

# Rigidity results for von Neumann algebras and Bernoulli actions

North British Functional Analysis Seminar

Lecture 2



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## Recall from Lecture 1

Three types of **equivalences** between  $\Gamma \curvearrowright (X, \mu)$  and  $\Lambda \curvearrowright (Y, \eta)$ .

- 1 **Conjugacy** :  $\Delta(x \cdot g) = \Delta(x) \cdot \delta(g)$ .
- 2 **Orbit equivalence** :  $\Delta(x \cdot \Gamma) = \Delta(x) \cdot \Lambda$ .
- 3 **Von Neumann equivalence** :  $L^\infty(X, \mu) \rtimes \Gamma \cong L^\infty(Y, \eta) \rtimes \Lambda$ .

**Remark:** 1  $\Rightarrow$  2  $\Rightarrow$  3

**Rigidity:** converse implications for families of group actions

### Theorem (Popa, see lecture 1)

Let  $\Gamma$  be  $w$ -rigid and  $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^f$  (and some assumptions). Any orbit equivalent essentially free action is actually conjugate.

- Lecture 2 :
- Explain **group measure space construction**  $L^\infty(X) \rtimes \Gamma$ .
  - Von Neumann rigidity results : 3  $\Rightarrow$  1
  - Invariants of  $\text{II}_1$  factors.

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## Recall : Von Neumann algebras

### Definition

A **von Neumann algebra** is a weakly closed unital  $*$ -subalg. of  $B(H)$ .

### Examples :

- ▶  $B(H)$  itself, in particular  $M_n(\mathbb{C})$ .
- ▶  $L^\infty(X, \mu)$  (acting on  $L^2(X, \mu)$ ).
- ▶ Let  $\Gamma$  be a countable group. Consider the left regular rep.  
 $\lambda : \Gamma \rightarrow \mathcal{U}(\ell^2(\Gamma)) : (\lambda_g \xi)(h) = \xi(g^{-1}h)$ .

$$\begin{aligned}\mathcal{L}(\Gamma) &= \text{weakly closed linear span of } \{\lambda_g \mid g \in \Gamma\} \\ &= \text{weak closure of } \mathbb{C}\Gamma \text{ represented in } B(\ell^2(\Gamma))\end{aligned}$$

is the **group von Neumann algebra**.



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## Group measure space construction (Murray - von Neumann)

Let  $\Gamma \curvearrowright (X, \mu)$ , probability measure preserving, free, ergodic.

Let  $\Omega \in Z^2(\Gamma, S^1)$  be a scalar 2-cocycle.

### The $II_1$ factor $L^\infty(X) \rtimes_{\Omega} \Gamma$

- ▶ contains a copy of  $L^\infty(X)$ ,
- ▶ contains unitaries  $(u_g)_{g \in \Gamma}$ ,

in such a way that

- ▶  $u_g u_h = \Omega(g, h) u_{gh}$ ,
- ▶  $u_g F(\cdot) u_g^* = F(\cdot g)$ ,
- ▶  $\tau(F u_g) = \begin{cases} \int_X F d\mu & \text{if } g = e, \\ 0 & \text{if } g \neq e. \end{cases}$



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## Popa's von Neumann rigidity theorem

### Theorem (Popa, 2005)

Let  $\Lambda$  be  $w$ -rigid and ICC. Take  $\Lambda \curvearrowright (Y, \eta)$  free ergodic.

Let  $\Gamma$  be ICC and  $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$  **plain Bernoulli action**.

If  $\theta$  is an isomorphism between  $L^\infty(X) \rtimes \Gamma$  and  $L^\infty(Y) \rtimes \Lambda$ , then

- ▶ the groups are isomorphic and the actions conjugate,
- ▶  $\theta$  is given by the conjugation of the actions and a character of  $\Gamma$ .

~ First theorem in the literature  
 deducing conjugacy from von Neumann equivalence.

~ **Attention** : plain Bernoulli action.

→ The isomorphism  $\theta$  has the following form.

