

## Rudimentary recursion, provident sets and forcing.

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Many important set-theoretical functions, such as rank or transitive closure, are defined by a recursion of the form

$$F(x) = G(F \upharpoonright x)$$

where  $G$  is a rudimentary function, and  $F \upharpoonright x$  denotes the restriction of  $F$  to (the members of)  $x$ .

Others become similarly definable when we permit parameters. If  $p$  is a set, call  $F$  *p-rud-rec* if there is a rud function  $G$  such that for all  $x$ ,

$$F(x) = G(p, F \upharpoonright x),$$

and call a set  $A$  *provident* if it is transitive and closed under all *p-rud-rec* functions with  $p$  a member of  $A$ . \*

If  $\zeta$  is the least ordinal not in a provident set  $A$ , then  $\zeta$  is indecomposable, that is, that the sum of two ordinals less than  $\zeta$  is less than  $\zeta$ . Conversely, if  $\zeta$  is indecomposable,  $\eta \geq \zeta$  and  $p \in J_\zeta$ , then the Jensen set  $J_\eta$  is closed under all *p-rud-rec* functions; in particular  $J_\zeta$  is provident.

Let  $c$  be a transitive set. A modification of the usual hierarchy defining the constructible closure  $L(c)$  of  $c$  proves desirable: define, by a simultaneous rudimentary recursion on ordinals, sets  $c_\nu$ ,  $P_\nu^c$  thus:

$$\begin{array}{lll} c_0 & = & \emptyset & c_{\nu+1} & = & c \cap \{x \mid x \subseteq c_\nu\} & c_\lambda & = & \bigcup_{\nu < \lambda} c_\nu \\ P_0^c & = & \emptyset & P_{\nu+1}^c & = & \{c_\nu\} \cup c_{\nu+1} \cup \mathbb{T}(P_\nu^c) & P_\lambda^c & = & \bigcup_{\nu < \lambda} P_\nu^c \end{array}$$

$\mathbb{T}$  being the rudimentary function introduced in [M] such that for each transitive set  $u$ ,  $u \subseteq \mathbb{T}(u) \subseteq \mathcal{P}(u)$ ,  $u \in \mathbb{T}(u)$ , and the rudimentary closure of  $u \cup \{u\}$  equals  $\bigcup_{n \in \omega} \mathbb{T}^n(u)$ . Then  $P_\theta^c$  will be provident whenever  $\theta$  is indecomposable and  $> 1$ .

Let  $\theta \geq \omega$  be indecomposable, and  $C$  a collection of transitive sets, each of rank strictly less than  $\theta$ , with the property that any two elements of  $C$  are members of a third. Then  $\bigcup_{c \in C} P_\theta^c$  is provident, and every provident set is of this form.

Another characterization is that a provident set is a transitive set  $A$  closed under ordinal addition and pairing, under the functions rank, transitive closure and  $\mathbb{T}$ , and with the property that whenever  $c$  is a transitive member of  $A$  and  $\nu$  is an ordinal in  $A$ , the set  $P_\nu^c$  is in  $A$ . This remark gives a finitely axiomatisable set theory, which we call **MSF**, for “minimal for Shoenfield forcing”, of which the transitive models are precisely the provident sets.

**Theorem:** *Let  $A$  be provident and  $\mathbb{P}$  a separative poset which is a member of  $A$ . Then forcing over  $A$  with  $\mathbb{P}$  can be defined in  $A$ , and the forcing relation for  $\Delta_0$  formulae is in a precise sense close to  $\mathbb{P}$ -rud-rec; if  $A$  is provident and if  $X$  is  $(A, \mathbb{P})$  generic, then the generic extension  $A[X]$  is provident.*

Special cases of the above for certain *J*-like fragments were known to Hauser [H] and to Steel, cf. [S].

Each *p-rud-rec* function is primitive recursive in the sense of Jensen and Karp [JK]. Many years ago Jensen (unpublished) verified that set forcing over a primitive recursively closed set is definable and that the extension will be primitive recursively closed. Presumably there are many intermediate classes of function closure under which is similarly preserved under set-generic extension.

Finally, Gandy in [G] mentions an unsatisfactory attempt to study fragments of the infinitary language  $L_{\omega_1\omega}$  using the class of rudimentary functions; it appears that the fragments  $A \cap L_{\omega_1\omega}$ , where  $A$  is provident and countably infinite, will have the properties he was seeking.

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- [S] J. Steel, Scales in  $K(\mathbb{R})$ , preprint dated September 11 2003.

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\* Originally I called such sets *progressive*, in ignorance of that term’s established use in PCF theory.