Degree Spectra of Structures

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Outline

- Enumeration degrees
- Degree spectra and co-spectra
- Representing the countable ideals as co-spectra
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Enumeration degrees

Definition (Enumeration operator)
Γ_z : \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N}):

\[ x \in \Gamma_z(B) \iff \exists v (\langle v, x \rangle \in W_z \land D_v \subseteq B). \]

\( D_v \) – the finite set having canonical code \( v \),
\( W_0, \ldots, W_z, \ldots \) – the Gödel enumeration of the c.e. sets.

▶ \( A \) is enumeration reducible to \( B \), \( A \leq_e B \),
if \( A = \Gamma_z(B) \) for some enumeration operator \( \Gamma_z \).

▶ \( A \equiv_e B \iff A \leq_e B \land B \leq_e A \).

▶ \( d_e(A) = \{ B : B \equiv_e A \} \)

▶ \( d_e(A) \leq d_e(B) \iff A \leq_e B. \)

▶ \( \mathcal{D}_e = (\mathcal{D}_e, \leq, 0_e) \) – the structure of e-degrees.
Let $A \oplus B = \{2x \mid x \in A\} \cup \{2x + 1 \mid x \in B\}$.

Definition (A total set)

- $A^+ = A \oplus (\mathbb{N} \setminus A)$.
- $A$ is total iff $A \equiv_e A^+$.
- A degree is total if it contains a total set.

The substructure $\mathcal{D}_T$ of $\mathcal{D}_e$ consisting of all total degrees is isomorphic to the structure of the Turing degrees.

- $A \leq_T B$ iff $A^+ \leq_e B^+$.
- $A \leq_{c.e.} B$ iff $A \leq_e B^+$.
The enumeration jump operator is defined by Cooper:

**Definition (Enumeration jump)**

Given a set $A$, let

- $K_A^0 = \{ \langle x, z \rangle : x \in \Gamma_z(A) \}$.
- $A' = (K_A^0)^+$.
- $A^{(n+1)} = (A^{(n)})'$.

- If $A \leq_e B$, then $A' \leq_e B'$.
- $A$ is $\Sigma^0_{n+1}$ relatively to $B$ iff $A \leq_e (B^+)^{(n)}$. 
The enumeration jump is consistent with the Turing jump on the total enumeration degrees.

Let \( a = d_e(A) \) and \( \alpha < \omega_1^{CK} \).

By \( A^{(\alpha)} \) we shall denote the \( \alpha \)-th iteration of the \( e \)-jump of \( A \) and let \( a^{(\alpha)} = d_e(A^{(\alpha)}) \).

\[
\begin{align*}
E_0^A &= A; \\
E_{\beta+1}^A &= (E_\beta^A)'; \\
E_\alpha^A &= \{ \langle p, x \rangle \mid x \in E_{\alpha(p)}^A \}. \\
E_\alpha^A &= \{ \langle p, x \rangle \mid x \in E_{\alpha(p)}^A \}. \\
A^{(\alpha)} &= E_\alpha^A.
\end{align*}
\]

(Selman) If for all total \( X \) \( (B \leq_e X^{(\alpha)} \Rightarrow A \leq_e X^{(\alpha)}) \), then \( A \leq_e B \oplus 0^{(\alpha)}_e \).
Enumeration of a structure

Let $\mathcal{A} = (\mathbb{N}; R_1, \ldots, R_k, =, \neq)$ be a countable abstract structure.

- **An enumeration** $f$ of $\mathcal{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.
- For any $A \subseteq \mathbb{N}^a$ let
  \[ f^{-1}(A) = \{ \langle x_1 \ldots x_a \rangle : (f(x_1), \ldots, f(x_a)) \in A \}. \]
- $f^{-1}(\mathcal{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$. 
Degree spectra of structures

Definition

The Degree spectrum of $\mathcal{A}$ is the set

$$DS(\mathcal{A}) = \{ d_e(f^{-1}(\mathcal{A})) \mid f \text{ is an enumeration of } \mathcal{A} \}.$$  

- If $a$ is the least element of $DS(\mathcal{A})$, then $a$ is called the degree of $\mathcal{A}$.
Definition

Let $\alpha < \omega_1^{CK}$. Then the $\alpha$-th jump spectrum of $\mathcal{A}$ is the set

$DS_\alpha(\mathcal{A}) = \{d_e((f^{-1}(\mathcal{A}))^{(\alpha)}) \mid f \text{ is an enumeration of } \mathcal{A}\}$.