- Conjugacy :  $\Delta : X \rightarrow Y : \Delta(x \cdot g) = \Delta(x) \cdot \delta(g)$
- Group character  $\omega : \Gamma \rightarrow S^1$
- Then,  $\theta(F) = F \circ \Delta^{-1}$  and  $\theta(u_g) = \omega(g) u_{\delta(g)}$



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## Invariants for $\text{II}_1$ factors

Murray and von Neumann introduced two invariants for  $\text{II}_1$  factors.

- ▶ A qualitative one: **property ( $\Gamma$ )**.
- ▶ A quantitative one: the **fundamental group**  $\mathcal{F}(M) \subset \mathbb{R}_+^*$ .

$$\mathcal{F}(M) = \{\tau(p)/\tau(q) \mid pMp \cong qMq\}$$

~ (Murray, von Neumann, 1943)  
 $\mathcal{L}(S_\infty) \not\cong \mathcal{L}(\mathbb{F}_2)$  using property ( $\Gamma$ ).

~ Only very recently,  $\mathcal{F}(M)$  became effective to distinguish  $\text{II}_1$  factors.

*Two invariants of  $\text{II}_1$  factors are formed : the fundamental group and the property ( $\Gamma$ ). The first one is probably of greater general significance, but the last one has so far been put to greater practical use.*

(Murray, von Neumann, On rings of operators. IV)



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## Known results on the fundamental group

- 1 (Murray, von Neumann, 1943)  $\mathcal{F}(\mathcal{R}) = \mathbb{R}_+^*$ .
- 2 (Connes, 1980) If  $\Gamma$  is an ICC prop. (T) group,  $\mathcal{F}(\mathcal{L}(\Gamma))$  is countable.
- 3 (Popa, 2001) Take  $M = \mathcal{L}(\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})) = L^\infty(\mathbb{T}^2) \rtimes \mathrm{SL}(2, \mathbb{Z})$ .  
Then,  $\mathcal{F}(M)$  is trivial.

- 4 (Popa, 2003) Define the  $\mathrm{II}_1$  factor  $M$  by the equiv. relation  $\mathcal{R}$  :

- $\Gamma = \mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ .
- Atomic probability space  $(X_0, \mu_0)$  and  $(X, \mu) = (X_0, \mu_0)^\Gamma$ .
- Countable equivalence relation  $\mathcal{R}$  on  $(X, \mu)$  generated by  $\Gamma$ -orbits and by  $x \sim y$  if  $x_g = y_g$  for almost all  $g \in \Gamma$  and

$$\prod_{g, x_g \neq y_g} \mu_0(x_g) = \prod_{g, x_g \neq y_g} \mu_0(y_g).$$

Then,  $\mathcal{F}(M)$  is the group generated by the atoms of  $\mu_0$ .

- 5 (Ioana, Peterson, Popa, 2004 and Houdayer, 2007)

Other constructions involving free products.

Open problem : can  $\mathcal{F}(M)$  be uncountable without being  $\mathbb{R}_+^*$  ?

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## Outer automorphism group of a $\mathrm{II}_1$ factor

Let  $M$  be a  $\mathrm{II}_1$  factor.

- ▶  $\mathrm{Aut}(M)$  = the group of automorphisms of  $M$ .
- ▶  $\mathrm{Inn}(M)$  = the subgroup of automorphisms of the form  
 $\mathrm{Ad} u : (\mathrm{Ad} u)(x) = uxu^*$ .
- ▶  $\mathrm{Out}(M) = \frac{\mathrm{Aut}(M)}{\mathrm{Inn}(M)}$ , the outer automorphism group.

~ An invariant for  $\mathrm{II}_1$  factors, **but extremely hard to compute.**

### Theorems

- ▶ If  $M$  = the hyperfinite  $\mathrm{II}_1$  factor,  $\mathrm{Out}(M)$  is enormous.
- ▶ (Connes, 1980) If  $M = \mathcal{L}(\Gamma)$  where  $\Gamma$  is ICC with property (T), then  $\mathrm{Out}(M)$  is countable.

