

Analysis and operators

Research and training network (5th PCRDT)

Contract HPRN-CT-2000-00116

June 1, 2000 to May 31, 2004

Full title : [Classical analysis, operator theory, geometry of Banach spaces, their interplay and their applications](#)

Budget : **1 494 000 euro**

Coordinating team : Université Bordeaux 1

Homepage :

<http://maths.leeds.ac.uk/pure/analysis/rtn.html>

Nodes, subnodes and node coordinators :

1. Université Bordeaux 1 (Lille, Metz) N.Nikolski
2. Vrije Universitet Amsterdam (Delft, Leiden), R.Kaashoek
3. Universitat autonoma de Barcelona (Barcelona, La Laguna), J.Bruna
4. University College Dublin (Belfast, Dusseldorf, Maynooth, London), S.Gardiner
5. Leeds University (Cambridge, Lancaster, Newcastle, Sheffield), J.Partington
6. Université Paris 6 (Besançon, Cergy-Pontoise, Lyon, Marne la Vallée), Y.Raynaud
7. Norwegian University of Science and Technology at Trondheim (Bergen, Lund, Uppsala, Stockholm), K.Seip
8. TU Vienna (Bremen, Regensburg), H.Langer
9. Tel-Aviv University, A.Atzmon
10. St Petersburg branch of the Steklov Institute (University of St Petersburg), S.Kislyakov₂

Research objectives (from contract)

1) Function theory

For [Bergman](#) and related [spaces of holomorphic functions](#), develop factorization theory and characterize "inner-outer" functions in terms of their growth near the boundary. In one and multi-dimensional situations, characterize [interpolating](#) and [sampling sequences](#), and use interpolating Blaschke products to find new invertibility criteria for [Toeplitz operators](#).

Develop approximation theory ([harmonic approximation](#) on general sets, tangential approximation) and find further relations between quadrature identities and best approximation problems. Improve understanding of [capacities](#) in metric and geometric terms and, more generally, improve understanding of the [Cauchy](#) and [Hilbert transforms](#).

2) Operator theory

Develop the theory of function models, and clarify the relations between spectral properties of **Hankel and Toeplitz operators** and function theoretical properties of their symbols.

Develop new **operator theoretical** methods to analyze problems arising from concrete classes of integral differential and delay equations.

Describe the spaces spanned by generalized eigenvectors for nonselfadjoint operators arising from delay equations.

Find new applications of the **Brown approximation scheme**, and use function theoretical tools to study **bi-invariant subspaces** of $l^2_{\omega}(\mathbb{Z})$.

3) Geometry of Banach spaces, Convex geometry

Develop theories emerging from Banach spaces geometry and related to function theory and operator theory, ([operator spaces](#), noncommutative analysis) and continue the "transfer of technology of Banach spaces" to these areas.

Find new applications to analysis, convex geometry and statistical mechanics of the principle of concentration of measure and of the majorizing measure theorem. Improve estimates for contractive approximation algorithms of convex bodies by polytopes.

Develop [variational principles](#) and pursue their applications to differential equations.

Monographs by network members directly related to the training program (1)

(team 4) D.H. Armitage and S.J. Gardiner, [Classical potential theory](#). Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2001. xvi+333 pp.

H. Hedenmalm (team 7), B. Korenblum and K.Zhu, [Theory of Bergman spaces](#). Graduate Texts in Mathematics, 199. Springer-Verlag, New York, 2000. x+286 pp.

(team 4) J.R.Partington, [Linear operators and linear systems](#) : An Analytical approach to Control Theory, London Math. Soc. Student texts 60, Cambridge University Press, 2004, 176 pp.

(team 6) G. Pisier, [Introduction to operator space theory](#). London Mathematical Society Lecture Note Series, 294. Cambridge University Press, Cambridge, 2003. viii+478 pp.

Monographs by network members directly related to the training program (2)

(team 1) N.K. Nikolski, [Operators, functions, and systems](#) : an easy reading.

Vol. 1. Hardy, Hankel, and Toeplitz. Mathematical Surveys and Monographs, 92. American Mathematical Society, Providence, RI, 2002. xiv+461 pp.

Vol. 2. Model operators and systems. Mathematical Surveys and Monographs, 93. American Mathematical Society, Providence, RI, 2002. xiv+439 pp.

K.Seip (team 7), [Interpolation and Sampling in Spaces of Analytic Functions](#) University Lect. Notes Series 33, AMS, Providence, R.I., 2004, xii+139 pp.

Other relevant monographs

(team 5) H.G.Dales, [Banach algebras and automatic continuity](#). London Mathematical Society Monographs. New Series, 24. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 2000. xviii+907 pp.

P.Ara and M. Mathieu (team 4) [Local multipliers of \$C^*\$ -algebras](#). Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2003. xii+319 pp.