- If $a$ is the least element of $DS_\alpha(\mathcal{A})$, then $a$ is called the $\alpha$-th jump degree of $\mathcal{A}$.
Definition

A structure $\mathcal{A}$ is total if all elements of $\text{DS}(\mathcal{A})$ are total.

- The definition of the pullback:
  $\mathcal{A}^+ = (\mathbb{N}, R_1, \ldots, R_k, \neg R_1, \ldots, \neg R_k)$.
- $\text{DS}(\mathcal{A}^+)$ consists only total enumeration degrees.
- Only bijective enumerations are considered.
- Example $\mathcal{A} = (\mathbb{N}; =, \neq)$.
  - only the bijective enumerations: $\text{DS}(\mathcal{A}) = \{0_e\}$,
  - all surjective enumerations: $\text{DS}(\mathcal{A})$ will consist of all total enumeration degrees.
Proposition

Let $f$ be an arbitrary enumeration. There exists a bijective enumeration $g$ s.t. $g^{-1}(\mathcal{A}) \leq_e f^{-1}(\mathcal{A})$.

The Degree Spectra are upwards closed with respect to the total degrees:

$$a \in DS(\mathcal{A}), b \text{ is total and } a \leq b \Rightarrow b \in DS(\mathcal{A}).$$

Proposition

Let $g$ be an enumeration, $\alpha < \omega_1^{CK}$ and let $F$ be a total set s.t. $g^{-1}(\mathcal{A})^{(\alpha)} \leq_e F$.

There exists an enumeration $f$ s.t.

$$g^{-1}(\mathcal{A}) \leq_e f^{-1}(\mathcal{A}) \text{ and } f^{-1}(\mathcal{A})^{(\alpha)} \equiv_e F.$$
Co-spectra of structures

Definition
Let $\emptyset \neq A \subseteq D_e$. The *co-set of* $A$ is the set $co(A)$ of all lower bounds of $A$:

$$co(A) = \{ b : b \in D_e \land (\forall a \in A)(b \leq a) \}.$$  

Example
Fix a $d \in D_e$ and let $A_d = \{ a : a \geq d \}$. Then $co(A_d) = \{ b : b \leq d \}$.

$\implies$ $co(A)$ is a countable ideal.

Definition
The *Co-spectrum of* $\mathfrak{A}$ is the co-set of $DS(\mathfrak{A})$:

$$CS(\mathfrak{A}) = \{ b : (\forall a \in DS(\mathfrak{A}))(b \leq a) \}.$$
Definition

The $\alpha$th co-spectrum of $\mathcal{A}$ is the set $CS_\alpha(\mathcal{A}) = co(DS_\alpha(\mathcal{A}))$.

- If $CS_\alpha(\mathcal{A})$ contains a greatest element $a$, then $a$ is called the $\alpha$-th jump co-degree of $\mathcal{A}$.

- **Observation:** If $\mathcal{A}$ has $\alpha$-th jump degree $a$, then $a$ is also $\alpha$-th jump co-degree of $\mathcal{A}$. The opposite is not always true.
Normal Form

Definition
Let $A \subseteq \mathbb{N}$, $\alpha < \omega_1^{CK}$ and let $f$ be an enumeration of $\mathcal{A}$. The set $A$ is called $\alpha$-admissible in the enumeration $f$ if $A \leq_e f^{-1}(\mathcal{A})(\alpha)$.

The set $A$ is $\alpha$-admissible in $\mathcal{A}$ if $A$ is $\alpha$-admissible in all enumerations of $\mathcal{A}$.