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## Outer automorphisms for Bernoulli $\text{II}_1$ factors

**Obvious automorphisms**  $\theta$  of  $L^\infty(X) \rtimes \Gamma$  :

- ▶ **Self-conjugacies** :  $\Delta \in \text{Norm}_{\text{Aut}(X,\mu)}(\Gamma)$  and  
 $\theta(F) = F \circ \theta^{-1}$  ,  $\theta(u_g) = u_{\delta(g)}$ .
- ▶ **Characters**  $\omega : \Gamma \rightarrow S^1$  :  $\theta(F) = F$  ,  $\theta(u_g) = \omega(g)u_g$ .

### Corollary to Popa's von Neumann rigidity theorem

Let  $\Gamma$  be **w-rigid ICC** and  $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$  **Bernoulli action**.

Then,  $\text{Out}(L^\infty(X) \rtimes \Gamma) \cong \text{Char} \Gamma \rtimes \frac{\text{Norm}_{\text{Aut}(X,\mu)}(\Gamma)}{\Gamma}$ .

**But** : it is an open problem to actually compute  $\text{Norm}_{\text{Aut}(X,\mu)}(\Gamma)$  for **plain** Bernoulli actions.

## The normalizer of generalized Bernoulli actions

### (Easy) proposition

Let  $\Gamma \curvearrowright I$  be **transitive** such that  $\text{Stab } i$  acts **with infinite orbits** on  $I - \{i\}$ . Consider  $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^I$ .

Then,  $\text{Norm}_{\text{Aut}(X,\mu)}(\Gamma) = \text{Norm}_{\text{Perm } I}(\Gamma) \times \text{Aut}(X_0, \mu_0)$ .

### Reformulation :

Let  $\Gamma_0 < \Gamma$  with  $\text{Comm}_\Gamma(\Gamma_0) = \Gamma_0$  and let  $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^{\Gamma/\Gamma_0}$ .

Then,  $\frac{\text{Norm}_{\text{Aut}(X,\mu)}(\Gamma)}{\Gamma} \cong \frac{\text{Aut}(\Gamma_0 \subset \Gamma)}{\Gamma_0} \times \text{Aut}(X_0, \mu_0)$ .

$$\text{Comm}_G(\Gamma) = \left\{ g \in G \mid \begin{array}{l} [\Gamma : \Gamma \cap g\Gamma g^{-1}] < \infty \\ [\Gamma : \Gamma \cap g^{-1}\Gamma g] < \infty \end{array} \right\}$$

➤ We need a von Neumann rigidity theorem for generalized Bernoulli actions

## Von Neumann rigidity for generalized Bernoulli $\text{II}_1$ factors

### Theorem (V, 2007, extending Popa - V, 2006)

Let  $\Gamma \curvearrowright I$  and  $\Lambda \curvearrowright J$  be good actions of good groups.

$\rightsquigarrow L^\infty((X_0, \mu_0)') \rtimes \Gamma$  is isomorphic with  $L^\infty((Y_0, \eta_0)') \rtimes \Lambda$   
iff  $\Gamma \curvearrowright I$  is conjugate to  $\Lambda \curvearrowright J$  and  $(X_0, \mu_0) \cong (Y_0, \eta_0)$ .

$\rightsquigarrow \text{Out}(L^\infty((X_0, \mu_0)') \rtimes \Gamma) \cong \frac{\text{Norm}_{\text{Perm} I}(\Gamma)}{\Gamma} \times \text{Aut}(X_0, \mu_0)$

- ▶ Leads to the first explicit computations of  $\text{Out}(M)$ .
- ▶  $\text{Aut}(X_0, \mu_0)$  is trivial for atomic  $\mu_0$  with distinct weights.
- ▶ With  $\text{PSL}_n(\mathbb{Z}) \curvearrowright \mathbb{P}(\mathbb{Q}^n)$ ,  $n \geq 3$ , we get uncountably many  $\text{II}_1$  factors  $M$  with **trivial outer automorphism group**.

### Good actions of good groups

A **condition on the group  $\Gamma$**  :

- ▶  $\Gamma$  admits an infinite, almost normal subgroup with relative (T).

Some **conditions on the action  $\Gamma \curvearrowright I$**  :

- ▶ Transitivity.
- ▶  $\text{Stab } i_0$  acts with infinite orbits on  $I - \{i_0\}$ .
- ▶ Minimal condition on stabilizers :  
no infinite sequence  $(i_n)$  with  $\text{Stab}(i_0, \dots, i_n)$  strictly decreasing.
- ▶ A faithfulness condition of  $\Gamma \rightarrow \text{Perm } I$ .