M.Talagrand, (team 6) [Spin glasses : A challenge for mathematicians. Cavity and mean field models](#). Erg. Math. 46, Springer-Verlag Berlin, 2003, ix, 586 pp.

Relevant surveys

R. Deville ([team 1](#)), G. Godefroy, B. Maurey, A. Pajor, G. Pisier, S. J. Szarek, Q. Xu ([team 6](#)), V. Milman ([team 9](#)) and S. Kislyakov ([team 10](#)) made one or several contributions to the two volumes of the [Handbook of the Geometry of Banach spaces](#), edited by W. B. Johnson and J. Lindenstrauss, North-Holland, Volume 1, 2001 and volume 2, 2003.

Invited speakers at 4ecm

M. Sodin ([team 9](#)) and X. Tolsa ([team 3](#)) gave an invited talk at 4ecm, and X. Tolsa got one of the ten ECM prizes for his (outstanding) proof of semiadditivity of analytic and continuous analytic capacities.

Besides individual training provided in the various nodes, training at network level was organized through 3 types of events

1) Series of morning conferences at annual meetings.

First general meeting, **St Petersburg**, May 13-17, 2001, (44 network participants + 4 participants exterior to the network)

Second general meeting, **Biarritz**, May 2-7, 2002, (68+3)

Third general meeting, **Tenerife**, May 21-26, 2003, (67+14)

Fourth general meeting, **Dalfsen**, the Netherlands, May 1-7, 2004, (50+2)

2) More specialized workshops on specific topics

Spaces of Analytic Functions and Applications, **Trondheim**, July 2-4, 2003

Invariant Subspaces, **Leeds**, July 3-5, 2003

Operator Theory and Applications, **Amsterdam**, August 20-22, 2003

Bergman Spaces and Related Topics in Complex Analysis, **Barcelona**, November 20-23, 2003

3) Pre/postdoc meetings

Postdoc workshop, **Bordeaux**, January 17-18, 2002

Pre/postdoc workshop, **Paris**, November 21-22, 2002

Pre/postdoc workshop, **Paris**, January 22-23, 2004.

Series of Morning Lectures at annual meetings
(1)

St Petersburg (May 2001) Capacities and harmonic approximation, by S.Gardiner and A.O'Farrell (**Dublin**)- Linear approximation on Krein spaces and applications, by H.Langer (**Vienna**), Spectral Analysis of selfadjoint Jacobi matrices, by S. Naboko (**St Petersburg**)

Biarritz (May 2002) The semiadditivity of analytic capacity, by X.Tolsa (**Barcelona**), Interpolation of Hardy type spaces, by S.Kislyakov (**St Petersburg**)- Geometric aspects of approximation in high dimension and connections of convex geometry with complexity theory, by V.Milman (**Tel-Aviv**), Local theory of operator spaces, by G.Pisier (**Paris**)

Tenerife (May 2003) Bergman function theory, by H. Hedenmalm (**Stockholm**)- Control theory for analysts, by N.Nikolski (**Bordeaux**)-Translation invariant subspaces, by A.Atzmon (**Tel-Aviv**), and J.E. (**Bordeaux**)

Series of morning lectures at annual meetings
(2)

Dalfsen (May 2004) **Singular integrals and capacities**, by G.David (Orsay) and J.Verdera (**Barcelona**)-**Delay equations and infinite dimensional systems**, by J.Partington (**Leeds**) and S.Verduyn Lunel (**Leiden**)- **Toeplitz operators on Bergman spaces**, by N.Vasilevski (Mexico).

Possible improvements :

- 1) Put on the network webpage an access to the slides used for the series of morning lectures (this was partially done for the lectures at the third and fourth meetings).
- 2) Give in advance a **relevant bibliography** and arrange within the nodes **training seminars** related to these morning lectures before and after the annual meetings.

$$\{f \in \text{Hol}D \mid \sup_{0 \leq r \leq 1} \int_0^{2\pi} |f(re^{it})|^2 dt < +\infty\}$$

$$= H^2(D) \simeq \{f \in L^2(T) \mid \hat{f}(n) = 0 \ \forall n < 0\}$$

$$\{F \in \text{Hol}(P^+) \mid \sup_{b>0} \int_{-\infty}^{+\infty} |F(x+ib)|^2 dx < +\infty\}$$

$$= H^2(P^+) \simeq \{F \in L^2(\mathbb{R}) \mid \hat{F}|_{\mathbb{R}^-} = 0 \text{ a.e.}\}$$

D open unit disc, P^+ open upper half-plane.

$H^\infty(D)$, $H^\infty(P^+)$: bounded holomorphic functions on D and P^+ .

$$f^*(e^{it}) := \lim_{r \rightarrow 1^-} f(re^{it}) \text{ a.e. for } f \in H^2(D).$$

$$P_+(f)(z) = \sum_{n=0}^{+\infty} \hat{f}(n) z^n = \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for $f \in L^2(\mathbb{T})$, $|z| < 1$.

