable metric space with rational distances such that any isometry between finite subspaces extends to an isometry of the whole space. It follows that the full isometry group has ample homogeneous

generic automorphisms. The result is slightly extended in [139] to yield that the isometry group has a locally finite dense subgroup – see also the remarks after Theorem 4.2.5 above. Solecki in [139] also proves a slight strengthening of Theorem 5.3.3, which yields the following strengthening of Theorem 4.2.5 above:

if L is a finite relational language, $\mathcal{F}$ a finite set of finite L-structures, and $\mathcal{C}$ an amalgamation class of finite $\mathcal{F}$-free L-structures with Fraissé limit M, then $\mathrm{Aut}(M)$ has a locally finite dense subgroup.

The class of finite tournaments is not a class of the form $\mathcal{K}_{\mathcal{PF}}$ as in Theorem 5.3.6. In fact, the following question, relevant to Question 5.2.7(i), remains open. I find it very natural, in view of the many common features of the random graph and the random tournament (see e.g. Section 3.2). At the end of [78], the authors give a Ribes-Zalesskii-like equivalent of (EP) for the class of finite tournaments.

Question 5.3.9 Does the class of finite tournaments satisfy (EP)?

Towards a negative answer, we note the following.

Proposition 5.3.10 *Suppose that the class $\mathcal{C}_T$ of finite tournaments has (EP). Then for every finite tournament T_1 there is a finite tournament T_2 such that*

(i) $T_1 \leq T_2$;

(ii) every isomorphism between subtournaments of T_1 extends to an automorphism of T_2;

(iii) T_2 has primitive automorphism group.

Proof. Write VT for the vertex set of a tournament T. Given T_1, by (EP) we may choose T_2 satisfying (i) and (ii) with $|VT_2|$ as small as possible. Then $\mathrm{Aut}(T_2)$ is transitive on VT_2. For since singletons of T_1 are isomorphic, by (ii) they all live in the same $\mathrm{Aut}(T_2)$-orbit. Any automorphism of T_2 induces an automorphism of this orbit, so by minimality this orbit equals VT_2.

Suppose for a contradiction $\mathrm{Aut}(T_2)$ preserves a proper non-trivial congruence E on VT_2. Then VT_1 does not lie in a single E-class, since otherwise we could replace T_2 by the subtournament on this E-class, contradicting minimality of $|VT_2|$. Thus, each E-class meets VT_1 in at most a singleton: for otherwise, as any two arcs of T_1 lie in the same $\mathrm{Aut}(T_2)$-orbit on arcs, by (ii) T_2 would have an automorphism taking two E-equivalent elements to inequivalent elements.

Observe that as $\mathrm{Aut}(T_2)$ has no involutions, it has odd order, so no element can interchange two E-classes. Let U_1 be the $\mathrm{Aut}(T_2)$-orbit on arcs which contains all T_1-arcs. We now define a tournament structure T_2' on $VT_2' := VT_2/E$. If $B_1, B_2 \in VT_2'$ and there is an arc of U_1 from a vertex in B_1 to a vertex in B_2, write $B_1 \to B_2$. For each other orbit W on pairs of E-classes, either decide that $B_1 \to B_2$ for all $(B_1, B_2) \in W$, or that $B_2 \to B_1$ for all $(B_1, B_2) \in W$.

Then $\mathrm{Aut}(T_2)$ acts as a group of automorphisms of T_2'. Furthermore, T_1 embeds into T_2' under the map $x \mapsto x/E$. If we identify T_1 with its image under this map, we see that every isomorphism between subtournaments of T_1 extends to an automorphism of T_2', again contradicting the minimality of VT_2. $\square$

Remark 5.3.11 As noted in the above proof, the automorphism group of any finite tournament has odd order, so, by the Feit-Thompson Theorem, is soluble. By a theorem of Pálfy [123], any soluble primitive permutation group on a set of size n has order bounded by a polynomial in n. Furthermore, by a familiar line of argument which leads to the O'Nan-Scott Theorem, if G is a primitive soluble permutation group on a finite set X, then X has prime power size, and indeed, can be identified with, for some n, an n-dimensional vector space over a finite field $\mathbb{F}_p$, with G a subgroup of the affine group $\mathrm{AGL}_n(p)$. Thus, in Theorem 5.3.10, we may suppose that $\mathrm{Aut}(T_2)$ has this form. We may take T_1 to be a small tournament with many partial isomorphisms, like a 5-element total order. It seems likely that current information on odd order primitive permutation groups could now be used to show that the class $\mathcal{C}_T$ of finite tournaments does not have (EP).

5.4 Rubin's approach to reconstruction.

We discuss here an alternative approach to the problem of reconstruction of an ω-categorical structure from its automorphism group, initiated by M. Rubin, and generalised slightly and used by Barbina. This approach may be more powerful than that from the small index property, and it gives additional information, namely a first order interpretation of the structure in its automorphism group. The presentation below is from [9]. Below, if G is a group and $g \in G$ then g^G denotes the conjugacy class of g in G.

Definition 5.4.1 (i) If G is a group, and $\bar{g} = (g_1, \ldots, g_n) \in G^n$, then a formula $\phi(\bar{g}, x, y)$ in the language of groups (with parameters $\bar{g}$) is an *equivalence formula* if:

(a) ϕ is $\forall\exists$, i.e. has the form $\forall \bar{u} \exists \bar{v}(\psi(\bar{u}, \bar{v}, x, y))$ where ψ is quantifier-free;

(b) for any group H and $\bar{h} \in H^n$, $\phi(\bar{h}, x, y)$ defines an equivalence relation on the conjugacy class h_1^H, and

(c) the equivalence relation, denoted E^ϕ, on g_1^G defined by $\phi(\bar{g}, x, y)$ is invariant under G-conjugation.

(ii) Let M be a transitive ω-categorical structure. A *weak $\forall\exists$-interpretation* is a triple $\langle \phi, \bar{g}, \tau \rangle$ where $\phi = \phi(\bar{g}, x, y)$ is an equivalence formula for $G := \mathrm{Aut}(M)$, $\bar{g} \in G^n$, and $\tau : g_1^G/E^\phi \to M$ is a bijection such that for all $g, h \in G$, $g(\tau(h/E^\phi)) = \tau(h^g/E^\phi)$ (so τ induces a permutation group isomorphism between $(\mathrm{Aut}(M), g_1^G/E^\phi)$ and $(\mathrm{Aut}(M), M)$).

Theorem 5.4.2 (Rubin [131]) *Let M and N be ω-categorical structures such* *that* Aut(M) $\cong$ Aut(N) *as abstract groups, and suppose that* M *has a weak* $\forall\exists$-*interpretation. Then* M *and* N *are bi-interpretable.*

Rubin's proof uses a set-theoretic forcing argument.

Remark 5.4.3 1. In fact, Rubin's conclusion is slightly different: he assumes in addition that M and N have trivial algebraic closure (that is, the pointwise stabiliser of any finite set A has no finite orbits outside A), and the conclusion is then that M and N are bi-definable – they have the same $\emptyset$-definable sets. The statement in 5.4.2 is mentioned in [131], with details given in [8].

2. The transitivity condition in Definition 5.4.1 (ii) can be dispensed with, if we work orbit-by-orbit.

3. In the formula $\phi(\bar{g}, x, y)$, one can allow that x and y are tuples, not necessarily singletons.

4. If M (with automorphism group G) has a weak $\forall\exists$-interpretation then we may identify M with a definable quotient g^G/E of a conjugacy class g^G. The relations of M are then identifiable with finite unions of orbits of G on n-tuples from g^G/E, so are definable in G. Thus, the structure M is parameter-interpretable in the group G.

5. A weak $\forall\exists$-interpretation for M gives an interpretation in Aut(M) *with parameters* of M and the action of Aut(M) on M. Such an interpretation *without parameters* gives additional group-theoretic information, namely that Aut(M) has no non-trivial outer automorphisms. For any automorphism α of Aut(M) will respect this interpretation, so induce an automorphism of M which must also be induced by some $g \in$ Aut(M), and it follows that α and g induce the same automorphism of Aut(M). For example, if M is a pure set, then the set X of transpositions is $\emptyset$-definable in $S = \mathrm{Sym}(M)$; for $g \in S \setminus \{1\}$ is a transposition if and only if, for all $h \in S$, gg^h has order at most three. Now if Y is the set of $(g_1, g_2) \in X^2$ such that $|\mathrm{supp}(g_1) \cap \mathrm{supp}(g_2)| = 1$, then Y is definable without parameters, and M may be identified with an $\emptyset$-definable quotient of Y.

Constructions of a weak $\forall\exists$-interpretation have been given for various structures M. The idea in general is to identify a suitable conjugacy class C of automorphisms of M which have a single fixed point, and show that there is an appropriate formula $\phi(\bar{g}, x, y)$ in the language of (abstract) groups which holds precisely when automorphisms x and y (chosen in C) have the *same* fixed point. The problem is that, in general, this equivalence relation is easily definable in the language of permutation groups, but not in the language of abstract groups.

In [131], Rubin showed that $(\mathbb{Q}, <)$ and the universal homogeneous partial order have weak $\forall\exists$-interpretations, essentially by identifying a conjugacy class C of automorphisms with a single fixed point, for which 'has the same fixed point' is easily describable. For $(\mathbb{Q}, <)$, C consists of elements g with a single fixed point a, such that for any $b > a$ $\{g^n(b) : n \in \mathbb{Z}\}$ is coterminal in $\{z \in$

$\mathbb{Q} : a < z\}$, and likewise for any $b < a$, $\{g^n(b) : n \in \mathbb{Z}\}$ is terminal in $\{z \in \mathbb{Q} : z < a\}$. It can be shown that if $g, h \in C$, then they have the same fixed point if and only if $gh \in C$, a group-theoretically expressible property. Rubin also showed that binary homogeneous relational structures with a very simple form of amalgamation (analogous to freeness) have a weak $\forall\exists$-interpretation. Extending this, Singerman (unpublished) proved that an n-ary analogue of the universal homogeneous tournament has a weak $\forall\exists$-interpretation, as does the Cherlin-Hrushovski structure mentioned after Theorem 4.2.9. Note that the latter does *not* have the small index property.