### About the minimal condition on stabilizers.

↳ Automatic if  $\Gamma \curvearrowright I$  embeds in  $\text{GL}(V) \rtimes V \curvearrowright V$  for fin.dim.  $V$ .

↳ For left-right action  $\Lambda \times \Lambda \curvearrowright \Lambda$  : equivalent with minimal condition on centralizers of  $\Lambda$ .

## Every countable group arises as $\text{Out}(M)$

Let  $\Lambda$  be an arbitrary

ICC group satisfying the minimal condition on centralizers and assume moreover that  $\Lambda$  is indecomposable as a direct product.

With  $(\text{PSL}(n, \mathbb{Z}) \times \Lambda \times \Lambda) \curvearrowright \mathbb{P}(\mathbb{Q}^n) \times \Lambda$ , we get  
 $\text{Out}(M) \cong \mathbb{Z}/2\mathbb{Z} \times \text{Out } \Lambda$ .

- (Bumagin, Wise, 2004)  
Every countable group arises as  $\text{Out } \Lambda$  for such kind of group  $\Lambda$ .  
     $\rightsquigarrow$   $\Lambda$  is given quite concretely as a subgroup of a  $C'(1/6)$ -small cancellation group (Rips' construction).
- Being a little smarter, one can kill  $\mathbb{Z}/2\mathbb{Z}$ .  
     $\rightsquigarrow$  Every countable group arises as the outer automorphism group of a  $\text{II}_1$  factor.

## Existence theorems about $\text{Out}(M)$

We just saw :

$\text{Out}(M)$  can be any countable group in a constructive manner.

But, we also have the following existence theorems :

- 1 Ioana, Peterson, Popa, 2004  
 $\text{Out}(M)$  can be any compact abelian, second countable group.
- 2 Falguières - V, 2007  
 $\text{Out}(M)$  can be any compact, second countable group.

## Final aim : compute fusion algebras of $\text{II}_1$ factors

We discussed the invariants  $\mathcal{F}(M)$  and  $\text{Out}(M)$ .

~> Both are part of a finer invariant, called **fusion algebra**.

~> Intimately connected to **Jones' subfactor theory**.

**Final aim of my lecture :**

- ▶ Explain this invariant  $\text{FAlg}(M)$ .
- ▶ Present the first explicit computations of  $\text{FAlg}(M)$ .
- ▶ Make the link with **Hecke fusion algebras**.

## Connes' correspondences

A representation theory of  $\text{II}_1$  factors

Let  $M$  be a type  $\text{II}_1$  factor with trace  $\tau$ .

- ▶ A right  **$M$ -module** is a Hilbert space with a right action of  $M$ .

~> Example :  $L^2(M, \tau)_M$  by completing  $M$  (being obviously a right  $M$ -module) for the scalar product  $\langle x, y \rangle = \tau(x^*y)$ .

- ▶ Always,  $H_M \cong \bigoplus_{i \in I} p_i L^2(M)$

and one defines  $\dim(H_M) = \sum_i \tau(p_i) \in [0, +\infty]$ .

~> Complete invariant of right  $M$ -modules.

### Definition

An  **$M$ - $M$ -bimodule of finite Jones index**, is an  $M$ - $M$ -bimodule  ${}_M H_M$  satisfying  $\dim(H_M) < \infty$  and  $\dim({}_M H) < \infty$ .

Denote  $\text{FAlg}(M)$  the set of (equiv. classes) of finite index bimodules.

## The structure of the fusion algebra $\text{FAlg}(M)$

**Example :** let  $\alpha \in \text{Aut}(M)$  and define the  $M$ - $M$ -bimodule  $H(\alpha)$  on  $L^2(M)$  by  $a \cdot \xi \cdot b = a\xi\alpha(b)$

~> Precisely those  ${}_M H_M$  with  $\dim({}_M H) = 1 = \dim(H_M)$ .

