Degree Spectra of Structures

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Outline

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- Degree spectra and co-spectra
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Enumeration degrees

Definition (Enumeration operator)

$\Gamma_z : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$:

$$x \in \Gamma_z(B) \iff \exists v(\langle v, x\rangle \in W_z \;\&\; D_v \subseteq B).$$

D_v – the finite set having canonical code v,
$W_0, \ldots, W_z, \ldots$ – the Gödel enumeration of the c.e. sets.

- A is *enumeration reducible* to B, $A \leq_e B$, if $A = \Gamma_z(B)$ for some enumeration operator Γ_z.
- $A \equiv_e B \iff A \leq_e B \;\&\; B \leq_e A$.
- $\mathrm{d}_e(A) = \{B : B \equiv_e A\}$
- $\mathrm{d}_e(A) \leq \mathrm{d}_e(B) \iff A \leq_e B$.
- $\mathcal{D}_e = (\mathcal{D}_e, \leq, \mathbf{0}_e)$ – the structure of e-degrees.

Let $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$.

Definition (A total set)

- $A^+ = A \oplus (\mathbb{N} \setminus A)$.
- A is *total* iff $A \equiv_e A^+$.
- A degree is *total* if it contains a total set.

The substructure $\mathcal{D}_T$ of $\mathcal{D}_e$ consisting of all total degrees is isomorphic of the structure of the Turing degrees.

- $A \leq_T B$ iff $A^+ \leq_e B^+$.
- $A \leq_{c.e.} B$ iff $A \leq_e B^+$.

The enumeration jump operator is defined by **Cooper**:

Definition (Enumeration jump)

Given a set A, let

- $K_A^0 = \{\langle x, z\rangle : x \in \Gamma_z(A)\}$.
- $A' = (K_A^0)^+$.
- $A^{(n+1)} = (A^{(n)})'$.

- If $A \leq_e B$, then $A' \leq_e B'$.
- A is Σ^0_{n+1} relatively to B iff $A \leq_e (B^+)^{(n)}$.

- The enumeration jump is consistent with the Turing jump on the total enumeration degrees.
- Let $\mathbf{a} = \mathrm{d_e}(A)$ and $\alpha < \omega_1^{CK}$.
- By $A^{(\alpha)}$ we shall denote the α-th iteration of the e-jump of A and let $\mathbf{a}^{(\alpha)} = \mathrm{d_e}(A^{(\alpha)})$.
  - $E_0^A = A$;
  - $E_{\beta+1}^A = (E_\beta^A)'$;
  - If $\alpha = \lim \alpha(p)$, then $E_\alpha^A = \{\langle p, x\rangle \mid x \in E_{\alpha(p)}^A\}$.
  - Set $A^{(\alpha)} = E_\alpha^A$.
- (Selman) If for all total X $(B \leq_{\mathrm{e}} X^{(\alpha)} \Rightarrow A \leq_{\mathrm{e}} X^{(\alpha)})$, then $A \leq_{\mathrm{e}} B \oplus 0_e^{(\alpha)}$.

Enumeration of a structure

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k, =, \neq)$ be a countable abstract structure.

- **An enumeration** f of $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.
- for any $A \subseteq \mathbb{N}^a$ let $f^{-1}(A) = \{\langle x_1 \ldots x_a \rangle : (f(x_1), \ldots, f(x_a)) \in A\}$.
- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$.

Degree spectra of structures

Definition

The Degree spectrum of $\mathfrak{A}$ is the set

$$\mathrm{DS}(\mathfrak{A}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

- If **a** is the least element of $\mathrm{DS}(\mathfrak{A})$, then **a** is called the *degree of* $\mathfrak{A}$.
- L. Richter [1981] — degrees of structures.

Definition

Let $\alpha < \omega_1^{CK}$. Then the **α-th jump spectrum of $\mathfrak{A}$** is the set

$$\mathrm{DS}_\alpha(\mathfrak{A}) = \{\mathrm{d_e}((f^{-1}(\mathfrak{A}))^{(\alpha)}) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

- If **a** is the least element of $\mathrm{DS}_\alpha(\mathfrak{A})$, then **a** is called the *α-th jump degree of $\mathfrak{A}$*.
- J. Knight [1986] — jump degrees of structures.