Barbina [8] showed that infinite-dimensional projective spaces over a finite field, possibly equipped with a non-degenerate sesquilinear form, have weak $\forall\exists$-interpretations. Transvections play a key role.

In [9] it is shown that a large class of homogeneous relational structures M, including free monotone homogeneous structures, have a weak $\forall\exists$-interpretation. The key step is to show that, for any $c \in M$, if $\mathcal{D}_c$ is the complete metric space whose elements are those pairs of automorphisms of M which each have the unique fixed point c, then $\mathrm{Aut}(M)_c$, acting diagonally by conjugation, has a comeagre orbit on $\mathcal{D}_c$. Given this, let $\mathcal{D} := \bigcup_{c \in M} \mathcal{D}_c$. So $\mathcal{D} \subset \mathrm{Aut}(M) \times \mathrm{Aut}(M)$. Let C be the projection of $\mathcal{D}$ to the first coordinate. Then C is a conjugacy class of automorphisms of M each with a single fixed point. If $g, h \in C$, then g and h have the same fixed point if and only if there is $k \in C$ such that $(g, k), (h, k) \in \mathcal{D}$. This equivalence relation is expressible in the language of groups, and gives the weak $\forall\exists$-interpretation.

5.5 More on ample homogeneous generic automorphisms

We mention some other group-theoretic consequences of the existence of ample homogeneous generic automorphisms. For more on this material, see [93].

First, if G is any group which is not finitely generated, we define the *cofinality* $\mathrm{cf}(G)$ of G to be the smallest cardinal κ such that G is the union of a chain of length κ of proper subgroups. This invariant arose in connection with group actions on trees and Serre's property (FA) - see [135] and the end of this section.

Many ω-categorical structures are known to have automorphism groups with uncountable cofinality. We first note

Theorem 5.5.1 ([110]) *Let S be the full symmetric group on a set X of infinite cardinality κ. Then* $\mathrm{cf}(S) > \kappa$*.*

Sketch proof. One first shows that if $Y \subset X$ with $|Y| = |X \setminus Y| = \kappa$ (that is, Y is a *moiety* of X), and G is a subgroup of S such that every permutation of Y is induced by an element of G, then there is $h \in S$ such that $\langle G, h\rangle = S$. Now, suppose for a contradiction that $(G_i : i < \lambda)$ is a chain of proper subgroups of S of length $\lambda \leq \kappa$, with $\bigcup_{\mu<\lambda} G_\mu = S$; so $G_i \leq G_j$ for all i, j with $i < j < \lambda$. Choose a partition $(Y_\mu : \mu < \lambda)$ of X into moieties. By the above observation, for each $\mu < \lambda$ there is $g_\mu \in \mathrm{Sym}(Y_\mu)$ which is not induced by an element of

G_μ; for otherwise, there would be $g \in S$ with $\langle G_\mu, g\rangle = S$, and if $g \in G_\nu \geq G_\mu$ then $G_\nu = S$, contradicting that G_ν is a proper subgroup. Let g be the unique permutation of X which induces g_μ on Y_μ for each μ. Then $g \in S \setminus \bigcup_{\mu<\lambda} G_\mu$, which is a contradiction. □

The same result was proved for $\mathrm{Aut}(\mathbb{Q}, <)$ by Gourion by an analogous 'piecewise-patching' argument, and reproved in a broader context by Droste and Holland [52] . We also have the following extension of Theorem 5.5.1.

Theorem 5.5.2 *[81] Let M be an ω-categorical structure such that* $\mathrm{Aut}(M)$ *has ample $\mathcal{B}(M)$-generic automorphisms for some base $\mathcal{B}(M)$. Then* $\mathrm{cf}(\mathrm{Aut}(M)) > \aleph_0$.

In fact, in [93] it is shown that any Polish group with ample homogeneous generic automorphisms has uncountable cofinality.

There is another group-theoretic condition, which is closely related to cofinality (and possibly, at least in view of evidence from examples, to the small index property).

Definition 5.5.3 Let G be a group which is not finitely generated.

(i) The *strong cofinality* of G, denoted $\mathrm{scf}(G)$, is the smallest cardinal λ such that there is a chain $(U_i)_{i<\lambda}$ of proper subsets of G with $U_i \subseteq U_j$ for $i < j$, with $\bigcup(U_i : i < \lambda) = G$, and such that for each $i < \lambda$, $U_i = U_i^{-1}$ and $U_i.U_i \subseteq U_j$ for some $j \in \lambda$.

(ii) The group G has the *Bergman Property* if for each subset E of G such that $1 \in E = E^{-1}$ and $\langle E\rangle = G$, there is $k \in \mathbb{N}$ such that $E^k = G$, where $E^k := \{x_1...x_k : x_1, \ldots, x_k \in E\}$.

(iii) The group G has the *k-Bergman property* if there is a positive integer k such that for any chain $(U_i)_{i<\omega}$ of subsets of G, ordered by inclusion, such that $G = \bigcup(U_i : i < \omega)$, there is $n \in \mathbb{N}$ such that $U_n^k = G$.

Condition (i) was introduced in [52], and condition (ii) by Bergman [12], where it was shown that the symmetric group on any infinite set has the Bergman property. Easily, condition (iii) implies that G has uncountable strong cofinality. In addition, we have

Proposition 5.5.4 *[52] Let G be a group which is not finitely generated. Then G has uncountable strong cofinality if and only if G has the Bergman property and has uncountable cofinality.*

Droste and Göbel [53] have given sufficient conditions for a group G to have uncountable cofinality or to have the Bergman property. These conditions are applicable mainly, it seems, to permutation groups with good piecewise-patching properties. As with the small index property, ample homogeneous generics provide another approach, by the following result.

Theorem 5.5.5 *[93] Let M be an ω-categorical structure which has ample homogeneous generic automorphisms. Then* $\mathrm{Aut}(M)$ *has the 21-Bergman property, so has uncountable strong cofinality and the Bergman property.*

There is an extensive discussion of these conditions in [93], much of it in the more general context of Polish groups. We note the following.

Definition 5.5.6 *(Serre[135])* (i) A group G is said to have property (FA) if whenever G acts on a tree (i.e. a connected graph without cycles) without inversions (so no element of G reverses an edge) then G has a global fixed vertex, i.e., there is a vertex v of T such that for all $g \in G$, $g(v) = v$.

(ii) A group G has property (FH) if any action of G by isometries on a real Hilbert space has a global fixed point.

By a well-known result from [135], a group G which is not countable has property (FA) if and only if it satisfies all three of the following properties:

(i) G is not a non-trivial free product with amalgamation,

(ii) $\mathbb{Z}$ is not a homomorphic image of G,

(iii) $\mathrm{cf}(G) > \aleph_0$.

Theorem 5.5.7 *[93] Let M be an ω-categorical structure with ample homogeneous generic automorphisms. Then* $\mathrm{Aut}(M)$ *has properties (FA) and (FH).*

Property (FA) follows from the above characterisation, together with Theorem 5.5.2 and Proposition 4.2.12 (i),(v). To obtain (FH), first note that, by [44], a group G has uncountable strong cofinality if and only if every action by isometries on a metric space has a bounded orbit. There is extensive further recent work in this area, which we cannot cover here.

6 Further topics

6.1 Jordan groups and treelike structures

This section concerns a very old topic in permutation group theory (initiated by Jordan), for which substantial progress was made recently. Since many of the examples are homogenizable in th sense of Section 3.1, and since the treelike structures which arise keep recurring in this paper, it seems to deserve treatment here. For a detailed study of the treelike structures, see [4] and also [28], and for an overview of this subject see [14].

Definition 6.1.1 If G is a permutation group on a set X, then a subset Y of X is a *Jordan set* if $|Y| > 1$ and $G_{(X \setminus Y)}$ is transitive on Y. We say that Y is *improper* if there is $k \in \mathbb{N}$ such that $|X \setminus Y| \leq k$ and G is $(k+1)$-transitive on X, and that Y is *proper* otherwise. A transitive permutation group with a proper Jordan set is called a *Jordan group.*

Mainly, we shall be interested in *primitive* Jordan groups.

Example 6.1.2 We describe some important families of relational structures whose automorphism groups are Jordan groups.

1. If V is an n-dimensional vector space over F and W is a proper subspace of V, then $\mathrm{GL}_n(F)_{(W)}$, acting on V, is transitive on $V \setminus W$. Indeed, if $v_1, v_2 \in V \setminus W$, we may pick a basis $\bar{w}$ of W, and extend $\bar{w}v_1$ and $\bar{w}v_2$ to ordered bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of V. Then the unique element $g \in \mathrm{GL}_n(F)$ taking $\mathcal{B}_1$ to $\mathcal{B}_2$ fixes W pointwise and maps v_1 to v_2. The group $\mathrm{GL}_n(F)$ is intransitive on V (it fixes 0), and (assuming $|F| > 2$) is imprimitive on $V \setminus \{0\}$ (which is partitioned into 1-spaces). However, both the projective group $\mathrm{PGL}_n(F)$ on $\mathrm{PG}_{n-1}(F)$ and the affine group $\mathrm{AGL}_n(F)$ on $\mathrm{AG}_n(F)$ are 2-transitive Jordan groups; likewise in the infinite-dimensional case.