For $\phi \in L^\infty(T)$, the Toeplitz operator of symbol ϕ is the operator $T_\phi : H^2(D) \rightarrow H^2(D)$ defined by $T_\phi(f) = P_+(f^*\phi)$.

It is known that if $|\phi| = 1$ a.e., T_ϕ is invertible iff $\text{dist}(\phi, H^\infty) < 1$ and $\text{dist}(\bar{\phi}, H^\infty) < 1$. $S = T_z$ is the **shift operator** and $R : f \mapsto \frac{f-f(0)}{z}$ is the **backward shift**, and $R = S^*$. If T_ϕ is Toeplitz, $\text{Ker}T_\phi$ is **nearly R-invariant** : if $f \in \text{Ker}T_\phi$, and if $f(0) = 0$, then $Rf = \frac{f}{z} \in \text{Ker}(T_\phi)$.

Let g be the **extremal function** for $M := \text{Ker}T_\phi$, i.e. $\|g\| = 1$, $\text{Re}g(0)$ is maximum. Then multiplication by g is an **isometry** on $N := \{f/g : f \in M\}$, and N is **R-invariant**. Hence $N = H^2 \ominus IH^2$, where $I \in H^\infty$ is **inner**, i.e. $|I^*(e^{it})| = 1$ a.e.

Also g is **outer** : $g(z) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|g^*(e^{it})| dt\right)$.

One can write $g = \frac{a}{1-b}$, a, b in unit ball of H^∞ , where $\frac{1+b(z)}{1-b(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} |g(e^{it})|^2 dt$. Then $|a|^2 + |b|^2 = 1$ a.e. , and I is a divisor of b in H^∞ .

Theorem (Hartmann-Sarason-Seip) : If ϕ unimodular and $\text{Ker}T_\phi \neq \{0\}$ then T_ϕ is onto iff $g_0 := \frac{a}{1-b/I}$ satisfies $g_0^2 = \exp(u + \tilde{v})$, $u, v \in L^\infty_{\mathbb{R}}(\mathbb{T})$, $\|v\| < \frac{\pi}{2}$ (Helson-Szego weight)

A.Hartmann(team 1) was a postdoc at Trondheim(team 7) from September 2001 to August 2002. This paper is the second joint paper with the coordinator of team 7 initiated during the postdoc position. A.Hartmann is now preparing his habilitation at Bordeaux.

The space $B_2 = \{f \in \text{Hol}D \mid \int_D |f(x+iy)|^2 dx dy < +\infty\}$ is the **Bergman space**, natural analog of the **Hardy space** $H^2(D)$.

On B_2

-analogs of the notions of **inner and outer functions**

-elaborate theory of **contractive divisors** of z -invariant (i.e. shift-invariant) subspaces of B_2 due to Hedenmalm, based on the notion of **extremal function**, see the monograph by Hedenmalm, Korenblum and Zhu.

-no progress during network activity concerning the characterization of inner and outer functions in terms of their behavior near the boundary.

-no characterization of **zero sets**.

-very large lattice of **z -invariant subspaces**

-if M z -invariant, **$M = \bigvee_{n \geq 0} z^n (M \ominus zM)$**

For $\omega \in L^2[0, 1]$, strictly positive, set

$$B_2(\omega) = \{f \in \text{Hol}(D) \mid \int_D |f(z)|^2 \omega^2(|z|) dm(z) < +\infty\},$$

$dm(x + iy) = \frac{dx dy}{\pi}$. This is a closed subspace of

$$L^2(\omega) = \{f \text{ meas.} \mid \int_D |f(z)|^2 \omega^2(|z|) dm(z) < +\infty\}.$$

Set $\sigma(n) = [2 \int_0^1 r^{2n+1} \omega^2(r) dr]^{\frac{1}{2}}$, $\hat{f}(n) = \frac{f^{(n)}(0)}{n!}$
for $n \geq 0$, $f \in \text{Hol}(D)$.

$$B_2(\omega) = H^2(\sigma)$$

$$:= \{f \in \text{Hol}(D) \mid \sum_{n=0}^{+\infty} |\hat{f}(n)|^2 \sigma(n)^2 < +\infty\}$$

Set $k_\lambda^\omega(z) = \sum_{n=0}^{+\infty} \bar{\lambda}^n \sigma^{-2}(n) z^n$, so that k_λ^ω is the reproducing kernel for $B_2(\omega)$:

$$f(\lambda) = \langle f, k_\lambda^\omega \rangle \quad \text{for } f \in B_2(\omega), |\lambda| < 1.$$

For $\phi \in L^\infty(D)$, $f \in B_2(\omega)$, $\lambda \in D$, define the Toeplitz operator of symbol ϕ by the formula

$$T_\phi(f)(\lambda) = \langle \phi f, k_\lambda^\omega \rangle, \quad (1)$$

so that $T_\phi(f) = P_+(\phi f)$, where P_+ is the orthogonal projection of $L^2(\omega)$ onto $B_2(\omega)$.