Definition

A structure $\mathfrak{A}$ is *total* if all elements of $\mathrm{DS}(\mathfrak{A})$ are total.

- The definition of the pullback: $\mathfrak{A}^+ = (\mathbb{N}, R_1, \ldots, R_k, \neg R_1, \ldots, \neg R_k)$.
- $\mathrm{DS}(\mathfrak{A}^+)$ consists only total enumeration degrees.
- Only bijective enumerations are considered.
- Example $\mathfrak{A} = (\mathbb{N}; =, \neq)$.
    - only the bijective enumerations: $\mathrm{DS}(\mathfrak{A}) = \{\mathbf{0}_e\}$,
    - all surjective enumerations: $\mathrm{DS}(\mathfrak{A})$ will consist of all total enumeration degrees.

Proposition

Let f be an arbitrary enumeration. There exists a bijective enumeration g s.t. $g^{-1}(\mathfrak{A}) \leq_e f^{-1}(\mathfrak{A})$.

The Degree Spectra are upwards closed with respect to the total degrees:

$$\mathbf{a} \in \mathrm{DS}(\mathfrak{A}), \mathbf{b} \text{ is total and } \mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{b} \in \mathrm{DS}(\mathfrak{A}).$$

Proposition

Let g be an enumeration, $\alpha < \omega_1^{CK}$ *and let F be a total set s.t.* $g^{-1}(\mathfrak{A})^{(\alpha)} \leq_e F$.
There exists an enumeration f s.t.

$$g^{-1}(\mathfrak{A}) \leq_e f^{-1}(\mathfrak{A}) \text{ and } f^{-1}(\mathfrak{A})^{(\alpha)} \equiv_e F.$$

Co-spectra of structures

Definition

Let $\emptyset \neq \mathcal{A} \subseteq \mathcal{D}_e$.
The *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$:

$$co(\mathcal{A}) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_{\mathrm{e}} \ \& \ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq \mathbf{a})\}.$$

Example

Fix a $\mathbf{d} \in \mathcal{D}_{\mathrm{e}}$ and let $\mathcal{A}_{\mathbf{d}} = \{\mathbf{a} : \mathbf{a} \geq \mathbf{d}\}$. Then $co(\mathcal{A}_{\mathbf{d}}) = \{\mathbf{b} : \mathbf{b} \leq \mathbf{d}\}$.

- $co(\mathcal{A})$ is a countable ideal.

Definition

The Co-spectrum of $\mathfrak{A}$ is the co-set of $\mathrm{DS}(\mathfrak{A})$:

$$\mathrm{CS}(\mathfrak{A}) = \{\mathbf{b} : (\forall \mathbf{a} \in \mathrm{DS}(\mathfrak{A}))(\mathbf{b} \leq \mathbf{a})\}.$$

Definition

The αth co-spectrum of $\mathfrak{A}$ is the set $\mathrm{CS}_\alpha(\mathfrak{A}) = co(\mathrm{DS}_\alpha(\mathfrak{A}))$.

- If $\mathrm{CS}_\alpha(\mathfrak{A})$ contains a greatest element **a**, then **a** is called *the α-th jump co-degree* of $\mathfrak{A}$.
- **Observation:** If $\mathfrak{A}$ has α-th jump degree **a**, then **a** is also α-th jump co-degree of $\mathfrak{A}$. The opposite is not always true.

Normal Form

Definition

Let $A \subseteq \mathbb{N}$, $\alpha < \omega_1^{CK}$ and let f be an enumeration of $\mathfrak{A}$. The set A is called α-*admissible* in the enumeration f if $A \leq_e f^{-1}(\mathfrak{A})^{(\alpha)}$.
The set A is α-admissible in $\mathfrak{A}$ if A is α-admissible in all enumerations of $\mathfrak{A}$.

Theorem

$\mathbf{a} \in \mathrm{CS}_\alpha(\mathfrak{A})$ iff $\mathbf{a}$ contains an α-admissible in $\mathfrak{A}$ set iff all elements of $\mathbf{a}$ are α-admissible in $\mathfrak{A}$.