2. If $G := \mathrm{Aut}(\mathbb{Q}, <)$ and U is an open proper interval of $\mathbb{Q}$ (or any infinite proper convex open subset) then U is a Jordan set for G. To see this, note that *any* order automorphism of $(U, <)$, extended by the identity elsewhere, lies in G. There are similar Jordan sets for automorphism groups of linear betweenness relations, circular orders, and separation relations (see Theorem 6.2.1 below).

3. Let $(T, \leq)$ be one of the 2-homogeneous trees of Droste [48]. These are all countable lower semilinear orders such that each maximal chain is isomorphic to $(\mathbb{Q}, <)$. These are classified by two parameters. One parameter indicates whether or not each *ramification point* (i.e. infimum in the Dedekind-MacNeille completion of two incomparable elements) lies in T; we say T has *positive type* if it contains its ramification points, and *negative type* otherwise. For each ramification point a, there is an equivalence relation E_a on $\{x \in T : a \leq x\}$, where xE_ay if and only if there is $z \in T$ with $a < z \leq x, y$. By 2-homogeneity, the number of equivalence classes, which belongs to $\{n \in \mathbb{N} : n > 1\} \cup \{\aleph_0\}$, is independent of a. It is the second parameter for T, called the 'ramification number'. The E_a-classes, called *cones at* a, are isomorphic to $(T, \leq)$, and are Jordan sets. In fact, if a is a ramification point then any union of cones at a is Jordan set. The group G acts primitively on T but not 2-transitively or even 2-homogeneously as a permutation group, although the trees themselves are 2-homogeneous.

These trees are all *homogenizable*, in the sense of Section 3.1. For example, consider any 2-homogeneous tree of infinite ramification order and negative type. This can be viewed as a reduct of a Fraissé limit as follows. Let L be a language with a binary relation $<$ and a ternary relation R. Let $\mathcal{C}$ be the class of all finite L-structures $(A, <, R)$ such that $(A, <)$ is a semilinear order, the relation R is irreflexive, $R(x, y, z)$ implies that x, y, z are incomparable, and for any incomparable $x, y, z \in A$, if there is $w \in A$ incomparable to z with $w < x, y$, then $R(x, y, z)$. It can be verified that $\mathcal{C}$ is an amalgamation class, and $(T, <, R)$ is its Fraissé limit.

4. Let $(T, \leq)$ be one of the 2-homogeneous trees in (3), and let M be the set of maximal chains in T, that is, maximally totally ordered subsets of T. We view M as a structure with a single ternary relation C as follows. Define $C(\alpha; \beta, \gamma)$

to hold if and only if $\alpha \cap \beta$ is a proper subset of $\beta \cap \gamma$. Put $H := \mathrm{Aut}(M, C)$. Then H acts 2-transitively on M, and there are several kinds of Jordan sets. Indeed, as in (3) there is a notion of 'cone' – if $a \in T$, then the cones of M at a are the classes of the natural equivalence relation on the set of maximal chains which contain a. If $a \in T$ and Y is a cone at a, then $(Y, C) \cong (M, C)$, so admits a 2-transitive automorphism group; and any automorphism of (Y, C) can be extended by $\mathrm{id}_{M \setminus Y}$ to an automorphism of (M, C), so Y is a Jordan set. Again, the union of any set of cones at a is a Jordan set. Furthermore, $\{x \in M : a \notin x\}$ is also a Jordan set for H.

Of course, $|M| = 2^{\aleph_0}$, but M has a countable elementary substructure (N, C) whose automorphism group is a Jordan group with similar transitivity properties. This can be proved by the downward Löwenheim-Skolem Theorem (in a 2-sorted language with a sort for M and a sort for the automorphism group). Alternatively, we may simply choose N to be a countable *dense* subset of M, that is, ensure that

$$\forall x \in T \exists \alpha \in N (x \in \alpha).$$

Alternatively, again, observe that the target structure (N, C) is homogeneous, so can be built as a Fraissé limit.

In [4], the following axioms for a C-relation are identified, analogous to those for dense linear orders.

(C1) $\forall x, y, z (C(x; y, z) \to C(x; z, y))$;

(C2) $\forall x, y, z (C(x; y, z) \to \neg C(y; x, z))$;

(C3) $\forall x, y, z, w (C(x; y, z) \to (C(x; y, w) \vee C(w; y, z)))$;

(C4) $\forall x, y (x \neq y \to \exists z C(x; y, z))$.

(C5) $\forall y, z \exists x C(x; y, z)$.

It is shown in [4, Theorem 12.4] that if (M, C) satisfies (C1)–(C5) then there is a tree $(T, \leq)$ such that M is a dense set of maximal chains of T, with C interpreted as above.

5. Let $(T, \leq)$ be a 2-homogeneous tree. Then there is a natural ternary *general betweenness relation* B on the elements of T. For details (and for axioms for a *general betweenness relation*) see [4, Section 15]. The group $\mathrm{Aut}(T, B)$ is a 2-transitive Jordan group.

6. Let (M, C) be a C-relation obtained from a 2-homogeneous tree $(T, \leq)$ as in (4) above. Put $M' := M \cup \{\infty\}$ where $\infty \notin M$. We may view the elements of M' as the *directions* of the general betweenness relation (T, B), rather like the ends of a tree in the sense of graph theory. We identify ∞ with the 'downwards' direction of $(T, \leq)$. There is a natural quaternary relation D on M'. Informally, $D(x, y; z, w)$ expresses that in the underlying general betweenness relation, the path from x to y (which, remember, are 'directions', or 'leaves') does not meet the path from z to w. The group $\mathrm{Aut}(M', D)$ is a 3-transitive but not 4-transitive Jordan group. Again, $|M'| = 2^{\aleph_0}$, but (M', D) has a countable elementary substructure which is homogeneous. For axioms for a D-relation, see [4, Section 22]. To see Jordan sets, observe that with (M', D)

as above, $\mathrm{Aut}(M', D)_\infty = \mathrm{Aut}(M, C)$, so has many Jordan sets.

There are connections between D-relations and ends of graphs – see e.g. Möller [120].

The main theorem of [2] is the following.

Theorem 6.1.3 *Let G be a primitive Jordan permutation group on an infinite set X, and suppose that G is not highly transitive. Then G is a group of automorphisms of a structure on X of one of the following types: a Steiner system (possibly with infinite blocks); a linear order, circular order, linear betweenness relation, or separation relation; a semilinear order; a general betweenness relation, or C-relation, or D-relation; or a limit of Steiner systems, general betweenness relations, or D-relations.*

Remark 6.1.4 1. The case of Steiner systems includes the projective and affine spaces in Example 6.1.2. Versions with an arbitrary finite degree of transitivity are constructed in [14], using a 'Hrushovski construction', a variant of Fraissé amalgamation. Other constructions, again with an arbitrary degree of transitivity, are given by Johnson in [89].

2. We do not give the definition of limits of Steiner systems, or of general betweenness relations and D-relations. Constructions of the last two were given in an unpublished manuscript of Adeleke. An ω-categorical structure (M, L) with L ternary (a 'limit of general betweenness relations') is constructed in [15] as a Fraissé limit. The automorphism group is 3-homogeneous, not 3-transitive, and the point stabiliser preserves a C-relation. Limits of general betweenness and D-relations seem to be important new examples of treelike structures, deserving further investigation.

A limit of Steiner systems is constructed by Adeleke in [1]. A version of the construction, with an arbitrary degree of transitivity, was provided by Johnson [89].

3. Theorem 6.1.3 was proved in [3] in the particular case when there is a *primitive* Jordan set, that is a proper Jordan subset Y of X such that $G_{(X\setminus Y)}$ acts primitively on Y. In this case, the only examples are the linear orders (and circular orders, linear betweenness relations, and separation relations), semilinear orders, general betweenness relations, and C and D-relations. Both in this case and in general, the proof is based on an analysis of the intersection properties of G-invariant families of Jordan sets. For example, if G is primitive on X and there is a G-invariant family $\mathcal{F}$ of subsets of X any two of which are comparable under inclusion, then there is a G-invariant linear order on X: define

$$x < y \Leftrightarrow (\exists A \in \mathcal{F})(x \in A \wedge y \notin A).$$

An important observation is that if A and B are Jordan sets and $A \cap B \neq \emptyset$, then $A \cup B$ is also a Jordan set.

I mention two open problems.

1. Give a full classification of (homogenizable) ω-categorical structures whose automorphism group is a primitive Jordan group.

2. Describe primitive structures M which are homogeneous over a finite relational language, and have the property that there is an infinite coinfinite subset A of M such that for all $x, y \in M \setminus A$, $\mathrm{id}_A \cup \{(x, y)\}$ is an isomorphism.

6.2 Reducts of omega-categorical structures

Suppose that M is an ω-categorical structure, and $G = \mathrm{Aut}(M)$. Consider structures M' with the same domain as M, such that the $\emptyset$-definable relations on M' are $\emptyset$-definable in M. We call such structures M' *reducts* of M, and identify two reducts if they have the same $\emptyset$-definable relations. This slightly extends the traditional use of the term, which would require that M' is obtained from M by restricting the language. It is clear that if M' is a reduct of M then $\mathrm{Aut}(M) \leq \mathrm{Aut}(M') \leq \mathrm{Sym}(M)$, and $\mathrm{Aut}(M')$ is closed. In fact, by the Ryll-Nardzewski Theorem, this is a Galois correspondence: the lattice of reducts of M (with the natural lattice operations) is isomorphic (after the lattice operations $\wedge$ and $\vee$ are swapped) to the lattice of closed subgroups of $\mathrm{Sym}(M)$ with contain $\mathrm{Aut}(M)$. We shall refer to the latter as *closed supergroups* of $\mathrm{Aut}(M)$.

It is challenging, and generally requires interesting techniques, to *classify* the reducts for certain M. We describe below some results. This topic has additional recent motivation through work of Bodirsky and coauthors on constraint satisfaction – see Section 6.6.

