Let $K$ be an algebraically closed field and $\bar{x} = (x_1, \ldots, x_n)$ a tuple of indeterminates. An affine variety over $K$ can be thought of as a set of the form
\[
\{ a \in K^n \mid f_1(a) = \cdots = f_m(a) = 0 \}
\]
for some polynomials $f_1, \ldots, f_m \in K[\bar{x}]$. A projective variety is defined similarly except that the polynomials $f_1, \ldots, f_m$ are homogeneous and $\bar{a} \notin K^n \setminus \{0\}$.

Affine and projective varieties are examples of abstract varieties. We say that a straight variety $Y$ is complete if for any abstract variety $X$, the projection map $\pi: X \times Y \to X$ is closed in the Zariski topology. Such a closed set is just a finite union of varieties called “Zariski closed”. The theorem we shall prove here is the following

**Theorem:** Every projective variety is complete.

This is classic result in algebraic geometry but here we show that it can easily be obtained by using some simple model theoretic techniques.

### Model Theoretic Aspects

Roughly speaking, we take “positive” to mean “without negations”. Then, in the language of rings ($\mathcal{L}_R = \{0, 1, +, -\}$) affine varieties are positively definable with parameters from $K$. In fact, they are just the sets that are defined by a conjunction of polynomial equations. Thus, Zariski closed sets are exactly the positively quantifier free definable sets. Furthermore, projective varieties are definable by homogeneous polynomials. Let $X$ be an affine variety and $Y$ a projective variety. Then the product $X \times Y$ is defined by some formula of the form
\[
\bigwedge_{i,j} f_i(x) = 0 \land \bigwedge_{i} g_i(y) = 0 \land y \neq 0
\]
where $f_i, g_i \in K[\bar{x}, y]$ and $g_i$ homogenous in $y$ for all $i, j$. Hence for any closed subset $Z \subseteq X \times Y$, $\pi(Z)$ is defined by
\[
\exists y \left( \bigwedge_{i,j} f_i(x) = 0 \land \bigwedge_{i} g_i(y) = 0 \land y \neq 0 \right)
\]
we shall denote this formula by $\phi(\bar{x})$ and show that it is positive quantifier free definable using the following theorem.

**Lyndon-Robinson Type Lemma:** Let $T$ be a first order theory defined in a language $\mathcal{L}$ and $\phi(\bar{x})$ be an $\mathcal{L}$-formula. Suppose that for each model $K$ of $T$ and each $\mathcal{L}$-homomorphism $\sigma: A \to K'$ where $A$ a substructure of $K$ and $K' \models T$ we have
\[
\text{if } \sigma \in A' \text{ and } K \models \phi(\bar{c}) \text{, then } K' \models \phi(\bar{\sigma}(\bar{c}));
\]
then there is a positive quantifier free formula $\psi(\bar{x})$ such that $T \vdash \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$.

In our case take $\mathcal{L} = \mathcal{L}_R(\bar{c})$ where $\bar{c}$ is a tuple of constant symbols which for the coefficients of $f_i$ and $g_i$ for all $i, j$ and we take $T = ACF$.  

### Valuation Theory

A valuation ring $\mathcal{O}$ of $K$ is a subring of $K$ such that for every $a \in K$ either $a \in \mathcal{O}$ or $a^{-1} \in \mathcal{O}$.

**Valuation Theory...Contd**

We wish to extend homomorphisms to valuations rings since valuation rings have the following property.

**Proposition:** Let $g \in K[\bar{y}]$ be a homogenous polynomial and $\mathcal{O}$ a valuation ring of $K$. Then for all $b \in K^m$, $b \neq 0$ there exists $\bar{y} \in \mathcal{O}^m$ such that
1. $b$ has a coordinate equal to 1.
2. $g(\bar{b}) = 0 \leftrightarrow g(\bar{b}') = 0$.

**Proof of Main Theorem:**

Recall the formula $\phi(\bar{z})$ from before (“Model Theoretic Aspects”). We prove that $\phi(\bar{z})$ is positively quantifier free definable using the Lyndon-Robinson type lemma. So let $K, L \models ACF$ and $\sigma: A \to L$ a homomorphism where $A \subseteq K$ is a substructure. By the Place Extension lemma we can assume that $A$ is a valuation ring. Assume $K \models \phi(\bar{a})$ for some $\bar{a} \in A^n$. Then there exists some $b \in K^n$ such that $g(b) = 0$ for $1 \leq i \leq m$. Since the polynomials $g_i$ are homogeneous, there is some $\bar{y} \in A$ such that $g(\bar{y}) = 0$ for all $i$ and one of the coordinates of $\bar{y}$ is 1. But then
\[
L \models \bigvee_{i=1}^m g_i(\bar{y}) = 0 \land \bigwedge_{i=1}^m g_i(\bar{a}'(\bar{y})) = 0 \land \sigma(\bar{a}') \neq 0
\]
Thus $L \models \phi(\bar{a}(\bar{y}))$.

**Semi-Algebraic Geometry**

In [1] L. van den Dries applies this same technique to semi-algebraic geometry. In this case $K$ is a real closed field and in place of varieties we consider semi-algebraic sets. An semi-algebraic set is made of finite boolean combinations of sets of the form
\[
\{ a \in K^n \mid f_1(a) \geq 0, \ldots, f_s(a) \geq 0 \}
\]
where $f_1, \ldots, f_s \in R[\bar{x}]$. In particular, positive semi-algebraic sets are made of positive boolean combinations of such sets. Parallel to projective varieties, a homogeneous positive semi-algebraic set is positive semi-algebraic set defined by homogeneous $f_1, \ldots, f_s$. Completeness of positive semi-algebraic sets is also defined as for varieties but instead we consider the euclidian topology. Then L. van Dries proved the following

**Theorem:** All homogeneous positive semi-algebraic are complete sets.

**Additional Notes**

- A. Prestel extended these results to the valued cases, using the same techniques in $ACF$ and $RCF$ in [2]. Here, as well as equations or inequations, we consider sets defined by $v(g(\bar{z})) < v(h(\bar{z}))$ where $g, h$ are polynomials. Projective/positive semi-algebraic sets are as before but we also require that the polynomials $h, g$ are homogeneous and $deg h = deg g$.
- W. Pong used these techniques in $DCF$ to obtain a “Valuating Criterion” for completeness of differential varieties (cf. [3]).
- The problem is still open for differentially closed valued fields.

**References**