Theorem (Ash, Knight, Manasse , Slaman, Soskov)

The α-admissible sets are the sets definable on $\mathfrak{A}$ by means of recursive (Σ_α^+) infinitary formulae.

Some Examples

1981 (Richter) Let $\mathfrak{A} = (\mathbb{N}; <, =, \neq)$ be a linear ordering.

- $\mathrm{DS}(\mathfrak{A})$ contains a minimal pair of degrees, $\mathrm{CS}(\mathfrak{A}) = \{\mathbf{0}_e\}$.
- If $\mathrm{DS}(\mathfrak{A})$ has a degree $\mathbf{a}$, then $\mathbf{a} = \mathbf{0}_e$.

1986 (Knight 1986) Consider again a linear ordering $\mathfrak{A}$.

- $\mathrm{CS}_1(\mathfrak{A})$ consists of all Σ^0_2 sets. The first jump co-degree of $\mathfrak{A}$ is $\mathbf{0}'_e$.

1990 (Ash, Jockush, Knight and [1992] Downey, Knight)
For every $\alpha < \omega_1^{CK}$ there exists a linear ordering $\mathfrak{A}$ with α-th jump degree $\mathbf{0}_e^{(\alpha)}$ and with no β-th jump degree for $\beta < \alpha$.

1998 (Slaman, Wehner)
$\mathrm{DS}(\mathfrak{A}) = \{\mathbf{a} : \mathbf{a}$ is total and $\mathbf{0}_e < \mathbf{a}\}$,

- $\mathrm{CS}(\mathfrak{A}) = \{\mathbf{0}_e\}$.
  The structure $\mathfrak{A}$ has co-degree $\mathbf{0}_e$ but has not a degree.

1998 (based on Coles, Downey, Slaman) Let G be a torsion free abelian group of rank 1, i.e. G is a subgroup of Q. Let $a \neq 0 \in G$ and let p be a prime number.

$$h_p(a) = \begin{cases} k & \text{if } k \text{ is the greatest s.t. } p^k | a, \\ \infty & \text{if } p^k | a \text{ for all } k. \end{cases}$$

Let $\chi(a) = (h_{p_0}(a), h_{p_1}(a), \dots)$ and

$$S_a = \{\langle i, j \rangle : j \leq \text{the } i\text{-th member of } \chi(a)\}.$$

For $a, b \neq 0 \in G$, $S_a \equiv_e S_b$.
Set $\mathbf{s}_G = d_e(S_a)$.

$$\mathrm{DS}(G) = \{\mathbf{b} : \mathbf{b} \text{ is total and } \mathbf{s}_G \leq \mathbf{b}\}.$$

- The co-degree of G is $\mathbf{s}_G$.
- G has a degree iff $\mathbf{s}_G$ is total
- If $1 \leq \alpha$, then $\mathbf{s}_G^{(\alpha)}$ is the α-th jump degree of G.

For every $\mathbf{d} \in \mathcal{D}_e$ there exists a G, s.t. $\mathbf{s}_G = \mathbf{d}$. Hence every principle ideal of enumeration degrees is $\mathrm{CS}(G)$ for some G.

2002 (Soskov) Representing all countable ideals as CS of structures.
Let $B_0, \dots, B_n, \dots$ be a sequence of sets of natural numbers. Set $\mathfrak{A} = (\mathbb{N}; G_\varphi; \sigma, =, \neq)$,

$$\varphi(\langle i, n\rangle) = \langle i+1, n\rangle;$$
$$\sigma = \{\langle i, n\rangle : n = 2k+1 \vee n = 2k \ \& \ i \in B_k\}.$$

- $\mathrm{CS}(\mathfrak{A}) = I(d_e(B_0), \dots, d_e(B_n), \dots)$
  - $I \subseteq \mathrm{CS}(\mathfrak{A})$: $B_k \leq_e f^{-1}(\mathfrak{A})$ for each k;
  - $\mathrm{CS}(\mathfrak{A}) \subseteq I$: if $d_e(A) \in \mathrm{CS}(\mathfrak{A})$, then $A \leq_e B_k$ for some k.