Theorem

$a \in CS_\alpha(\mathcal{A})$ iff $a$ contains an $\alpha$-admissible in $\mathcal{A}$ set iff all elements of $a$ are $\alpha$-admissible in $\mathcal{A}$.

Theorem (Ash, Knight, Manasse, Slaman, Soskov)

The $\alpha$-admissible sets are the sets definable on $\mathcal{A}$ by means of recursive ($\Sigma^+_\alpha$) infinitary formulae.
Some Examples

1981 (Richter) Let $\mathcal{A} = (\mathbb{N}; <, =, \neq)$ be a linear ordering.

- $DS(\mathcal{A})$ contains a minimal pair of degrees, $CS(\mathcal{A}) = \{0_e\}$.
- If $DS(\mathcal{A})$ has a degree $a$, then $a = 0_e$.

1986 (Knight 1986) Consider again a linear ordering $\mathcal{A}$.

- $CS_1(\mathcal{A})$ consists of all $\Sigma^0_2$ sets. The first jump co-degree of $\mathcal{A}$ is $0'_e$. 
1990 (Ash, Jockush, Knight and [1992] Downey, Knight) For every $\alpha < \omega_1^{CK}$ there exists a linear ordering $\mathcal{A}$ with $\alpha$-th jump degree $0_e^{(\alpha)}$ and with no $\beta$-th jump degree for $\beta < \alpha$.

1998 (Slaman, Wehner)
$DS(\mathcal{A}) = \{a : a$ is total and $0_e < a\}$,
$\triangleright \ CS(\mathcal{A}) = \{0_e\}$. 
The structure $\mathcal{A}$ has co-degree $0_e$ but has not a degree.
1998 (based on Coles, Downey, Slaman) Let $G$ be a torsion free abelian group of rank 1, i.e. $G$ is a subgroup of $\mathbb{Q}$. Let $a \neq 0 \in G$ and let $p$ be a prime number.

$$h_p(a) = \begin{cases} 
k & \text{if } k \text{ is the greatest s.t. } p^k | a, \\
\infty & \text{if } p^k | a \text{ for all } k.\end{cases}$$

Let $\chi(a) = (h_{p_0}(a), h_{p_1}(a), \ldots)$ and

$$S_a = \{\langle i, j \rangle : j \leq \text{the } i\text{-th member of } \chi(a)\}.$$  

For $a, b \neq 0 \in G$, $S_a \equiv_e S_b$.

Set $s_G = d_e(S_a)$. 
DS(G) = \{ b : b \text{ is total and } s_G \leq b \}.

- The co-degree of $G$ is $s_G$.
- $G$ has a degree iff $s_G$ is total
- If $1 \leq \alpha$, then $s_G^{(\alpha)}$ is the $\alpha$-th jump degree of $G$.

For every $d \in \mathcal{D}_e$ there exists a $G$, s.t. $s_G = d$. Hence every principle ideal of enumeration degrees is $CS(G)$ for some $G$. 
2002 (Soskov) Representing all countable ideals as CS of structures.

Let $B_0, \ldots, B_n, \ldots$ be a sequence of sets of natural numbers. Set $\mathcal{A} = (\mathbb{N}; G_\varphi; \sigma, =, \neq)$,

\[
\varphi(\langle i, n \rangle) = \langle i + 1, n \rangle;
\]

\[
\sigma = \{ \langle i, n \rangle : n = 2k + 1 \lor n = 2k & i \in B_k \}.
\]

- $\text{CS}(\mathcal{A}) = l(d_e(B_0), \ldots, d_e(B_n), \ldots)$
  - $l \subseteq \text{CS}(\mathcal{A}) : B_k \leq_e f^{-1}(\mathcal{A})$ for each $k$;
  - $\text{CS}(\mathcal{A}) \subseteq l :$ if $d_e(A) \in \text{CS}(\mathcal{A})$, then $A \leq_e B_k$ for some $k$. 
Properties of the degree spectra

Let \( A \subseteq \mathcal{D}_e \). Then \( A \) is \textit{upwards closed} if

\[
\text{a} \in A, \text{b is total and a} \leq \text{b} \Rightarrow \text{b} \in A.
\]

▶ The Degree spectra are upwards closed.
▶ General properties of upwards closed sets of degrees.