**Example :** let  $M \subset M_1$  be a finite index subfactor.  
Then,  ${}_M L^2(M_1)_M$  belongs to  $\text{FAlg}(M)$ .

**In general,  $\text{FAlg}(M)$  carries the following structure.**

- ▶ **Direct sum** of bimodules.
- ▶ Identity element :  ${}_M L^2(M)_M$ .
- ▶ **Connes' tensor product**  $H \otimes_M K$ .  
Example :  $H(\alpha) \otimes_M H(\beta) \cong H(\alpha \circ \beta)$ .
- ▶ **Contragredient**  ${}_M \bar{H}_M$  of  ${}_M H_M$ .

~>  $\text{FAlg}(M)$  consists of the **generalized symmetries** of  $M$ .



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## Group-like elements in $\text{FAlg}(M)$

We call  ${}_M H_M$  **group-like** if  $H \otimes_M \bar{H}$  is the identity.

~> Example of group-like element  ${}_M H_M$  :

- ▶  $H = L^2(M)p$ .
- ▶  $\alpha : M \rightarrow pMp$  an isomorphism.
- ▶  $a \cdot \xi \cdot b = a\xi\alpha(b)$ .

### Observation

We have a short exact sequence

$$e \rightarrow \text{Out}(M) \rightarrow \{\text{group-like bimodules}\} \rightarrow \mathcal{F}(M) \rightarrow e$$



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## Abstract fusion algebras

### Definition

A **fusion algebra**  $\mathcal{A}$  is a **free  $\mathbb{N}$ -module**  $\mathbb{N}[\mathcal{G}]$ , equipped with

- ▶ an associat. distribut. **product** :  $x * y = \sum_{z \in \mathcal{G}} \text{mult}(z, x * y)z$ ,
- ▶ a multiplicative **neutral element**  $e \in \mathcal{G}$ ,
- ▶ a **contragredient** map  $x \mapsto \bar{x}$  which is ...,

such that **Frobenius reciprocity** holds : for all  $x, y, z \in \mathcal{G}$ , we have

$$\text{mult}(z, x * y) = \text{mult}(x, z * \bar{y}) = \text{mult}(y, \bar{x} * z)$$

### Examples

- ▶  $\mathbb{N}[\Gamma]$  for a group  $\Gamma$ .
- ▶  $\text{Rep}(G)$ , the finite dim. unitary rep. of a compact group  $G$ .
- ▶  $\text{FAlg}(M)$  of a  $\text{II}_1$  factor  $M$ .

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## Fusion algebra of a Hecke pair

Let  $\Gamma < G$  be a **Hecke pair**, i.e.  $[\Gamma : g\Gamma g^{-1} \cap \Gamma] < \infty$  for all  $g \in G$ .

### Hecke fusion algebra

$$\mathcal{H}(\Gamma < G) = \{ \xi : \Gamma \backslash G / \Gamma \rightarrow \mathbb{N} \mid \xi \text{ has finite support} \}$$

$$(\xi * \eta)(g) = \sum_{h \in G/\Gamma} \xi(h) \eta(h^{-1}g)$$

↪  $\Gamma \backslash G / \Gamma$  is the set of irreducibles with fusion rules ...

### Example

Let  $T$  be  $k$ -valent tree and  $G < \text{Aut}(T)$  **countable dense subgroup**.

Choose a vertex  $e$  and set  $\Gamma = \text{Stab } e$ .

Identify  $\Gamma \backslash G / \Gamma = \Gamma \backslash T \cong \mathbb{N}$  via the **distance to  $e$** . Then,

**$\text{mult}(n, a * b) = \#\{t \in T \mid |et| = a, |ts| = b\}$  when  $|es| = n$ .**

↪ A way to understand the Hecke fusion algebra of  $\text{PSL}(2, \mathbb{Z}) < \text{PSL}(2, \mathbb{Q})$ .

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## Computations of $\text{FAlg}(M)$

We consider as before  $\Gamma \curvearrowright I$  and  $M = L^\infty((X_0, \mu_0)^I) \rtimes \Gamma$ .