Properties of the degree spectra

Let $\mathcal{A} \subseteq \mathcal{D}_e$. Then $\mathcal{A}$ is *upwards closed* if

$$\mathbf{a} \in \mathcal{A}, \mathbf{b} \text{ is total and } \mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{b} \in \mathcal{A}.$$

- The Degree spectra are upwards closed.
- General properties of upwards closed sets of degrees.

Theorem

Let $\mathcal{A}$ be an upwards closed set of degrees. Then

(1) $co(\mathcal{A}) = co(\{\mathbf{b} \in \mathcal{A} : \mathbf{b} \text{ is total}\})$.

(2) *Let* $1 \leq \alpha < \omega_1^{CK}$ *and* $\mathbf{c} \in \mathcal{D}_e$. *Then*

$$co(\mathcal{A}) = co(\{\mathbf{b} \in \mathcal{A} : \mathbf{c} \leq \mathbf{b}^{(\alpha)}\}).$$

Specific properties

Theorem

Let $\mathfrak{A}$ *be a structure,* $1 \leq \alpha < \omega_1^{CK}$*, and* $\mathbf{c} \in \mathrm{DS}_\alpha(\mathfrak{A})$*. Then*

$$\mathrm{CS}(\mathfrak{A}) = co(\{\mathbf{b} \in \mathrm{DS}(\mathfrak{A}) : \mathbf{b}^{(\alpha)} = \mathbf{c}\}).$$

Example

Let $B \not\leq_e A$ and $A \not\leq_e B'$. Set

$$\mathcal{D} = \{\mathbf{a} : \mathbf{a} \geq d_e(A)\} \cup \{\mathbf{a} : \mathbf{a} \geq d_e(B)\}.$$

$$\mathcal{A} = \{\mathbf{a} : \mathbf{a} \in \mathcal{D} \ \& \ \mathbf{a}' = d_e(B)'\}.$$

- $d_e(B)$ is the least element of $\mathcal{A}$ and hence $d_e(B) \in co(\mathcal{A})$.
- $d_e(B) \not\leq d_e(A)$ and hence $d_e(B) \notin co(\mathcal{D})$.

Minimal Pair Type Theorems

Theorem

Let $\mathbf{c} \in \mathrm{DS}_2(\mathfrak{A})$. *There exist total* $\mathbf{f}, \mathbf{g} \in \mathrm{DS}(\mathfrak{A})$, *such that* $\mathbf{f}'' = \mathbf{g}'' = \mathbf{c}$ *and* $\mathrm{CS}(\mathfrak{A}) = co(\{\mathbf{f}, \mathbf{g}\})$.

Theorem

Let $\alpha < \omega_1^{CK}$ *and let* $\mathbf{b} \in DS_\alpha(\mathfrak{A})$. *There exist elements* $\mathbf{f}_0$ *and* $\mathbf{f}_1$ *of* $\mathrm{DS}(\mathfrak{A})$ *such that*

- $\mathbf{f}_0^{(\alpha)} \leq \mathbf{b}$ *and* $\mathbf{f}_1^{(\alpha)} \leq \mathbf{b}$.
- *If* $\beta < \alpha$, *then* $\mathbf{f}_0^{(\beta)}$ *and* $\mathbf{f}_1^{(\beta)}$ *do not belong to* $\mathrm{CS}_\beta(\mathfrak{A})$.
- *If* $\beta + 1 < \alpha$, *then* $co(\{\mathbf{f}_0^{(\beta)}, \mathbf{f}_1^{(\beta)}\}) = \mathrm{CS}_\beta(\mathfrak{A})$.

Example

Finite lattice $L = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a} \wedge \mathbf{b}, \mathbf{a} \wedge \mathbf{c}, \mathbf{b} \wedge \mathbf{c}, \top, \bot\}$.

$$\mathcal{A} = \{\mathbf{d} \in \mathcal{D}_e : \mathbf{d} \geq \mathbf{a} \vee \mathbf{d} \geq \mathbf{b} \vee \mathbf{d} \geq \mathbf{c}\}.$$

The Quasi-minimal degree

Definition

Let $\mathcal{A}$ be a set of enumeration degrees. The degree **q** is quasi-minimal with respect to $\mathcal{A}$ if:

- $\mathbf{q} \notin co(\mathcal{A})$.
- If **a** is total and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$.
- If **a** is total and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