Since k_λ^ω is bounded on D , formula 1 defines an analytic function $T_\phi(f)$ on D for $\phi \in L^2(\omega)$, $f \in B_2(\omega)$, and for $\phi \omega^2 \in L^1(D)$, the Berezin transform of ϕ is defined on D by the formula

$$Ber_\omega(\phi)(\lambda) = \frac{\langle \phi k_\lambda^\omega, k_\lambda^\omega \rangle}{\|k_\lambda^\omega\|^2} \quad (2)$$

Set $\omega_\alpha(r) = (1 - r^2)^{\alpha/2}$ for $\alpha \in (-1, +\infty)$. Following partial answers by Stroethoff and Zheng to a question of Sarason concerning boundedness of the product of Toeplitz operators on the standard Bergman space, Sandra Pott (team 5) and Elizabeth Strouse (team 1) obtained the following result

Theorem : Let $\alpha \in (-1, +\infty)$, and let $\phi, \psi \in B_2(\omega_\alpha)$.

(i) If $T_\phi T_{\bar{\psi}}$ defines a bounded operator from $B_2(\omega_\alpha)$ into itself, then

$$\sup_{\lambda \in D} \text{Ber}_{\omega_\alpha}(|\phi|^2)(\lambda) \text{Ber}_{\omega_\alpha}(|\psi|^2)(\lambda) < +\infty.$$

(ii) If $\sup_{\lambda \in D} \text{Ber}_{\omega_\alpha}(|\phi|^2)(\lambda) \text{Ber}_{\omega_\alpha}(|\psi|^2)(\lambda) < +\infty$, then $T_\phi T_{\bar{\psi}}$ defines a bounded operator from $B_2(\omega_\beta)$ into itself for every $\beta > \alpha$.

This work was initiated during a three months postdoc appointment of S.Pott at Bordeaux in the spring 2003.

The existence of a nontrivial zero-free closed z -invariant subspace M of $B_2(\omega)$ such that $\dim(M \ominus zM) = 1$ **is an open problem**. Partial results go back to Nikolski (1974) and Atzmon obtained a positive answer with a mild regularity condition on the weight by using entire functions of zero exponential type in 1997. Borichev(**team 1**), Hedenmalm(**team 7**) and Volberg (**partially in team 6**) obtained the following result (JFA 207 (2004), 111-160) (their functions F has in some sense **"extremal growth"**)

Theorem Assume that $\omega(r)$ decreases to zero as $r \rightarrow 1^-$, and satisfies for some $\epsilon \in (0, 1)$

$$\lim_{r \rightarrow 1^-} (1 - r)^\epsilon \log \log \frac{1}{\omega(r)} = 0$$

Then there exists a non z -cyclic function $F \in B_2(\omega)$ without zeroes in D .

Other spaces of holomorphic functions include the Paley-Wiener space PW of all entire functions of exponential type at most π whose restrictions to \mathbb{R} are square-integrable.

A sequence $\Lambda = (\lambda_k)_{k \in \mathbb{Z}}$ is **sampling** for PW iff there exists $A, B > 0$ such that

$$\begin{aligned} A \int_{-\infty}^{+\infty} |g(x)|^2 dx &\leq \sum_{k=-\infty}^{+\infty} |g(\lambda_k)|^2 \\ &\leq B \int_{-\infty}^{+\infty} |g(x)|^2 dx \end{aligned} \quad (3)$$

holds for all $f \in PW$.

Set $\hat{f}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-izt} dt$ for $f \in L^2[-\pi, \pi]$. Then $\mathcal{F} : f \mapsto \hat{f}$ is an **isometry from $L^2[-\pi, \pi]$ onto PW** (Paley-Wiener theorem).

Condition (3) is equivalent to the fact that

$$\begin{aligned} A \int_{-\pi}^{\pi} |f(t)|^2 dt &\leq \sum_{k=-\infty}^{+\infty} \left| \int_{-\pi}^{\pi} f(t) e^{-i\lambda_k t} dt \right|^2 \\ &\leq B \int_{-\pi}^{\pi} |f(t)|^2 dt \end{aligned} \quad (4)$$

for all $f \in L^2[-\pi, \pi]$.

In other terms the sequence (λ_k) is sampling for PW iff the system $\{e^{i\lambda_k x}\}$ is a Fourier frame in the sense of Duffin and Schaeffer.

A nondecreasing sequence (λ_k) is separated if $\inf_k \lambda_{k+1} - \lambda_k > 0$, and (λ_k) is a complete interpolating sequence if the equation $f(\lambda_k) = a_k \forall k$ has a unique solution in PW for every square integrable sequence $(a_k)_{k \in \mathbb{Z}}$. Separated complete interpolating sequences were characterized by Pavlov and Hruschev-Nikolski-Pavlov, and these sequences are in some sense sampling sequences with no redundant points.