Theorem 6.2.1 (Cameron) *Let $G = \mathrm{Aut}(\mathbb{Q}, <)$. Then the proper non-trivial reducts of $(\mathbb{Q}, <)$ are the linear betweenness relation on $\mathbb{Q}$, the countable dense circular order induced from the linear order, and the countable dense separation relation.*

Proof. Since $\mathrm{Aut}(\mathbb{Q}, <)$ is highly homogeneous, this follows from Cameron's classification of highly homogeneous closed groups acting on a countable set (see Example 2.3.1(1) above). $\square$

Certain model-theoretic/group-theoretic properties are preserved (apart from obvious exceptions) under reducts, and are well-understood in the ω-categorical case. For example, a *strictly minimal set* is an ω-categorical structure M with primitive automorphism group such that any parameter-definable subset of M is finite or cofinite in M; equivalently, $\mathrm{Aut}(M)$ is primitive oligomorphic and for any finite $A \subset M$, $\mathrm{Aut}(M)_{(A)}$ has a cofinite orbit on M. Strictly minimal sets are the building blocks of ω-categorical ω-stable theories and have been classified. Indeed, the following major result is proved in [148, Theorem 3.0.1] and in [40], with other proofs appearing later.

Theorem 6.2.2 *Let M be a strictly minimal set, and $G = \mathrm{Aut}(M)$. Then one of the following holds.*

(i) $G = \mathrm{Sym}(M)$*;*

(ii) for some prime power q*,* $\mathrm{PGL}(\aleph_0, q) \leq G \leq \mathrm{P\Gamma L}(\aleph_0, q)$*, acting on projective space* $PG(\aleph_0, q)$*;*

(iii) for some prime power q*,* $\mathrm{AGL}(\aleph_0, q) \leq G \leq \mathrm{A\Gamma L}(\aleph_0, q)$*, acting on affine space* $\mathrm{AG}(\aleph_0, q)$*.*

Using this, it is feasible to characterise reducts of strictly minimal sets; for example, the projective and affine spaces $\mathrm{PG}(\aleph_0, p)$ and $\mathrm{AG}(\aleph_0, p)$ (where p is prime) have no proper non-trivial reducts.

In [143, Example 1.2], Thomas considered a variant of these examples, also ω-stable (and, unlike the above strictly minimal sets, also homogenizable). Fix an integer $k > 1$, and let Γ be the graph whose vertices are the k-element subsets of $\mathbb{N}$, with vertices A and B adjacent if and only if $|A \cap B| = k - 1$. This graph has totally categorical theory, of Morley rank k and Morley degree 1. An easy model-theoretic argument yields that Γ has no proper non-trivial reducts.

Similarly, if $G = \mathrm{Aut}(M)$ is a Jordan group and H is a closed supergroup of G, then except in trivial cases H will also be a Jordan group. Using the description of primitive Jordan groups in [2], this makes a full description of closed supergroups of Jordan groups feasible, at least in certain cases. We do not pursue this here, but for example, by Theorem 6.1.3 (or the results in [3]), the countable homogeneous D-relations and the general betweenness relations considered in Example 6.1.2(6) will have no proper non-trivial reducts.

Thomas [142, 143] has used structural Ramsey theory to classify reducts of certain homogeneous structures, and variants of his methods seem to have great further potential, using ideas from [22] and from Sections 6.5 and 6.6 below.

We first describe some reducts of the random graph R, first noted by Cameron. Let $G = \mathrm{Aut}(R)$.

(i) The graph R is isomorphic to its complement R^c. If $h : R \to R^c$ is an isomorphism, let $D(R) := \langle G, h \rangle$. This is a closed supergroup of G, preserving R up to complementation, and $|D(R) : G| = 2$. Note that $D(R)$ is 2-transitive but not 3-transitive.

(ii) Define a 3-hypergraph H with vertex set R, so that a 3-set $\{x, y, z\}$ is an edge of H if and only if it contains an odd number of graph-edges of R. Then H is a *two-graph*, that is, a 3-hypergraph such that every 4-set contains an even number of 3-edges (see Example 2.3.1(4)). In fact, H is the *universal homogeneous 3-hypergraph*. We define $S(R) := \mathrm{Aut}(H)$. Then $S(R)$ is 2-transitive, not 3-transitive.

(iii) The two-graph H is isomorphic to its complement. Let $k : H \to H$ be an anti-isomorphism of H witnessing this, and put $B(R) := \langle S(R), k \rangle$. Then $B(R)$ is 3-transitive but not 4-transitive.

Theorem 6.2.3 *(i) Any closed supergroup of* $\mathrm{Aut}(R)$ *is equal to one of:* $\mathrm{Aut}(R)$*,* $D(R)$*,* $S(R)$*,* $B(R)$*, or* $\mathrm{Sym}(R)$*.*

(ii) For any $n \geq 3$*, the generic* K_n*-free graph has no proper non-trivial*

reducts.

Note that the lattice of reducts of $(\mathbb{Q}, <)$ is isomorphic to the lattice of reducts of R. This reinforces the analogy between the circular order and the homogeneous two-graph mentioned in Examples 2.3.1(4).

The key tool in the original proof of Theorem 6.2.3 is the following result of Nešetřil and Rödl.

Theorem 6.2.4 *(i) Let A be a finite graph, and r a positive integer. Then there is a finite graph B such that if the edges of B are coloured with r colours, then B contains an induced subgraph A' isomorphic to A, such that all edges of A' have the same colour.*

(ii) For any integer $n \geq 3$, if A is K_n-free, then B can be chosen to be K_n-free.

A slightly different approach to such problems was developed by Bennett and Thomas in [11, 143]. In [11], classifications are given of the reducts of the homogeneous tournaments, and the reducts of generic k-coloured graphs. Then, in [143], the reducts of the homogeneous k-uniform hypergraphs are classified. I briefly describe the latter. The methods in [11] are similar.

First, there is a notion of 'switching' which generalises all the proper non-trivial reducts of the random graph. Let X, Y are k-hypergraphs, and let A be a i-subset of X, for some i with $0 \leq i \leq k-1$. Then a bijection $\pi : X \to Y$ is a *switch with respect to A* if for every k-subset B of X, $\pi|_B$ is an isomorphism if and only if $A \not\subseteq B$. So an anti-isomorphism is just a switch with respect to $\emptyset$.

Theorem 6.2.5 *Let Γ_k denote the universal homogeneous k-hypergraph, and let G be a closed permutation group such that $\mathrm{Aut}(\Gamma_k) \leq G < \mathrm{Sym}(\Gamma_k)$. Then there is $X \subseteq \{0, 1, \ldots, k-1\}$ such that G is generated (as a topological group), by $\mathrm{Aut}(\Gamma_k)$ together with all $\pi \in \mathrm{Sym}(\Gamma_k)$ such that, for some $i \in X$, π is a switch with respect to an i-subset of Γ_k.*

In the special case when $k = 2$ (so $\Gamma_k = R$), the group $S(R)$ arises by putting $X = \{1\}$. This arises from the well-known correspondence between two-graphs and switching classes.

The proof of Theorem 6.2.5 uses an analogue of Theorem 6.2.4 due independently to Abramson and Harrington, and to Nešetřil and Rödl [121]. It also uses Theorem 3.2.4 above.

It follows from Theorem 6.2.5 that Γ_k has finitely many reducts, and that each is homogeneous over a finite relational language. Thomas has conjectured that every homogeneous structure over a finite relational language has just finitely many reducts. An example is given in [142] of a homogeneous structure over a finite relational language which has a reduct which is not homogeneous over any finite relational language.

In the direction of Thomas's conjecture, we mention also the paper [90] by Junker and Ziegler. This contains a new proof of Cameron's Theorem, and also a proof that any homogeneous expansion of $(\mathbb{Q}, <)$ by finitely many convex unary predicates has finitely many reducts. It is also shown that the expansion of $(\mathbb{Q}, <)$ by a single constant has 116 reducts, and results are also obtained for expansions by dense predicates.

6.3 Growth rates for orbits on subsets

We discuss here aspects of the following question, investigated very heavily in papers by Cameron – see [29] as a core reference.

Question 6.3.1 *If M is an ω-categorical structure with automorphism group G, what can be said about the sequence $(f_k(G))$, where $f_k(G)$ denotes the number of orbits of G on the collection of* unordered *k-subsets of M?*

By the Ryll-Nardzewski Theorem, the numbers $f_k(G)$ are all finite. So too are the numbers $F_k(G)$ of orbits of G on *ordered* k-sets, and $F_k^*(G)$ of orbits of G on M^k. Results on the relationships between these sequences can be found in [32] and [29], but we focus on the $f_k(G)$.

As noted in several papers by Cameron, many important combinatorial sequences arise in the form $(f_k(G))$ for $G = \mathrm{Aut}(M)$, where M is a homogeneous ω-categorical structure. For example, if M is the random graph, then $f_k(G)$ is the number of unlabelled k-vertex graphs up to isomorphism. More generally, if M is the Fraissé limit of the age $\mathcal{C}$, then $f_k(G)$ is the number of k-element members of $\mathcal{C}$ up to isomorphism. Sequences counting *labelled* isomorphism types can also in many cases arise. For example, let M be the universal totally ordered graph, the Fraissé limit of an age consisting (in a language with two binary relations) of all finite totally ordered graphs. Then $f_k(G)$ is the number of isomorphism types of k-element labelled graphs. If M is the homogeneous structure consisting of two (independent) total orders, then $f_k(G) = k!$. In fact, this construction generalises: if $\mathcal{C}$ is an amalgamation class with disjoint amalgamation, and $\mathcal{C}^+$ is the corresponding class of all *ordered* members of $\mathcal{C}$ (in a language with an additional relation symbol $<$), then $\mathcal{C}^+$ is an amalgamation class, and if M^+ denotes its Fraissé limit, then $f_k(\mathrm{Aut}(M^+))$ is the number of *labelled* members of $\mathcal{C}$ (and equals $F_k(\mathrm{Aut}(M))$. Other well-known sequences, such as the Catalan numbers [28] can also arise.