### Theorem (V, 2007)

When  $\Gamma \curvearrowright I$  is a **good action of a good group** and if  $(X_0, \mu_0)$  **atomic with distinct weights**, we have

$$\text{Rep}_{\text{fin}}(\Gamma) \rightarrow \text{FAlg}(M) \xrightarrow{\cong} \mathcal{H}(\Gamma < G)$$

where  $G = \text{Comm}_{\text{Perm } I}(\Gamma) = \text{commensurator of } \Gamma \text{ inside Perm } I$ .

For every Hecke pair  $\Gamma < G$ , one defines  $\mathcal{H}_{\text{rep}}(\Gamma < G)$ . Then,

$$\text{FAlg}(M) \cong \mathcal{H}_{\text{rep}}(\Gamma < G).$$

### Compare with our computations of $\text{Out}(M)$

- ▶  $\text{Char } \Gamma \xrightarrow{\cong} \text{Rep}_{\text{fin}}(\Gamma)$ .
- ▶  $\text{Norm}_{\text{Perm } I}(\Gamma) \xrightarrow{\cong} \text{Comm}_{\text{Perm } I}(\Gamma)$ .

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## First examples with $\text{FAlg}(M)$ trivial

Take  $\Gamma = \text{SL}(2, \mathbb{Q}) \ltimes \mathbb{Q}^2$ , with  $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^{\mathbb{Q}^2}$ .

### Theorem (V, 2007)

The  $\text{II}_1$  factor  $M = L^\infty(X) \rtimes_{\Omega_\alpha} \Gamma$  has **trivial fusion algebra** if

- ▶  $(X_0, \mu_0)$  atomic with atoms of different weights,
- ▶  $\alpha \in \mathbb{R} \setminus \{0\}$  and  $\Omega_\alpha \in Z^2(\Gamma, S^1)$  defined by

$$\Omega_\alpha \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) = \exp(2\pi i \alpha (x_1 y_2 - x_2 y_1))$$

on  $\mathbb{Q}^2$  and extended to  $\Gamma$  by  $\text{SL}(2, \mathbb{Q})$ -invariance.

➤ Moreover, the  $\text{II}_1$  factor  $M$  **remembers**  $(X_0, \mu_0)$  and  $\alpha$ .

➤ First explicit examples having trivial fusion algebra

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## Further computations

**Recall :**  $M = L^\infty(X) \rtimes \Gamma$  and  $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^!$ .

| $\Gamma \curvearrowright I$  | $\text{FAlg}(M)$   |
|--|--|
| <p><math>\text{SL}_n(\mathbb{Z}) \rtimes \mathbb{Q}^n \curvearrowright \mathbb{Q}^n</math> for <math>n \geq 2</math>.</p> <p><math>\Lambda &lt; \text{PSL}_n(\mathbb{Q})</math> proper subgroup, relative ICC,</p> <p><math>\Gamma = \Lambda \rtimes \text{PSL}_n(\mathbb{Q}) \curvearrowright \text{PSL}_n(\mathbb{Q})</math> by left-right action.</p> <p><math>\mathbb{Z} &lt; R &lt; \mathbb{Q}</math> strict inclusions of rings, i.e. <math>R = \mathbb{Z}[\mathcal{P}^{-1}]</math></p> <p>Write some <math>\Lambda_0 &lt; \Lambda = \text{SL}_2(\mathbb{Q}) \rtimes \mathbb{Q}^2</math>. Take <math>\Lambda_0 \rtimes \Lambda \curvearrowright \Lambda</math>. Add 2-cocycle.</p> | <p style="text-align: center;"><math>\mathcal{H}_{\text{rep}}(\text{SL}_n(\mathbb{Z}) &lt; \text{GL}_n(\mathbb{Q}))</math></p> <p style="text-align: center;"><math>\mathcal{H}_{\text{rep}}(\Lambda &lt; \text{Comm}_{\text{PGL}_n(\mathbb{Q})}(\Lambda))</math>.</p><br><p style="text-align: center;"><math>\mathcal{H}_{\text{rep}}(R^* \rtimes R &lt; \mathbb{Q}^* \rtimes \mathbb{Q})</math></p> <p style="text-align: center;"><math>\rightsquigarrow</math> Relation with Bost-Connes Hecke algebra.</p> |