There exists sampling sequences for which no subsequence is complete interpolating, but every sampling sequence has a separated sampling subsequence.

A necessary condition for a separated sequence to be sampling is given by Landau's inequality.

Ortega-Cerda (Barcelona) and Seip (Trondheim) gave a characterisation of separated sampling sequences (Annals of Maths 155 (2002), 789-806).

U : set of all entire function E without zeroes in the upper half-plane such that $|E(z)| \geq |E(\bar{z})|$ for $Im(z) > 0$.

If $E \in U$, $H(E)$ is the set of entire functions f such that $\frac{f(z)}{E(z)}$ and $\frac{\bar{f}(\bar{z})}{E(z)}$ belong to $H^2(P^+)$.

In fact $H(E)$ is a Hilbert space with respect to the norm $\|f\|_E = \int_{-\infty}^{+\infty} \left| \frac{f(t)}{E(t)} \right|^2 dt$. The family of these spaces is exactly the class of de Branges Hilbert spaces of entire functions.

Theorem (Ortega-Cerda, Seip) A separated sequence Λ is sampling for PW iff there exists two entire functions E, F in U such that

(i) $H(E) = PW$

(ii) Λ is the zero sequence of the entire function $z \mapsto E(z)F(z) + \bar{E}(\bar{z})\bar{F}(\bar{z})$

Using this theorem it is in particular possible to deduce Landau's inequality from the Hruschev-Nikolski-Pavlov theorem by using the John-Nirenberg theorem for BMO functions.

Poisson kernel on D : $P(z, e^{it}) = \frac{1}{2\pi} \frac{1-|z|^2}{|z-e^{it}|^2}$

$h^\infty(D)$: space of real valued bounded harmonic functions on D .

Theorem (Bonsall, 1987) Let $E \subset D$. The following are equivalent

(a) For each $f \in L^1(\partial D)$ and $\epsilon > 0$ there exists sequences $(\lambda_k) \in \mathbb{R}$ and (x_k) in E such that

$$f = \sum \lambda_k P(x_k, \cdot) \text{ in } L^1(\partial D) \text{ and} \\ \sum |\lambda_k| < \|f\|_{L^1(\partial D)} + \epsilon$$

(b) $\sup_E h = \sup_D h$ for every $h \in h^\infty(\partial D)$.

(c) Almost every point of ∂D is the nontangential limit of some sequence in E .

Analogous result for $C^+(\partial D)$ by Hayman and Lyons (1990)

An extension of these results to all connected Greenian open subsets of \mathbb{R}^N , was obtained by S.Gardiner (team 4) and J.Pau (team 3) during the one year postdoctoral position of Jordi Pau at Dublin (Illinois J. Maths, 47 (2003) 1115-1136).

$$\text{Set } \begin{cases} U_y(x) = -\log(\|x - y\|) & x \neq y, N = 2 \\ U_y(x) = \|x - y\|^{2-N} & x \neq y, N \geq 3 \\ U_y(x) = +\infty & x = y \end{cases}$$

An open set Ω is **Greenian** when U_y has a subharmonic minorant on Ω for every $y \in \Omega$ (always true for $N \geq 3$). In this case U_y has a largest harmonic minorant h_y on Ω and the Green function G on Ω is defined on $\Omega \times \Omega$ by the formula

$$G(x, y) = U_y(x) - h_y(x)$$

For the general case of the Gardiner-Pau theorem the notion of **Martin boundary** and **sets of minimal boundary points** would be needed (see **Chap.8 of the Armitage-Gardiner monograph**).

The situation is simpler for **Lipschitz domains**. Let $\nu_0 \geq 0$ be a measure with compact support contained in Ω . The **Martin kernel** is then defined on $\Omega \times \partial\Omega$ by the formula

$$M(x, y) = \lim_{z \rightarrow y} \frac{G(x, z)}{\int_{\Omega} G(z, u) d\nu_0(u)}.$$

Theorem (Gardiner-Pau) Let Ω be a bounded Lipschitz domain of \mathbb{R}^N and let $\mu \geq 0$ be a measure on $\partial\Omega$. Set $H(x) = \int_{\partial\Omega} M(x, y) d\mu(y)$ ($x \in \Omega$). The following conditions are equivalent for $E \subset \Omega$:

a) For each $f \in L^1(\mu)$ and $\epsilon > 0$ there exists sequences $(\lambda_k) \in \mathbb{R}$ and (x_k) in E such that

$$f = \sum \lambda_k M(x_k, \cdot) \text{ in } L^1(\mu) \text{ and } \sum |\lambda_k| H(x_k) < \|f\|_{L^1(\mu)} + \epsilon$$

(b) $\sup_E \frac{h}{H} = \sup_D \frac{h}{H}$ for every harmonic function h such that $\frac{h}{H}$ is bounded on Ω .