Cameron [27] and Pouzet [126] showed independently that the sequence $(f_k(G))$ is non-decreasing. Both arguments are Ramsey-theoretic. The approach in [27] is to consider an incidence matrix with rows indexed by orbits on k-sets, columns by orbits on $(k+1)$-sets, and show that the rank is equal to the number of rows.

Furthermore, Cameron noted that arbitrarily fast growth rate is possible for $(f_k(G))$: for any prescribed function $h : \mathbb{N} \to \mathbb{N}$, just do a Fraissé amalgamation, in a language where the number of relation symbols of arity k is greater than $h(k)$.

Cameron has proved a number of results on the *local* behaviour, for example recovering strong structural information on M from the assumption $f_k(G) = f_{k+1}(G) > 1$. One relevant tool for local behaviour is a graded algebra $A^G = \Sigma_{k\geq 0} V_k^G$, whose homogeneous degree k elements consist of the G-invariant functions from the collection of k-subsets of M to $\mathbb{Q}$. The addition and scalar multiplication is pointwise. If $f \in V_k^G$ and $g \in V_l^G$, we define $h = fg \in V_{k+l}^G$ as follows, where, for any structure P and positive integer k, $\binom{P}{k}$ denotes the set of k-element substructures of P:

$$\text{for any } A \in \binom{M}{k+l}, \quad h(A) = \Sigma_{B\in\binom{A}{k}} f(B)g(A\setminus B).$$

The dimension of the k^{th} graded component is then $f_k(G)$. Recently, Pouzet [127] settled an old conjecture of Cameron, proving part (i) of the following. Part (ii) is derived by Cameron from (i).

Theorem 6.3.2 *Let M be an ω-categorical structure, and $G = \mathrm{Aut}(M)$, and suppose that G has no finite orbits on M. Then*

(i) (Pouzet [127]) A^G is an integral domain,

(ii) (Cameron [31]) for any $k, l \in \mathbb{N}$, $f_{k+l}(G) \geq f_k(G) + f_l(G) - 1$.

In fact, Pouzet's theorem is in the more general setting of the *profile* of a relational structure (see below), without symmetry assumptions. Cameron conjectures a stronger version of (i) (under the same assumption), namely that the element $e \in V_1$ which has constant value 1 is prime in A^G. This also has a local consequence for $(f_k(G))$. Cameron has shown in some natural cases that A^G is a polynomial ring.

Given that $(f_k(G))$ is non-decreasing, it is natural to investigate the asymptotic growth rate. In this direction, we have the following.

Theorem 6.3.3 *Let M be an ω-categorical structure, and $G = \mathrm{Aut}(M)$.*

(i) (Pouzet [125]) Either there is an integer t and constants c_1 and c_2 such that for all k, $c_1k^t \leq f_k(G) \leq c_2k^t$, or $(f_k(G))$ grows faster than any polynomial.

(ii) [106] If G is primitive on M, then either $f_k(G) = 1$ for all k, or $(f_k(G))$ grows at least exponentially (more precisely, if $1 < c < 2^{\frac{1}{5}}$, then $f_k(G) > c^k$ for large enough k).

(iii) [107] Either $(f_k(G))$ is polynomially bounded above, or for all $\epsilon > 0$, there is K such that $f_k(G) > e^{k^{\frac{1}{2}-\epsilon}}$ for all $k > K$.

(iv) [109] If M has the independence property (see Definition 6.4.1 below), then there is a polynomial p of degree at least 2 such that $f_k(G) > 2^{p(k)}$ for all k.

(v) [109] If M is homogeneous over a finite relational language, and there is $\epsilon > 0$ such that $f_k(G) > 2^{k^{1+\epsilon}}$ for sufficiently large k, then M has the independence property.

Note that in (ii), if $f_k(G) = 1$ for all k, then G is highly homogeneous on M. If G is not highly transitive on M, then, by Theorem 6.2.1, M is essentially a linear order, a linear betweenness relation, a circular order, or a separation relation.

I make some remarks on the proofs. For (ii), the proof is by induction on the unique k such that G is k-transitive but not $(k+1)$-transitive: the idea is that in the action of G_x on $M \setminus \{x\}$, which is $(k-1)$-transitive but not k-transitive, the corresponding growth rate should by inductive hypothesis be exponential – this implies easily that $(f_k(G))$ grows exponentially. The main problem is to start the induction, i.e. to handle the case when G is primitive but not 2-transitive (some work is also needed in the case when G is 2-transitive but G_x is imprimitive on $X \setminus \{x\}$). If G is 2-homogeneous but not 2-transitive then there is an $\emptyset$-definable tournament structure on M, with arcs given by a G-orbit on M^2, and it is fairly easy to show that the number of non-isomorphic k-element subtournaments (which is a lower bound for $f_k(G)$) grows at least exponentially. In the harder case, G is primitive but not 2-homogeneous, and there is a G-invariant (so $\emptyset$-definable) graph on M whose edge set is a G-orbit on 2-sets. One must show that the sequence enumerating the number of non-isomorphic k-element subgraphs grows at least exponentially. This can be done by coding trees into isomorphism types. Primitivity is used as follows: consider an equivalence relation $\equiv$ on M, where vertices x, y are equivalent if their neighbour sets (in the invariant graph) have finite symmetric difference. By primitivity, this equivalence relation is trivial or universal, and the latter case is easily eliminated. If the $\equiv$-classes are singletons, one can code trees into graph isomorphism types by heavy use of Ramsey's Theorem.

Given (ii), to prove (iii) we may assume G is imprimitive on M. We just consider here the particular case when there is a G-invariant equivalence relation $\equiv$ on M with infinitely many classes, all infinite. In this case, $f_k(G) \geq p(k)$, where $p(k)$ is the number of partitions of the number k: indeed, for each partition $k = a_1 + \ldots + a_k$, where the a_i are positive integers, one may pick distinct $\equiv$-classes $C_1, \ldots, C_k$ and form a k-subset U of M consisting of a_i elements of C_i for each i; distinct partitions give k-sets lying in different G-orbits. In this case, (iii) follows, since by a well-known result (see e.g. [73]), $p(k) \sim \frac{1}{4k\sqrt{3}} \exp(\pi\sqrt{(2k/3)})$.

Observe that by Theorem 6.3.3, there are several gaps in possible growth rates. For example, there are gaps between different integer degrees of polynomials, and between polynomial growth and growth related to the partition function. Furthermore, if M is homogeneous over a finite relational language, then there is a gap between growth faster than some function $2^{k^{1+\epsilon}}$ and growth slower than every function $2^{p(k)}$ (p a polynomial of degree 2), by parts (iv) and (v). It would be interesting to investigate this gappiness more. For example, we have the following conjecture.

Conjecture 6.3.4 *Let M be ω-categorical with $G := \mathrm{Aut}(M)$, and suppose that $f_k(G)$ grows faster than polynomially but slower than the function $e^{k^{1-\epsilon}}$ (for some $\epsilon > 0$). Then there is n such that for any $\epsilon > 0$, and for any sufficiently large k,*

$$e^{k^{\frac{n}{n+1}-\epsilon}} < f_k(G) < e^{k^{\frac{n}{n+1}+\epsilon}}.$$

It would also be interesting to investigate further those *primitive* structures M for which the growth is no faster than exponential. The only known examples are either treelike (see [28]), or closely related to linear or circular orders or the structures $S(n)$ of Example 2.3.1. A structure closely related to the homogeneous tournament $S(2)$ (the 'local order') yields the slowest known growth rate for an ω-categorical structure with primitive but not highly homogeneous automorphism group. By a result of Cameron (see e.g. [29, P. 58]) one has

$$f_k(\mathrm{Aut}(S(2))) = \frac{1}{2k}\Sigma_{d|k,d \text{ odd}}\phi(d)2^{k/d},$$

where $\phi(d)$ is Euler's totient function. Thus, $f_k(\mathrm{Aut}(S(2))) \sim \frac{2^{k-1}}{k}$. There is an isomorphism α from $S(2)$ to the tournament obtained by reversing all the arcs. Hence, if $G := \langle \mathrm{Aut}(S(2)), \alpha\rangle$, then $|G : \mathrm{Aut}(S(2))| = 2$ and $f_k(G) \sim \frac{2^{k-2}}{k}$.

There are other possible extensions of Theorem 6.3.3. The conclusion of (iii) holds also, for an infinite graph Γ, for the sequence counting the number of non-isomorphic k-element induced subgraphs. It would be interesting to generalise this to other relational structures. In general, a substantial proportion of the work on $(f_k(G))$ generalises to what Pouzet calls the *profile* of an age: namely, the number of k-element members up to isomorphism. Most of the results of Pouzet on the profile are in this combinatorial context, without an assumed oligomorphic group action.

6.4 Further model-theoretic conditions: NIP and simple theories, o-minimality and variations.

Independence property, strict order property, simplicity

Beyond ω-categoricity, the only model-theoretic restriction we have considered so far is stability. Here we discuss some model-theoretic ideas related to stability, in the light of various examples. The conditions have a combinatorial flavour and provide dividing lines between first order theories, which seem meaningful for homogeneous structures.

Definition 6.4.1 Let T be a complete theory.

(i) We say that T has the *independence property* if there is a formula $\phi(\bar{x}, \bar{y})$ (where $l(\bar{x}) = r$ and $l(\bar{y}) = s$), some $M \models T$, and some $\{\bar{a}_i : i \in \mathbb{N}\} \subset M^r$ such that for every $S \subset \mathbb{N}$ there is $\bar{b}_S \in M^s$ such that for all $i \in \mathbb{N}$, $M \models \phi(\bar{a}_i, \bar{b}_S)$ if and only if $i \in S$. The theory T is said to be *NIP*, or *dependent*, otherwise.

