Löwenheim-Skolem type properties in choiceless set theory

Ioanna M Dimitriou

We'll look at Löwenheim-Skolem type statements about first-order structures for countable languages and their possible elementary substructures. In particular we'll discuss the following statements. For cardinals $\kappa, \lambda, \kappa', \lambda'$,

$$(\kappa, \lambda) \to (\kappa', \lambda')$$

is the statement that every first-order structure $\langle \kappa, \lambda, \ldots \rangle$ with underlying set κ and distinguished predicate λ has an elementary substructure $\langle A, B, \ldots \rangle \prec \langle \kappa, \lambda, \ldots \rangle$ such that $\mathsf{card}(A) = \kappa'$ and $\mathsf{card}(B) = \lambda'$, [CK90, page 450]. Such a statement is called a higher Chang conjecture. The original conjecture by Cheng-Chung Chang states that $(\omega_2, \omega_1) \to (\omega_1, \omega)$.

These two-cardinal versions of the Löwenheim Skolem theorem ([Kan03, page 85]) are large cardinal statements, deeply connected to certain infinitary combinatorial principles. In the presence of the axiom of choice some large cardinal principles are so strong that their consistency strengths with respect to the standard set theoretic axiom system ZFC cannot be exactly determined by current techniques. Weakening or omitting the choice assumptions can, however, weaken principles so that their consistency strengths become tractable.

These higher Chang conjectures have been thoroughly studied under the axiom of choice and only partial answers have been given about their consistency strength. By dropping choice assumptions we managed to exactly determine the consistency strength for almost all possible values of $\kappa, \lambda, \kappa', \lambda'$.

References

[CK90] Chen Chung Chang and H. Jerome Keisler, *Model theory*, 3rd edition, Studies in Logic and the Foundations of Mathematics vol.58, North Holland 1990

[Kan03] Akihiro Kanamori, *The higher infinite*, 2nd edition, Springer Monographs in Mathematics, Springer 2003