Consider the state linear system

$$\begin{cases} z'(t) = Az(t) + Bu(t) \\ y(t) = Cz(t) \\ z(0) = z_0 \end{cases}$$

where $u(t), y(t), z(t)$ respectively belong to the separable Hilbert spaces U, Y, Z , where A is the generator of a C_0 -semigroup $T(t)_{t>0}$ on Z , and where $B : U \rightarrow Z$ and $C : Y \rightarrow Z$ are bounded linear operators. The trajectories are given by

$$\begin{cases} z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)ds \\ y(t) = Cz(t) \end{cases}$$

Given a bounded invertible positive operator $R : U \rightarrow U$ we want to minimize for $u \in L^2([0, \infty), U)$ the quantity

$$J(z_0, u) = \int_0^{+\infty} \|y(s)\|^2 ds + \int_0^{+\infty} \|Ru(s)\|^2 ds$$

This problem is discussed in Chapter 6 of the monograph by Ruth Curtain and Hans Zwart (Text in Applied Math. 21, Springer-Verlag, 1995). Assuming that the linear system is optimizable, which just means that for each $z_0 \in Z$ there exists an input function u such that $J(z_0, u) < +\infty$, there exists a selfadjoint bounded nonnegative operator $\Pi : Z \rightarrow Z$ such that $\min_{u \in L^2([0, \infty), U)} J(z_0, u) = \langle z_0, \Pi z_0 \rangle$.

The minimizing function $s \mapsto u_{min}(s, z_0)$ can be computed explicitly from Π , and Π happens to be the minimal nonnegative solution in $\mathcal{B}(Z)$ of the "weak algebraic Ricatti equation"

$$\begin{aligned} & \langle Az_1, \Pi z_2 \rangle + \langle \Pi z_1, Az_2 \rangle + \langle Cz_1, Cz_2 \rangle \\ & - \langle R^{-1}B^*\Pi z_1, R^{-1}B^*\Pi z_2 \rangle = 0 \end{aligned}$$

for $z_1, z_2 \in \text{Dom}(A)$.

Set $Q = C^*C, D = BR^{-2}B^*$, so that Q and D are bounded and nonnegative. In some situations studied by H.Langer (team 8), A.C.M.Ran (team 2) and B.A. Van de Rotten (team 2) the strong algebraic Ricatti equation

$$\Pi D \Pi - A^* \Pi - \Pi A - Q = 0$$

has a unique nonnegative bounded solution Π , which is the minimal symmetric solution of the weak equation. Set $\tilde{A} = \begin{bmatrix} A & -D \\ -Q & -A^* \end{bmatrix}$, which we view as a perturbation of $\tilde{A}_0 = \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix}$. If the closed densely defined operator A satisfies $A - zI$ invertible, $\|(A - zI)^{-1}\| \leq \frac{M}{1+|z|^\beta}$ for $|\operatorname{Re}z| < \omega_0$, with $\omega_0 > 0, M > 0, \beta > 1/2$, then there exists $\omega > 0$ such that

$$\begin{aligned} &\tilde{A} - zI \text{ is invertible for } |z| \leq \omega \\ &\lim_{|t| \rightarrow \infty} \sup_{|s| \leq \omega} \|(\tilde{A} - (s + it)I)^{-1}\| = 0. \end{aligned}$$

Moreover there exists two closed subspaces \tilde{H}_+ and \tilde{H}_- of $\tilde{H} := H \oplus H$, invariant for \tilde{A} such that

$D(\tilde{A}) \cap \tilde{H}_+$ dense in H_+ , $D(\tilde{A}) \cap \tilde{H}_-$ dense in \tilde{H}_- , \tilde{H} is the direct sum of \tilde{H}_+ and \tilde{H}_- , and such that if \tilde{A}_+ is the restriction of \tilde{A} to $D(\tilde{A}) \cap \tilde{H}_+$ and \tilde{A}_- the restriction of \tilde{A} to $D(\tilde{A}) \cap \tilde{H}_-$ we have $\inf_{z \in \text{Spec} \tilde{A}_+} \text{Re} z > 0$, $\sup_{z \in \text{Spec} \tilde{A}_-} \text{Re} z < 0$.