(ii) The theory T has the *strict order property* if there is $M \models T$ such that for some $r \in \mathbb{N}$, there is a definable partial order on M^r with an infinite totally ordered subset.

Proposition 6.4.2 (Shelah [137]) *A theory T is unstable if and only if it has the independence property or the strict order property.*

Example 6.4.3 (i) The following ω-categorical structures have the independence property but not the strict order property: the random graph (and analogues, such as the random bipartite graph, random tournament, random digraph, random k-uniform hypergraph, random two-graph); the homogeneous K_n-free graph (for $n \geq 3$); the Henson digraphs; an infinite dimensional vector space over a finite field equipped with a symplectic bilinear form (see e.g. [42]).

(ii) The following ω-categorical structures have the strict order property but not the independence property: $(\mathbb{Q}, <)$; the 'local order' or 'circular tournament' $S(2)$ and other circular structures $S(n)$; the 2-homogeneous trees of Droste [48], and the related treelike objects considered in Section 6.1.

(iii) Structures which are ω-categorical and have both the independence property and the strict order property include the universal homogeneous partial order and the countable atomless Boolean algebra.

There has been extensive recent work on the model theory of NIP structures, motivated by many important examples (such as the real and p-adic fields). We remark that if T is NIP and $\phi(\bar{x}, \bar{y})$ is a formula, where $l(\bar{x}) = r$ and $l(\bar{y}) = s$, then the family

$$\{\{\bar{x} \in M^r : M \models \phi(\bar{x}, \bar{a})\} : \bar{a} \in M^s\}$$

is a family of subsets of M^r of finite Vapnik-Cervonenkis dimension, a notion emanating from statistics and with wide applications.

There is another generalisation of stability due to Shelah, orthogonal to the NIP property, namely the notion of a *simple* first order theory (or structure). I will not define this, but refer to [147]. Simple unstable theories will all have the independence property, but not the strict order property. The 'random' graph, digraph, k-hypergraph, bipartite graph, etc., are all simple but unstable. On the other hand, the generic triangle-free graph does not have simple theory, though its arity three analogue, the 3-hypergraph not embedding a 4-set all of whose triples are edges, does have simple theory. There is a suggestion of a connection between simplicity and the finite model property.

I mention one conjecture, connecting ω-categoricity to the above notions and also to Section 3.2. We say a first order theory T is *finitely axiomatised* if it is the deductive closure of finitely many sentences (and hence of a single sentence). In the ω-categorical world, dense linear orders without endpoints are an obvious example, and another is the countable atomless Boolean algebra (as is the universal homogeneous distributive lattice). Treelike structures arising from 2-homogeneous trees with finite ramification number have finitely axiomatised theories. More surprisingly, the universal homogeneous partially ordered set is an example [7]. On the other hand, structures with the finite model property, such as the random graph, clearly do not have finitely axiomatised theory. In [111], based partly on the above examples, the following conjecture was made. It was proved there for ω-categorical structures M with trivial algebraic closure, that is, with the property that for any $A \subset M$, $\mathrm{Aut}(M)_{(A)}$ has no finite orbits outside A (cf. Lemma 2.1.4 above). This has been extended by work of Ivanov and then by Lippel [105]. The conjecture remains open in general, even for structures homogeneous in a binary relational language (where it seems very feasible).

Conjecture 6.4.4 *Let M be an ω-categorical structure such that* $\mathrm{Th}(M)$ *is finitely axiomatised. Then* $\mathrm{Th}(M)$ *has the strict order property.*

o-minimality and variations

A very active branch of model theory in recent years has been o-minimality. A first-order structure $(M, < \ldots)$ is *o-minimal* if $<$ is a total order on M, and every first-order definable subset of M is a finite union of open intervals and singletons. It was shown in the initial development of o-minimality [124] that there are no interesting ω-categorical o-minimal structures. Essentially, they are built from $(\mathbb{Q}, <)$ using definable order preserving or reversing bijections, in very limited ways.

However, there is a generalisation of o-minimality for which there are interesting ω-categorical (and homogeneous over a finite relational language) examples. A totally ordered structure $(M, <)$ is *weakly o-minimal* if every (parameter) definable subset of M is a finite union of convex sets, not necessarily with endpoints in M. In general, weak o-minimality, unlike o-minimality, is not preserved under elementary equivalence, but in the ω-categorical case it is.

One example of an ω-categorical weakly o-minimal but not o-minimal structure is a totally ordered set equipped with an equivalence relation whose classes are convex, such that the quotient order on the set of classes is also convex. This structure has the same automorphisnm group as two of the countably infinite homogeneous permutations classified by Cameron (see Section 2.2). The construction can clearly be iterated, with a finite nested chain of equivalence relations with convex classes. If M is ω-categorical weakly o-minimal with transitive but imprimitive automorphism group, then the invariant *binary* relations have to be of this form [79, Theorem 2.2].

A more subtle construction combines a C-relation (see Example 6.1.2(4)) with a total order in the natural way, so that cones of the C-relation are convex in the ordering. Such constructions, called below *ordered C-relations*, were noted in [28] and considered in more detail in [79]. If the constructions of such M are done in a sufficiently regular way, then $\mathrm{Aut}(M)$ will be 2-homogeneous on M (so primitive), cones will be Jordan sets, and the growth rate of the sequence $(f_k(\mathrm{Aut}(M)))$ (in the sense of the previous section) will be exponential but no faster than exponential. (In fact, it is for one of these structures that Cameron

[28] noted that $(f_k(\mathrm{Aut}(M)))$ is the sequence of Catalan numbers.) Apart from the axiomatisation and construction of various ordered C-relations, the main result from [79] is:

Theorem 6.4.5 *Let $(M, <, \ldots)$ be an ω-categorical weakly o-minimal structure with primitive automorphism group. Then* $\mathrm{Aut}(M)$ *is 2-homogeneous, and either* $\mathrm{Aut}(M)$ *is highly homogeneous on M, or* $\mathrm{Aut}(M)$ *preserves an ordered C-relation on M.*

In the proof, the possibilities for the system of invariant relations of arity at most three are completely described. Essentially, it consists of a finite sequence of 'nested' C-relations. It is shown that there are $2^{\aleph_0}$ ω-categorical weakly o-minimal structures with non-isomorphic (2-homogeneous) automorphism groups.

From the point of view of automorphism groups, it is rather natural to extend the notion of weak o-minimality to circular orders. If K is a circular order on a set M (so K has arity three), there is a natural notion of convex subset of M with respect to K. One then says that a circularly ordered structure $(M, K, \ldots)$ is *weakly circularly minimal* (*wcm*) if any definable subset of M is a finite union of convex sets. This notion was introduced in [94], where ω-categorical wcm structures are investigated. The connection to weak o-minimality is immediate: if M is wcm and $a \in M$, then there is an a-definable total order on $M \setminus \{a\}$, and the induced structure on $M \setminus \{a\}$ is weakly o-minimal; so the main interest is in issues of multiple transitivity and homogeneity, and the role of parameters matters. The main results in [94] are:

Theorem 6.4.6 *Let $(M, K, \ldots)$ be an ω-categorical wcm structure, and assume that* $\mathrm{Aut}(M)$ *acts primitively on M.*

(i) If $\mathrm{Aut}(M)$ *is not 2-transitive on M, then either there is an $\emptyset$-definable (so* $\mathrm{Aut}(M)$*-invariant) total order $<$ on M with respect to which M is weakly o-minimal, or the invariant binary structure on M is that of the circular structure $S(n)$ for some n.*

(ii) If $\mathrm{Aut}(M)$ *is 2-transitive, then it is 3-homogeneous (so there is no invariant C-relation on M), and either* $\mathrm{Aut}(M)$ *is highly homogeneous, or it is not 6-homogeneous (and the possible invariant relations of arity at most 4 are known).*

In (ii), the invariant quaternary structure essentially consists of a D-relation compatible with a circular order, though more generally one can have a sequence of finitely many 'nested' D-relations, compatible with the circular order.

We may view these theorems as results about oligomorphic closed groups of automorphisms of structures with a linear or circular ordering, with the following property: any orbit of the pointwise stabiliser of a finite set is convex. The main structures considered above (at least those where one considers the invariant ternary structure in the weakly o-minimal case, or the invariant quaternary structure in the wcm case) are all homogeneous in a finite relational language. For all such structures, the growth rates (see the previous section) are no faster than exponential, and the automorphism groups are Jordan groups. We remark that ω-categorical weakly o-minimal and wcm structures are NIP but do not have simple theory.

6.5 Structural Ramsey theory and topological dynamics

We briefly describe here some of the work of Nešetřil and coauthors on Ramsey classes, and its applications in [92]. This is a rich subject, only touched on here. For an overview, the survey [122] is helpful.

Fix a finite relational language L, and a class $\mathcal{C}$ of finite L-structures. Given $A, B \in \mathcal{C}$, let $\binom{B}{A}$ denote the set of substructures of B which are isomorphic to A. For $A, B, C \in \mathcal{C}$ and positive integer k, we write $C \to (B)^A_k$ if the following holds: for every partition $\binom{C}{A} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k$, there is $B' \in \binom{C}{B}$ and $i \in \{1, \ldots, k\}$ such that $\binom{B'}{A} \subset \mathcal{A}_i$. Theorem 6.2.4(i) says (with $\mathcal{C}$ the class of finite graphs) that if A is an edge, then for every finite graph B and k there is a finite graph C such that $C \to (B)^A_k$.

Definition 6.5.1 The class $\mathcal{C}$ above is a *Ramsey class* if for every $A, B \in \mathcal{C}$ and positive integer k, there is $C \in \mathcal{C}$ such that $C \to (B)^A_k$.