Theorem

\textit{Let A be an upwards closed set of degrees. Then}

(1) \( \text{co}(A) = \text{co}(\{b \in A : b \text{ is total}\}) \).

(2) Let \( 1 \leq \alpha < \omega_1^{\text{CK}} \) and \( c \in \mathcal{D}_e \). Then

\[
\text{co}(A) = \text{co}(\{b \in A : c \leq b^{(\alpha)}\}).
\]
Specific properties

Theorem

Let $\mathcal{A}$ be a structure, $1 \leq \alpha < \omega_1^{CK}$, and $c \in DS_\alpha(\mathcal{A})$. Then

$$CS(\mathcal{A}) = \text{co}(\{b \in DS(\mathcal{A}) : b^{(\alpha)} = c\}).$$

Example

Let $B \not\leq_e A$ and $A \not\leq_e B'$. Set

$$D = \{a : a \geq d_e(A)\} \cup \{a : a \geq d_e(B)\}.$$

$$A = \{a : a \in D \& a' = d_e(B)\}.$$

- $d_e(B)$ is the least element of $A$ and hence $d_e(B) \in \text{co}(A)$.
- $d_e(B) \not\leq d_e(A)$ and hence $d_e(B) \not\in \text{co}(D)$. 
Minimal Pair Type Theorems

**Theorem**
Let \( c \in DS_2(A) \). There exist total \( f, g \in DS(A) \), such that \( f'' = g'' = c \) and \( CS(A) = co(\{f, g\}) \).

**Theorem**
Let \( \alpha < \omega_1^{CK} \) and let \( b \in DS_\alpha(A) \). There exist elements \( f_0 \) and \( f_1 \) of \( DS(A) \) such that
- \( f_0^{(\alpha)} \leq b \) and \( f_1^{(\alpha)} \leq b \).
- If \( \beta < \alpha \), then \( f_0^{(\beta)} \) and \( f_1^{(\beta)} \) do not belong to \( CS_\beta(A) \).
- If \( \beta + 1 < \alpha \), then \( co(\{f_0^{(\beta)}, f_1^{(\beta)}\}) = CS_\beta(A) \).

**Example**
Finite lattice \( L = \{a, b, c, a \land b, a \land c, b \land c, \top, \bot\} \).

\[ A = \{d \in D_e : d \geq a \lor d \geq b \lor d \geq c\} \].
The Quasi-minimal degree

Definition
Let \( A \) be a set of enumeration degrees. The degree \( q \) is quasi-minimal with respect to \( A \) if:

- \( q \notin co(A) \).
- If \( a \) is total and \( a \geq q \), then \( a \in A \).
- If \( a \) is total and \( a \leq q \), then \( a \in co(A) \).

Theorem
If \( q \) is quasi-minimal with respect to \( A \), then \( q \) is an upper bound of \( co(A) \).

Theorem
For every structure \( \mathcal{A} \) there exists a quasi-minimal with respect to \( DS(\mathcal{A}) \) degree.
Definition
Let $B \subseteq A$ be sets of degrees. Then $B$ is a base of $A$ if

$$\forall a \in A \exists b \in B (b \leq a).$$

Theorem
Let $A$ be a set of degrees possessing a quasi-minimal degree. Suppose that there exists a countable base $B$ of $A$ such that all elements of $B$ are total. Then $A$ has a least element.

Corrolary
A total structure $\mathfrak{A}$ has a degree if and only if $DS(\mathfrak{A})$ has a countable base.
**Corollary**

Let \(a\) and \(b\) be incomparable Turing degrees. There does not exist a structure \(\mathcal{A}\) such that \(DS(\mathcal{A})\) is equal to the union of the cones above \(a\) and \(b\).

**Example**

An upwards closed set \(\mathcal{A}\) of degrees which does not possess a quasi-minimal degree.

