Finitary properties for a monoid S arising from model theory for S-acts

Victoria Gould

A *monoid* is a semigroup with identity: a *finitary property for monoids* is a property guaranteed to be satisfied by any finite monoid, such as chain conditions on the lattice of right ideals. A monoid S may be represented via mappings of sets, or, equivalently and more concretely, by *(right) S-acts*. Here an S-act is a set A together with a map $A \times S \to A$ where $(a,s) \mapsto as$, such that for all $a \in A$ and $s,t \in S$ we have $a1 = a$ and $(as)t = a(st)$. I will be speaking about finitary properties for monoids arising from model theoretic considerations for S-acts.

Let L_S be the first-order language of S-acts, so that L_S has no constant or relational symbols (other than $=$) and a unary function symbol ρ_s for each $s \in S$. Clearly Σ_S axiomatises the class of S-acts, where

$$\Sigma_S = \big\{(\forall x)(x\rho_s\rho_t = x\rho_{st}) : s,t \in S\big\} \cup \{(\forall x)(x\rho_1 = x)\}.$$

Model theory tells us that Σ_S has a model companion Σ_S^* precisely when the class $\mathcal{E}$ of existentially closed S-acts is axiomatisable and in this case, Σ_S^* axiomatises $\mathcal{E}$. An old result of Wheeler tells us that Σ_S^* exists if and only if for every finitely generated right congruence μ on S, every finitely generated S-subact of S/μ is finitely presented, that is, S is *right coherent*. Until recently, little was known about right coherent monoids, and in particular whether free monoids are (right) coherent. I will present some work of Gould, Hartmann, Ruškuc and Yang in this direction; specifically we answer positively the question for free monoids.

Where Σ_S^* exists, it is known to be stable, and is superstable if and only if S has the maximal condition on right ideals, that is, S is *weakly right noetherian*. By using a description of types over Σ_S^*, we can show that Σ_S^* is totally transcendental if and only if S is weakly right noetherian and S is *ranked*. The latter condition says that every right congruence possesses a finite Cantor-Bendixon rank with respect to the *finite type topology*. Our results show that there is a totally transcendental theory of S-acts for which Morley rank of types does not coincide with U-rank, contrasting with the corresponding situation for modules over a ring.