We shall only consider this notion for classes of totally ordered structures, so some relation symbol is always interpreted by a total order. Some such assumption is needed. The kinds of problems which arise without an ordering are indicated for example in [122, Theorem 5.1].

The following result, which has a short proof, makes the connection to homogeneous structures.

Proposition 6.5.2 *[84, Theorem 1.2] Let $\mathcal{C}$ be an age of ordered structures over a finite relational language, and suppose that $\mathcal{C}$ is a Ramsey class. Then $\mathcal{C}$ has the amalgamation property.*

If $\mathcal{C}$ is a class of finite L-structures, we let $(\mathcal{C}, \leq)$ denote the class of all structures $(A, \leq)$, where $A \in \mathcal{C}$ and $\leq$ is a total ordering (interpreting a binary relation symbol not in L). The following key theorem has a sketch proof in [84], resting on results in [121]. In fact, one does not need that $(\mathcal{C}, \leq)$ consists of *all* orderings of members of $\mathcal{C}$ – it is sufficient to work with a class of 'admissible' orderings.

Theorem 6.5.3 *Let $\mathcal{C}$ be a monotone class of finite structures over the relational language L, and suppose that $\mathcal{C}$ is closed under isomorphism and has (JEP). Then the following are equivalent.*

(i) The class $(\mathcal{C}, \leq)$ is a Ramsey class.

(ii) $\mathcal{C}$ is a free amalgamation class.

The role of the 'monotone' assumption above suggests the following question.

Problem 6.5.4 *Find a direct connection between Corollary 5.3.7 and Theorem 6.5.3.*

In the remarkable paper [92], Kechris, Pestov and Todorcevic exhibited a connection between these notions and topological dynamics. Let G be a Hausdorff topological group. A *G-flow* is a continuous action of G on a compact Hausdorff topological space. Every G-flow contains a *minimal* subflow, that is, one in which each G-orbit is dense. Furthermore, each such G has a (unique up to isomorphism) *universal minimal G-flow*, denoted $M(G)$, namely, a minimal G-flow which maps homomorphically onto any other minimal G-flow. The group G is *extremely amenable* if $M(G)$ is a singleton; that is, if every G-flow has a fixed point (a point x such that for all $g \in G$, $g(x) = x$). History and motivation for these notions are given at the start of [92]. In particular, it seems that until recently there was a dearth of examples of extremely amenable groups.

In [92] the following theorem is proved. An *order class* is just a class of finite L^+-structures $(A, \leq)$ where A is an L-structure, $L^+ = L \cup \{\leq\}$, and $\leq$ is a linear order on A.

Theorem 6.5.5 *[92] Let G be a closed permutation group on a countably infinite set M. Then G (as a topological group) is extremely amenable if and only if $G = \mathrm{Aut}(\mathcal{M})$ for some structure $\mathcal{M}$ which is the Fraissé limit of a Fraissé order class which is a Ramsey class.*

Note in particular that G preserves a linear order on M. It is easy to see that every extremely amenable closed permutation group on a countably infinite set X preserves a linear ordering. For G has a continuous action on the space of all linear orderings on X (a compact topological space, when parsed as $\{0,1\}^{X^2}$ with the product topology) so there must be a fixed point, i.e., an invariant linear ordering.

In [92], the authors compute the universal minimal G-flow for various homogeneous structures. For the random graph, it is the action on the space of all orderings of the graph. On the other hand, for some other structures which are not monotone, there are restrictions on the invariant total order. If M is the disjoint union of infinitely many infinite complete graphs (i.e. an equivalence relation with infinitely many classes, all infinite), then the universal minimal G-flow is the action on the space of all linear orderings of M such that each equivalence class is convex.

Throughout the paper, we have collected a number of consequences of various restrictions on amalgamation. We summarise these in the following two theorems.

Theorem 6.5.6 *Let L be a finite relational language, and let M be a free homogeneous L-structure, with $G = \mathrm{Aut}(M)$. Then* *(i) G has a comeagre conjugacy class,*

(ii) G has no proper normal subgroup of countable index,

(iii) if G acts without inversions on a combinatorial tree T, then every element of G fixes a vertex of T,

(iv) G cannot be written non-trivially as a free product with amalgamation,

(v) if G is transitive on M and not equal to $\mathrm{Sym}(M)$*, then G is simple.*

Proof. For (i) see Theorem 4.2.11, for (ii)–(iv) see Theorem 4.2.12, and for (v) see Theorem 4.2.7.

Theorem 6.5.7 *Let L be a finite relational language, and let M be a monotone free homogeneous L-structure, with* $G = \mathrm{Aut}(M)$*. Then the conclusions of Theorem 6.5.6 hold, and in addition*

(i) G has ample homogeneous generic automorphisms,

(ii) M has the small index property, and G has uncountable cofinality and the Bergman property,

(iii) G has properties (FA) and (FH),

(iv) M has a weak $\forall\exists$*-interpretation,*

(v) if $\mathcal{C} = \mathrm{Age}(M)$*, and* $\mathcal{C}^+$ *is the order class obtained from* $\mathcal{C}$ *(in a language with an extra binary relation* $<$*) by ordering members of* $\mathcal{C}$ *in all possible ways, then* $\mathcal{C}^+$ *is a Ramsey class,*

(vi) if M^+ *is the Fraissé limit of* $\mathcal{C}^+$*, then* $\mathrm{Aut}(M^+)$ *is extremely amenable.*

Proof. (i) and the first part of (ii) are given by Corollary 5.3.7, and Theorems 5.5.2 and 5.5.5 yield the rest of (ii). For (iii) see Theorem 5.5.7, and for (iv), see the last paragraph of Section 5.4. Parts (v) and (vi) follow from Theorems 6.5.3 and 6.5.5 respectively. □

Note that in the second theorem we obtain three fixed point properties, namely (FA), (FH), and extreme amenability. I believe it is consistent with the known examples that 'monotone' can be omitted in the second theorem above (at least for (i)–(iv)).

6.6 Connections to constraint satisfaction

I briefly sketch some interesting recent connections of homogeneous structures and ω-categoricity to *constraint satisfaction*, a topic in complexity theory. This work is due to Bodirsky and his coauthors. There is an initial presentation in [17], and some results are surveyed in [19].

Recall from the introduction the notion of a *homomorphism* $\pi : M \to N$, where M, N are relational structures. An *endomorphism* of M is a homomorphism $M \to M$, and an *embedding* $M \to N$ is an injective homomorphism $\pi : M \to N$ which is *strong*, meaning that the inverse map $\pi^{-1} : \pi(M) \to M$ is also a homomorphism. A *self-embedding* of M is an embedding $M \to M$. We will write $\mathrm{End}(M)$ for the monoid of endomorphisms of M, and $\mathrm{Emb}(M)$ for the monoid of self-embeddings of M.

Fix a finite relational language L, and an L-structure M. The *constraint satisfaction problem with template* M, denoted $\mathrm{CSP}(M)$, asks, for any finite L-structure P (as input) whether there is a homomorphism $P \to M$. This problem is considered from the viewpoint of complexity theory.

There is another way to present $\mathrm{CSP}(M)$. We say that an L-formula $\phi(\bar{x})$ is *positive primitive* (or *p.p.*) if it has the form $\exists \bar{y} \psi(\bar{x}, \bar{y})$, where ψ is a conjunction of atomic formulas. Now $\mathrm{CSP}(M)$ can be viewed as asking, for any positive primitive *sentence* σ (as input), whether $M \models \sigma$. To see the connection (in one direction), let P be a finite structure with domain $\{a_1, \ldots, a_n\}$, and let $\psi(a_1, \ldots, a_n)$ be a conjunction of all the (distinct) atomic formulas true of P. Then there is a homomorphism $P \to M$ if and only if $M \models \exists x_1 \ldots \exists x_n \psi(x_1, \ldots, x_n)$.

It is thus clear that many natural questions in complexity theory can be posed as constraint satisfaction problems. A remarkable underlying conjecture states that if M is finite, then $\mathrm{CSP}(M)$ is either in P or is NP-complete. Considerable work, including a universal-algebraic approach via polymorphism clones, has been devoted to this.

In the context of constraint satisfaction, a key definability notion is p.p. definability, or at least positive existential definability (or interpretability). Indeed, we have the following – for a proof see for example [19, Proposition 3].

Proposition 6.6.1 *Let M and N be structures over finite relational languages. If there is a p.p. interpretation of N in M, then there is a polynomial-time reduction of CSP(N) to CSP(M).*

In [17, 18, 19], the authors give several very natural computational problems which can be parsed as a constraint satisfaction problem with ω-categorical (in fact, homogeneous) template, but not with finite template. For example, the problem of deciding whether a finite graph is triangle-free can be viewed as a constraint satisfaction problem with template R_3 (the universal homogeneous triangle-free graph). A finite digraph has a directed cycle if and only if it does not map homomorphically to $(\mathbb{Q}, <)$, so testing existence of a directed cycle reduces to $CSP((\mathbb{Q}, <))$. The problem of deciding whether, given a directed graph Γ, its vertex set can be partitioned into two pieces, each carrying the induced structure of a directed acyclic graph, is just $\mathrm{CSP}(S(2))$, where $S(2)$ is the homogeneous circular tournament (Example 2.3.1(2)). Other constraint satisfaction problems, coming from phylogenetic reconstruction (reconstruction of evolutionary trees), have the form $\mathrm{CSP}(M)$ where M is a homogeneous C-relation or D-relation (see Section 6.1). We emphasise that the CSP is very sensitive to the choice of language – for example to whether a partial or total order is strict – since the notion of homomorphism is sensitive to this.