Let \(a\) and \(b\) be two incomparable total degrees.

Let \(\mathcal{A} = \{c : c \geq a \lor c \geq b\}\).

Clearly \(\mathcal{A}\) has a countable base of total degrees, but it has not a least element. So, \(\mathcal{A}\) has no quasi-minimal degree.
Relatively \( \alpha \)-intrinsic sets

1989 Ash, Knight, Manasse, Slaman, Chisholm.

- The set \( A \) is relatively \( \alpha \)-intrinsic on \( \mathcal{A} \) if for every enumeration \( f \) of \( \mathcal{A} \) the set \( f^{-1}(A) \leq_e f^{-1}(\mathcal{A})^{(\alpha)} \) (\( \alpha \)-constructive ordinal).

2002 Soskov, Baleva.

- Let \( \{ B_\alpha \}_{\alpha \leq \zeta} \) be a sequence of subset of \( \mathbb{N} \) and \( \zeta \) be a constructive ordinal.
  - Add each set \( B_\alpha \) to the structure \( \mathcal{A} \) as a new predicate which is relatively \( \alpha \)-intrinsic on \( \mathcal{A} \).
  - Restrict the class of all enumerations of \( \mathcal{A} \) to the class of those enumerations \( f \) of \( \mathcal{A} \) for which \( f^{-1}(B_\alpha) \leq_e f^{-1}(\mathcal{A})^{(\alpha)} \).
Relative Spectra of Structures

Let $A_1, \ldots, A_n$ be arbitrary abstract structures on $\mathbb{N}$, $k \leq n$.

An enumeration $f$ of $A$ is **k-acceptable** with respect to the structures $A_1, \ldots, A_k$, if

$$f^{-1}(A_1) \leq_e (f^{-1}(A))^' \ldots f^{-1}(A_k) \leq_e (f^{-1}(A))^{(k)}.$$ 

Denote by $\mathcal{E}_k$ the class of all $k$-acceptable enumerations.

**Definition**

The **Relative spectrum** of the structure $A$ with respect to $A_1, \ldots, A_n$ is the set

$$RS(A, A_1, \ldots, A_n) = \{d_e(f^{-1}(A)) \mid f \in \mathcal{E}_n\}$$
Lemma

If $F$ is a total set, $f \in \mathcal{E}_n$ and $f^{-1}(\mathcal{A}) \leq_e F$, then there exists an enumeration $g \in \mathcal{E}_n$, such that

1. $g^{-1}(\mathcal{A}) \equiv_e F \oplus f^{-1}(\mathcal{A}) \equiv_e F$;
2. $g^{-1}(B) \leq_e F \oplus f^{-1}(B)$, for every $B \subseteq \mathbb{N}$.

Corollary

The Relative spectrum $\text{RS}(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n)$ is upwards closed.
Let $k \leq n$. The \textbf{kth Jump Relative spectrum} of $\mathcal{A}$ with respect to $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is the set

$$RS_k(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n) = \{ a^{(k)} | a \in RS(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n) \}.$$ 

\textbf{Proposition}

\textit{The kth Jump Relative spectrum} $RS_k(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n)$ is \textit{upwards closed}.
Relative Co-spectra of Structures

Definition
The Relative co-spectrum of $\mathcal{A}$ with respect to $\mathcal{A}_1, \ldots, \mathcal{A}_n$, is the co-set of $RS(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n)$, i.e.

$$CRS(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n) = \{ b \mid (\forall a \in RS(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n))(b \leq a) \}.$$ 

Let $k \leq n$. The Relative $k$th co-spectrum of $\mathcal{A}$ with respect to $\mathcal{A}_1, \ldots, \mathcal{A}_n$, is the co-set of $RS_k(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n)$, i.e.

$$CRS_k(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n) = \{ b \mid (\forall a \in RS_k(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n))(b \leq a) \}.$$
The jump set

The jump set $\mathcal{P}_k^f$ of $\mathcal{A}$ with respect to $\mathcal{A}_1, \ldots, \mathcal{A}_n$:

1. $\mathcal{P}_0^f = f^{-1}(\mathcal{A})$.
2. $\mathcal{P}_{k+1}^f = (\mathcal{P}_k^f)' \oplus f^{-1}(\mathcal{A}_{k+1})$.