Theorem

If **q** *is quasi-minimal with respect to* $\mathcal{A}$*, then* **q** *is an upper bound of* $co(\mathcal{A})$*.*

Theorem

For every structure $\mathfrak{A}$ *there exists a quasi-minimal with respect to* $\mathrm{DS}(\mathfrak{A})$ *degree.*

Definition

Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of degrees. Then $\mathcal{B}$ is a base of $\mathcal{A}$ if

$$(\forall \mathbf{a} \in \mathcal{A})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a}).$$

Theorem

Let $\mathcal{A}$ be a set of degrees possessing a quasi-minimal degree. Suppose that there exists a countable base $\mathcal{B}$ of $\mathcal{A}$ such that all elements of $\mathcal{B}$ are total. Then $\mathcal{A}$ has a least element.

Corrolary

A total structure $\mathfrak{A}$ has a degree if and only if $\mathrm{DS}(\mathfrak{A})$ has a countable base.

Corrolary

Let **a** *and* **b** *be incomparable Turing degrees. There does not exist a structure* $\mathfrak{A}$ *such that* $\mathrm{DS}(\mathfrak{A})$ *is equal to the union of the cones above* **a** *and* **b***.*

Example

An upwards closed set $\mathcal{A}$ of degrees which does not possess a quasi-minimal degree.
Let **a** and **b** be two incomparable total degrees.
Let $\mathcal{A} = \{\mathbf{c} : \mathbf{c} \geq \mathbf{a} \vee \mathbf{c} \geq \mathbf{b}\}$.
Clearly $\mathcal{A}$ has a countable base of total degrees, but it has not a least element. So, $\mathcal{A}$ has no quasi-minimal degree.

Relatively α-intrinsic sets

1989 Ash, Knight, Manasse, Slaman, Chisholm.

- The set A is *relatively α-intrinsic on* $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$ the set $f^{-1}(A) \leq_e f^{-1}(\mathfrak{A})^{(\alpha)}$,(α - constructive ordinal).

2002 Soskov, Baleva.

- Let $\{B_\alpha\}_{\alpha \leq \zeta}$ be a sequence of subset of $\mathbb{N}$ and ζ be a constructive ordinal.
- Add each set B_α to the structure $\mathfrak{A}$ as a new predicate which is relatively α-intrinsic on $\mathfrak{A}$.
- Restrict the class of all enumerations of $\mathfrak{A}$ to the class of those enumerations f of $\mathfrak{A}$ for which $f^{-1}(B_\alpha) \leq_e f^{-1}(\mathfrak{A})^{(\alpha)}$.

Relative Spectra of Structures

Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be arbitrary abstract structures on $\mathbb{N}$, $k \leq n$.
An enumeration f of $\mathfrak{A}$ is **k-acceptable** with respect to the structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$, if

$$f^{-1}(\mathfrak{A}_1) \leq_{\mathrm{e}} (f^{-1}(\mathfrak{A}))' \ldots f^{-1}(\mathfrak{A}_k) \leq_{\mathrm{e}} (f^{-1}(\mathfrak{A}))^{(k)}.$$

Denote by $\mathcal{E}_k$ the class of all k-acceptable enumerations.

Definition

The Relative spectrum of the structure $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set

$$\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f \in \mathcal{E}_n\}$$

Lemma

If F is a total set, $f \in \mathcal{E}_n$ and $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$, then there exists an enumeration $g \in \mathcal{E}_n$, such that

1. $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F \oplus f^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$;
2. $g^{-1}(B) \leq_{\mathrm{e}} F \oplus f^{-1}(B)$, *for every* $B \subseteq \mathbb{N}$.

Corollary

The Relative spectrum $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ is upwards closed.

Let $k \leq n$. **The kth Jump Relative spectrum** of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set

$$\mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \{\mathbf{a}^{(\mathbf{k})} \mid \mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)\}.$$

Proposition

The kth Jump Relative spectrum $\mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots \mathfrak{A}_n)$ is upwards closed.