Assume $\{(A - zI)^{-1}Dx : x \in H, |\text{Re} z| \leq \omega_0\}^- = H$ (the pair (A, D) is approximately controllable) and $\{(A^* - zI)^{-1}Qx : x \in H, |\text{Re} z| \leq \omega_0\}^- = H$ (the pair (Q, A) is approximately observable) then there exists a (possibly unbounded) positive one-to-one selfadjoint operator Π_+ and a (possibly unbounded) negative selfadjoint one-to-one operator Π_- such that

$$\tilde{H}_- = \left\{ \begin{bmatrix} x \\ \Pi_-(x) \end{bmatrix} \right\}_{x \in D(\Pi_-)} \quad \tilde{H}_+ = \left\{ \begin{bmatrix} x \\ \Pi_+(x) \end{bmatrix} \right\}_{x \in D(\Pi_+)}$$

These results use in particular two structures of **Krein spaces** on \tilde{H} . Recall that an operator A is said to be μ -sectorial if $A - zI$ is invertible for $\operatorname{Re} z > 0$ and if there exists $\theta \in (0, \pi/2)$ and $\beta > 0$ such that $\pi/2 + \theta \leq \operatorname{Arg}(Ax, x) \leq 3\pi/2 - \theta$ and $\operatorname{Re}(Ax, x) \leq -\beta\|x\|^2$, $x \in D(A)$.

Langer, Ran and van de Rotten proved (Operator theory : Advances and Applications 130 (2001), 235-254) the following result

Theorem : Assume that A is μ -sectorial, that the pair (A, D) is approximately controllable, and that the pair (Q, A) is approximately observable. Then the positive operator Π_+ is bounded, and it is the unique nonnegative bounded solution of the algebraic Riccati equation

$$\Pi D \Pi - A^* \Pi - \Pi A - Q = 0$$

Recall that the **Hadamard product** $M \circ A$ of two matrices $M = (m_{ij})_{i,j \geq 0}$ and $A = (a_{ij})_{i,j \geq 0}$ is given by the formula $M \circ A = (m_{i,j} a_{i,j})$ and M is a **Hadamard-Schur multiplier (HSM)** if the map $A \mapsto M \circ A$ is a bounded map from $\mathcal{B}(l^2)$ into itself.

L.N.Nikolskaia (**Bordeaux**) and Yu.B.Farforovskaya (**St Petersburg**) obtained recently interesting results on this very classical subject. For example let $\phi : Z^+ \times Z^+$ be a map, and set $\sigma_\phi = \{(i, j) \in Z^+ \times Z^+ \mid j \geq \phi(i)\}$

It is in particular shown in their paper to appear in 2004 in St Petersburg Math.J. that if $M(\phi)$ is the matrix associated to the characteristic function of $\sigma(\phi)$ (i.e. $M(\phi)_{i,j} = 1$ if $j \geq \phi(i)$, $M(\phi)_{i,j} = 0$, otherwise then $M(\phi)$ is HSM iff $\phi(Z^+)$ is finite. In this case we have

$$c \cdot \log(n + 1) \leq \|M(\phi)\|_{HSM} \leq 1 + \log(n)$$

where $n = \text{card}(\phi(Z^+))$, and where c is an absolute constant.

There was some important progress in **convex geometry** in joint works by the **Paris** and **Tel-Aviv** teams, see for example the joint paper "Essential uniqueness of M-ellipsoids of a given convex body", by Milman and Pajor, to appear in GAFA. The links between high dimensional convex geometry, complexity theory and the phenomenon of concentration of measure led to a very interesting RTN project, involving people originating from different areas (Brenier, Gromov, Milman, Pastour ,etc...) coordinated by Pajor, which was **one of the very few accepted in the November call of 6th PCRDT** (see Pajor's homepage to get a link to the webpage of this project).

A. Atzmon (Tel-Aviv) and G. Godefroy (Paris) obtained an application of variational principles to invariant subspaces (CRAS Paris t.332, Série 1, 151-156). If a Banach space X admits an equivalent Gateaux smooth norm (true for all separable Banach spaces) then given a function $G : X \rightarrow \mathbb{R} \cup \{\infty\}$ which is lower-semicontinuous and bounded below, if $\epsilon > 0$ and $y \in X$ satisfy

$$G(y) < \inf(G) + \frac{\epsilon^2}{12}$$

then there exists a Lipschitzian and Gateaux smooth function $g : X \rightarrow \mathbb{R}$ such that

$$1) \sup_{x \in X} (\|g(x)\| + \|g'(x)\|) < \epsilon$$

2) $G + g$ attains its minimum on X at w such that $\|y - w\| < \epsilon$.

An operator A is said to have a *moment sequence* if there exists $x_0 \in X \setminus \{0\}$, $x_0^* \in X^* \setminus \{0\}$ and a positive Borel measure μ on \mathbb{R} such that

$$\langle x_0^*, A^n(x) \rangle = \int_{\mathbb{R}} t^n d\mu(t)$$

The variational principle described before allowed Atzmon and Godefroy to prove **for all Banach spaces** the following result previously obtained by Atzmon in the reflexive case

Theorem (Atzmon-Godefroy) Let X be a real Banach space, and let $A : X \rightarrow X$ be a bounded linear operator. If A has a moment sequence, then A has a nontrivial invariant subspace.