Theorem

For every $A \subseteq \mathbb{N}$ and $k \leq n$, the following are equivalent:

1. $d_e(A) \in \text{CRS}_k(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n)$.
2. $A \leq_e \mathcal{P}_k^f$, for every $k$-acceptable enumeration $f$ of $\mathcal{A}$ with respect to $\mathcal{A}_1, \ldots, \mathcal{A}_k$. 
The set $A$ is \emph{formally $k$-definable} on $\mathcal{A}$ with respect to $\mathcal{A}_1, \ldots, \mathcal{A}_n$ if there exists a recursive sequence
\[
\{ \Phi^\gamma(x)(W_1, \ldots, W_r) \} \text{ of } \Sigma_k^+ \text{ formulae and elements}
\]
t_1, \ldots, t_r \text{ of } \mathbb{N} \text{ such that:}
\[
x \in A \iff (\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n) \models \Phi^\gamma(x)(W_1/t_1, \ldots, W_r/t_r).
\]

\begin{itemize}
  \item $\Sigma^+_0 : (\exists Y)(\beta_1 & \ldots & \beta_k)$ ;
  \item $\Sigma^+_k + 1 : \text{c.e. disjunction of } (\exists Y)\Phi(X, Y) ,
  \Phi = (\phi_1 & \ldots & \phi_l & \beta)$.
\end{itemize}

\textbf{Theorem}
\[
d_e(A) \in \text{CRS}_k(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n) \text{ if and only if } A \text{ is formally } k\text{-definable on } \mathcal{A} \text{ with respect to } \mathcal{A}_1, \ldots, \mathcal{A}_n.
\]
The connection with the Joint Spectra

**Definition**

The Joint spectrum of \( \mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n \) is the set

\[
\text{DS} \ (\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n) = \\
\{ a : a \in \text{DS}(\mathcal{A}), a' \in \text{DS}(\mathcal{A}_1), \ldots, a^{(n)} \in \text{DS}(\mathcal{A}_n) \}. 
\]

1. \( \text{CS}(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n) = \text{CRS}(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n). \)

2. There are structures \( \mathcal{A} \) and \( \mathcal{A}_1 \), for which
   \( \text{CS}_1(\mathcal{A}, \mathcal{A}_1) \neq \text{CRS}_1(\mathcal{A}, \mathcal{A}_1). \)

3. The difference:
   - \( A \leq_e \mathcal{P}(f^{-1}(\mathcal{A}), f_1^{-1}(\mathcal{A}_1), \ldots, f_n^{-1}(\mathcal{A}_n)) \) for every enumerations \( f \) of \( \mathcal{A} \), \( f_1 \) of \( \mathcal{A}_1 \), \ldots, \( f_n \) of \( \mathcal{A}_n \).
   - in the normal form \( (\mathcal{A}, \mathcal{A}_1 \ldots, \mathcal{A}_n) \) — as a many-sorted structure with separated sorts.
Minimal Pair Theorem

Theorem

For any structures $\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n$, there exist enumeration degrees $f$ and $g$ in $\text{RS}(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n)$, such that for any enumeration degree $a$ and each $k \leq n$:

$$a \leq f^{(k)} \& a \leq g^{(k)} \Rightarrow a \in \text{CRS}_k(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n).$$
Quasi-Minimal Degree

Theorem

For any structures $A, A_1, \ldots, A_n$ there exists an enumeration degree $q$ such that:

1. $q \notin \text{CRS}(A, A_1, \ldots, A_n)$;
2. If $a$ is a total degree and $a \geq q$, then $a \in \text{RS}(A, A_1, \ldots, A_n)$;
3. If $a$ is a total degree and $a \leq q$, then $a \in \text{CRS}(A, A_1 \ldots A_n)$. 