Relative Co-spectra of Structures

Definition

The Relative co-spectrum of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, is the co-set of $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$, i.e.

$$\mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \{\mathbf{b} \mid (\forall \mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n))(\mathbf{b} \leq \mathbf{a})\}.$$

Let $k \leq n$. **The Relative kth co-spectrum** of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, is the co-set of $\mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$, i.e.

$$\mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \{\mathbf{b} \mid (\forall \mathbf{a} \in \mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n))(\mathbf{b} \leq \mathbf{a})\}.$$

The jump set

The jump set $\mathcal{P}^f_k$ of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$:

1. $\mathcal{P}^f_0 = f^{-1}(\mathfrak{A})$.
2. $\mathcal{P}^f_{k+1} = (\mathcal{P}^f_k)' \oplus f^{-1}(\mathfrak{A}_{k+1})$.

Theorem

For every $A \subseteq \mathbb{N}$ and $k \leq n$, the following are equivalent:

1. $d_e(A) \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.
2. $A \leq_e \mathcal{P}^f_k$*, for every k-acceptable enumeration f of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$.*

The Normal Form Theorem

The set A is *formally k-definable* on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ if there exists a recursive sequence $\{\Phi^{\gamma(x)}(W_1, \ldots, W_r)\}$ of Σ^+_k formulae and elements $t_1, \ldots, t_r$ of $\mathbb{N}$ such that:
$x \in A \iff (\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \models \Phi^{\gamma(x)}(W_1/t_1, \ldots, W_r/t_r)$.

- Σ^+_0 : $(\exists \overline{Y})(\beta_1 \ \& \ \ldots \ \& \ \beta_k)$;
- Σ^+_{k+1}: c.e. disjunction of $(\exists \overline{Y})\Phi(\overline{X}, \overline{Y})$, $\Phi = (\phi_1 \ \& \ \ldots \ \& \ \phi_l \ \& \ \beta)$.

Theorem

$d_e(A) \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ *if and only if A is formally k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$.*

The connection with the Joint Spectra

Definition

The Joint spectrum of $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set

$$\mathrm{DS}\quad(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{a} : \mathbf{a} \in \mathrm{DS}(\mathfrak{A}), \mathbf{a}' \in \mathrm{DS}(\mathfrak{A}_1), \dots, \mathbf{a}^{(\mathbf{n})} \in \mathrm{DS}(\mathfrak{A}_n)\}.$$

1. $\mathrm{CS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.
2. There are structures $\mathfrak{A}$ and $\mathfrak{A}_1$, for which $\mathrm{CS}_1(\mathfrak{A}, \mathfrak{A}_1) \neq \mathrm{CRS}_1(\mathfrak{A}, \mathfrak{A}_1)$.
3. The difference:
    - $A \leq_{\mathrm{e}} \mathcal{P}(f^{-1}(\mathfrak{A}), f_1^{-1}(\mathfrak{A}_1), \dots, f_n^{-1}(\mathfrak{A}_n))$ for every enumerations f of $\mathfrak{A}$, f_1 of $\mathfrak{A}_1$,..., f_n of $\mathfrak{A}_n$.
    - in the normal form $(\mathfrak{A}, \mathfrak{A}_1 \dots, \mathfrak{A}_n)$ — as a many-sorted structure with separated sorts.

Minimal Pair Theorem

Theorem

For any structures $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$*, there exist enumeration degrees* $\mathbf{f}$ *and* $\mathbf{g}$ *in* $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$*, such that for any enumeration degree* $\mathbf{a}$ *and each* $k \leq n$*:*

$$\mathbf{a} \leq \mathbf{f^{(k)}} \ \& \ \mathbf{a} \leq \mathbf{g^{(k)}} \Rightarrow \mathbf{a} \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n).$$

Quasi-Minimal Degree

Theorem

For any structures $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ *there exists an enumeration degree* $\mathbf{q}$ *such that:*

1. $\mathbf{q} \notin \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_\mathbf{1}, \ldots, \mathfrak{A}_\mathbf{n})$;
2. *If* $\mathbf{a}$ *is a total degree and* $\mathbf{a} \geq \mathbf{q}$*, then* $\mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$;
3. *If* $\mathbf{a}$ *is a total degree and* $\mathbf{a} \leq \mathbf{q}$*, then* $\mathbf{a} \in \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1 \ldots \mathfrak{A}_n)$.

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