Let $\omega : \mathbb{Z} \rightarrow (0, +\infty)$ and assume that

$$0 < \inf_{n \in \mathbb{Z}} \frac{\omega(n+1)}{\omega(n)} \leq \sup_{n \in \mathbb{Z}} \frac{\omega(n+1)}{\omega(n)} < +\infty$$

Set $l_{\omega}^2(\mathbb{Z}) = \{u = (u_n)_{n \in \mathbb{Z}} \mid \sum_{n \in \mathbb{Z}} |u_n|^2 < +\infty\}$.

The bilateral shift $S : (u_n)_{n \in \mathbb{Z}} \rightarrow (u_{n-1})_{n \in \mathbb{Z}}$ is bounded and invertible on $l_{\omega}^2(\mathbb{Z})$, and a closed subspace M of $l_{\omega}^2(\mathbb{Z})$ is said to be **translation invariant** if $S(M) \cup S^{-1}(M) \subset M$. **The existence of nontrivial translation invariant subspaces is an open problem.**

Atzmon's Hilbert space version of the moment sequence theorem gives a positive answer for **all symmetric weights**. For "antisymmetric weights", i.e. weights satisfying $\omega(n) = \frac{1}{\omega(-n)}$ Domar obtained in 1997 a positive answer, with some regularity assumptions, using entire functions of exponential type.

The Borichev-Hedenmalm-Volberg theorem gives other types of translation invariant subspaces for antisymmetric weights which are log-convex on \mathbb{Z}^+ , and Atzmon obtained in 1997 a positive answer for new classes of weights by using entire functions of zero exponential type. The problem is still open in the case where the spectrum of the bilateral shift is an annulus.

Inspired by the solution by Borichev and Hedenmalm of Levin's problem and **using minimum principles for almost holomorphic functions**, A.Volberg and J.E. showed (Ann.Sci. ENS 35 (2002), 185-230) that if the spectrum of the bilateral shift equals the unit circle, and $\omega(n)$ tends to infinity sufficiently quickly and regularly as $n \rightarrow -\infty$ then all translation invariant subspaces M are generated by their "analytic part" $M^+ = \{u = (u_n)_{n \in \mathbb{Z}} \in M \mid u_n = 0 \ \forall n < 0\}$.

For example if $\omega(n) = 1$ for $n \geq 0$, $\omega(n) = \frac{|n|}{e^{\log(1+|n|)^2}}$ for $n < 0$, then all translation invariant subspaces of $l^2_\omega(\mathbb{Z})$ are generated by the Fourier sequence of a singular inner function.

Also if there exists a weighted Hardy space, for which the spectrum of the shift and of the backward shift equals the closed unit disc, such that $\dim(M \ominus zM) \geq 2$ for every zero-free z -invariant subspace, then there is a counterexample to the translation invariant subspace problem.

During network activity there was some joint work by S.Verduyn Lunel (team 2) and members of the Chemistry department of Leiden University, using functional analysis tools to study periodic chemical processes. Also R.Gay, a retired Professor from Bordeaux, published a couple of papers with engineers on problem on signal theory arising from industry.

At Bordeaux F.Turcu, who was preparing a thesis on a very abstract subject, also worked with Professor N.Najim, from an engineering laboratory (see IEEE Transactions on Information Theory 49 (2003), 3099-3106) on problems on $2 - D$ random field modelling. He actually helped solving some very concrete problems by using sophisticated operator-theoretical tools recently developed at Timisoara. **He now holds a permanent CNRS research position in this engineering laboratory .**

Altogether the potential for applications was not exploited as it could have been at network level.

What about employment ?

A precise report of the job situation of previous pre-post doc appointed by the network is not yet available. There was unfortunately at least one example of unemployment this year. There is also a success story of employment.

Starting annual salary (June 1, 2004) of a post-doc whose network appointment ended on May 31, 2004 :

Basic salary : £ 105 000

Guaranteed bonus : £ 20 000

Stock-Options : £ 20 000

plus a free car after six months of employment.

Open Problems (not suitable for network work-plan milestones) :

1) Let $p \geq 2$, let $F : \mathbb{C}^p \rightarrow \mathbb{C}^p$ be holomorphic and let F^n be the n th iterate of F .

Is $\bigcap_{n \geq 1} F^n(\mathbb{C}^p)$ always nonempty ?

More generally is $\bigcap_{n \geq 1} F_1 \circ F_2 \dots \circ F_n(\mathbb{C}^p)$ always nonempty if (F_n) is a sequence of holomorphic functions from \mathbb{C}^p into itself (a negative answer would imply that characters on Fréchet algebras are continuous).

2) Discontinuous algebra norms on $\mathcal{C}[0, 1]$ do exist if $2^{\aleph_0} = \aleph_1$, and their existence is not decidable if $2^{\aleph_0} = \aleph_2$.

What about $2^{\aleph_0} \geq \aleph_3$?