A finite L-structure M is said to be a *core* if any endomorphism of M is an automorphism, and M is a *core of* N if M is a core and is the image of an endomorphism of N. For infinite structures, the definitions are the same, except that 'automorphism' is replaced by 'self-embedding'. Easily, every finite L-structure N has a core: choose a homomorphic image of least size, and with as few tuples as possible satisfying relations. In fact, such a core will be unique up to isomorphism.

We shall say that L-structures M and N are *homomorphically equivalent* if there are homomorphisms $M \to N$ and $N \to M$. If M and N are homomorphically equivalent, then they give the same constraint satisfaction problem; that is, for any finite P, there is a homomorphism $P \to M$ if and only if there is a homomorphism $P \to N$. It is easily seen that any finite structure is homomorphically equivalent to a core of it. Thus, given a finite template, there is a finite core with the same constraint satisfaction problem.

A natural first question concerning CSPs of ω-categorical structures concerns the existence of cores. In [18], Bodirsky proves the following. Recall that a theory T is *model complete* if all embeddings between models of T are elementary, or equivalently, if every formula is equivalent modulo T to an existential formula. If M is homogeneous over a finite relational language then its theory has quantifier elimination, so is model complete; but there are many ω-categorical model complete theories which are not homogeneous – for example an equivalence relation with two classes, one infinite and one of size one.

Theorem 6.6.2 (Bodirsky [18]) *Let M be ω-categorical over a finite relational language. Then M is homomorphically equivalent to a model complete core M^c, (a core* of *M), and M^c is unique up to isomorphism. Also M^c is ω-categorical or finite, and its $\emptyset$-definable relations (i.e. the orbits on tuples) are p.p. definable in M^c.*

This theorem is reminiscent of an old result of Saracino [132], that if M is ω-categorical, then there is a unique model complete ω-categorical structure N (namely, one whose theory is a model companion of $\mathrm{Th}(M)$) such that $\mathrm{Age}(M) = \mathrm{Age}(N)$. We note also the following nice characterisation of model completeness, proved in [21], which follows easily from some definability characterisations in [22].

Proposition 6.6.3 (Bodirsky, Pinsker) *Let M be ω-categorical over a language without function symbols. Then* $\mathrm{Th}(M)$ *is model complete if and only if* $\mathrm{Emb}(M)$ *is dense in* $\mathrm{Aut}(M)$.

The following result ensures that for most homogeneous structures that we have considered, the CSP is in NP. For the notion 'finitely bounded', see the paragraph before Remark 2.1.5.

Proposition 6.6.4 *[17, Proposition 3,1] Let M be homogeneous and finitely bounded over a finite relational language. Then $CSP(M)$ is in NP.*

For such M, a non-deterministic algorithm for $\mathrm{CSP}(M)$, given a structure A, will guess a homomorphic image A' of A and check in polynomial time whether $A' \in \mathrm{Age}(M)$ by checking whether any of the finitely many minimal forbidden configurations which determine $\mathrm{Age}(M)$ embed in A'.

The authors in [17] also use Cherlin's classification of homogeneous digraphs to show that finitely bounded homogeneous digraphs M satisfy the dichotomy conjecture for CSPs: that is, $\mathrm{CSP}(M)$ is in P or is NP-complete. They conjecture that this holds for all finitely bounded ω-categorical homogeneous structures. They note that since there are continuum many homogeneous digraphs, some such are templates with undecidable CSP.

Some of the theory connecting bi-interpretabiliity and the topology on the automorphism group (see Theorem 5.1.2 above) has analogues for these notions of definability, with the automorphism group replaced by more universal-algebraic objects. The next few results are taken from [21]. The monoids $\mathrm{Emb}(M)$ and $\mathrm{End}(M)$ carry the topology of pointwise convergence, so basic open sets have the form $U_{\bar{a},\bar{b}} := \{g : g(\bar{a}) = \bar{b}\}$. Below, by a positive existential formula we mean one of the form $\exists \bar{y}\phi(\bar{x}, \bar{y})$ where ϕ is quantifier-free and has no negations. An *existential interpretation* is one in which all the defining formulas can be taken to be existential; similarly for *positive existential interpretation*, and *p.p. interpretation*. See [21, Section 3.3] for the corresponding bi-interpretability definitions.

Theorem 6.6.5 *Let M be ω-categorical over a language without function symbols.*

(i) A structure N is existentially interpretable in M if and only if there is a continuous monoid homomorphism $f : \mathrm{Emb}(M) \to \mathrm{End}(N)$ such that N is covered by finitely many cosets of the image $f(\mathrm{Aut}(M))$.

(ii) If M is non-contractible (i.e. it has no constant endomorphism) then a structure N is positive existentially interpretable in M if and only if there is a continuous monoid homomorphism $f : \mathrm{End}(M) \to \mathrm{End}(N)$ such that N is covered by finitely many orbits under $f(\mathrm{Aut}(M))$.

Theorem 6.6.6 *Let M, N be ω-categorical over a language without function symbols.*

(i) If M and N are positive existentially bi-interpretable then $\mathrm{End}(M)$ *and* $\mathrm{End}(N)$ *are isomorphic as topological monoids. The converse also holds if M and N are non-contractible.*

(ii) If $\mathrm{Emb}(M)$ *and* $\mathrm{Emb}(N)$ *are isomorphic as topological monoids, then M and N are existentially bi-interpretable.*

Due to Proposition 6.6.1, a major task is to investigate p.p. interpretability in ω-categorical structures. In this context, the right algebraic notion is that of *polymorphism* of a structure M; namely, a homomorphism from some power M^n to M, where M^n is equipped with the product structure. A set $X \subset M^k$

is *closed under polymorphisms* if, for all n, every polymorphism $\sigma : M^n \to M$ and all $\bar{a}_1, \ldots, \bar{a}_n \in X$, we have $(\bar{a}_1, \ldots, \bar{a}_n)^\sigma \in X$. Here, if $\bar{a}_i = (a_{i1}, \ldots, a_{ik})$ for each i, then $(\bar{a}_1, \ldots, \bar{a}_n)^\sigma := (\bar{b}_1, \ldots, \bar{b}_k)$, where $\bar{b}_j = (a_{1j}, \ldots, a_{nj})^\sigma$.

Theorem 6.6.7 (Theorem 5.1 of [17]) *Let M be ω-categorical over a language with no function symbols and let $X \subseteq M^k$. Then X is p.p. definable in M if and only if X is closed under polymorphisms of M.*

If D is a set, we consider the union O, over all $k \geq 1$, of the sets of functions $D^k \to D$. These functions are called *operations* on D. There are natural notions of composition of operations and of a projection operation. A *clone* on D is a subset of O which contains all projections and is closed under compositions. It can be checked that the set $\mathrm{Pol}(M)$ of all polymorphisms of a relational structure M is a clone.

There is a clone-theoretic analogue of the easy Lemma 4.1.1 about permutation groups. Let $\mathcal{C} \subset O$ be a clone. If $k \in \mathbb{N}$, and $f \in O$, with $f : D^k \to D$, then we say that f is *interpolated* by $\mathcal{C}$ if the following holds: for every finite $B \subset D$, there is $g \in \mathcal{C}$ such that $g(\bar{a}) = f(\bar{a})$ for every $\bar{a} \in B^k$. Now $\mathcal{C}$ is *locally closed* if it contains all operations which it interpolates. In particular, a clone $\mathcal{C}$ is locally closed if and only if, for every n, the set of n-ary operations in $\mathcal{C}$ is a closed subset of the topological space D^{D^n} (with the product topology, where D has the discrete topology).

Proposition 6.6.8 *[17, Proposition 4.1] Let $\mathcal{C} \subset O$ be a clone. Then $\mathcal{C}$ is locally closed if and only if $\mathcal{C} = \mathrm{Pol}(M)$ for some structure M on D.*

These results from [17] yield the following.

Theorem 6.6.9 ([17]) *Let M be ω-categorical. Then there is an anti-isomorphism $\Gamma \mapsto \mathrm{Pol}(\Gamma)$ from the lattice of all reducts of M which are closed under positive primitive definability, to the lattice of all locally closed clones on M containing $\mathrm{Aut}(M)$.*

We now have some very natural questions generalising the study of reducts of ω-categorical structures in Section 6.2. We aim to understand reducts of certain homogeneous structures M (e.g. those for which $\mathrm{CSP}(M)$ is interesting) up to p.p.-interdefinability. This is equivalent to describing the locally closed clones on M which contain $\mathrm{Aut}(M)$.

This problem is hard, even in the case when M is a pure set: in [20], it is shown that there are $2^{\aleph_0}$ locally closed clones on M containing $\mathrm{Sym}(M)$. However, in this particular case (M a pure set) a description of the clones is given. Continuing in this vein, there is a description in [22] of the 'locally closed' transformation monoids which contain the automorphism group of the random graph R. This problem corresponds to describing reducts of the random graph up to *existential* (rather than p.p.) interdefinability, but it is a step towards describing the p.p. reducts. In particular, the authors describe the minimal members of the lattice of locally closed clones on R containing $\mathrm{Aut}(R)$. The methods in [22] also give a new proof, which should have further uses, of Theorem 6.2.3 of Thomas.

We conclude by recording a result of Lascar related both to this section (Theorems 6.6.5 and 6.6.6) and to Section 5. See [101] for the definition of G-finiteness, a condition which holds for most familiar ω-categorical structures (conceivably, for all homogenizable structures). The statement below follows from [101, Theorem 2.4] via Theorem 5.1.2 above.

Theorem 6.6.10 *[101] Let M be an ω-categorical G-finite structure, let N be any countable structure, and suppose that semigroup of elementary embeddings $M \to M$ is isomorphic as an abstract semigroup to that for N. Then M and N are bi-interpretable.*

If M is homogeneous then $\mathrm{Th}(M)$ has quantifier elimination (see the paragraph after Example 3.1.5), so every embedding $M \to M$ is elementary, that is, the above semigroup is exactly $\mathrm{Emb}(M)$.

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