Products of projections in Hilbert space

Eva Kopecká

University of Innsbruck
Austria
To converge, or not to converge, that is the question:

- $K$ fixed, e.g. $K = 5$
- $L_1, L_2, \ldots, L_K \subset \mathbb{R}^d$ or $\ell_2$
- closed convex sets

$k_1, k_2, \ldots \in \{1, 2, \ldots, K\}$ be arbitrary

$z_n = P_{k_n}z_{n-1}$ sequence of projections

DO THE ITERATES CONVERGE?

≡ stay bounded, converge weakly, or converge in norm?
Affine subspaces

$L_1, L_2, \ldots, L_K$ closed affine subspaces of a Hilbert space $H$
$z_n = P_{k_n}z_{n-1}$ iterates of orthonormal projections of a point $z$

In $\mathbb{R}^d$, the sequence $\{z_n\}$ is always bounded.
[Aharoni, Duchet, Wajnryb ’84], [Meshulam ’96]

In $\ell_2$, there exist two closed affine subspaces $L_1, L_2$ and a sequence $\{z_n\}$ of iterates of nearest point projections which is not bounded.
[Bauschke, Borwein ’94]
Proof: Take two closed subspaces the sum of which is not closed, and translate one of them.
Linear subspaces

$L_1, L_2, \ldots, L_K$ closed subspaces of a Hilbert space $H$

$z_n = P_{k_n}z_{n-1}$ iterates of orthonormal projections of a point $z$

$Pz =$ orthogonal projection of $z$ onto $L_1 \cap L_2 \cap \cdots \cap L_K$

If $H = \mathbb{R}^d$, then $\{z_n\}$ converges to $Pz$. [Práger '60]

If $H = \ell_2$, then $\{z_n\}$ converges to $Pz$ weakly. [Amemiya, Ando '65]

$(\sum_1^N z_n)/N$ converges in norm for almost all $\{k_n\} \in \{1, \ldots, K\}^\mathbb{N}$.

If $L_1, L_2 \subset \ell_2$, then $\{z_n\}$ converges to $Pz$ in norm.
[von Neumann '49]

ASSUME $L_1, \ldots, L_K \subset \ell_2$. DOES $\{z_n\}$ CONVERGE IN NORM???

Yes, if the iterates are (quasi)cyclic e.g. $P_1P_2P_3P_1P_2P_3\ldots$
[Halperin '62], [Sakai '95]
Random projections on 3 subspaces do not have to converge

Let $H$ be an infinite dimensional Hilbert space. There exist 3 closed subspaces $L_1, L_2, L_3$ with the following property. For every $0 \neq w_0 \in H$ there is a sequence $k_1, k_2, \cdots \in \{1, 2, 3\}$ so that the sequence of iterates defined by $w_n = P_{k_n}w_{n-1}$ does not converge in norm.

[Eva Kopecká & Vladimír Müller & Adam Paszkiewicz]

If $L_1, L_2$ are closed subspaces of $H$, then any sequence $\{z_n\}$ of projections on $L_1, L_2$ converges in norm.

[John von Neumann 1949]
Cyclic case

Let $L_1, L_2, \ldots, L_K$ be closed subspaces of a Hilbert space $H$, let $T = P_k P_{k-1} \ldots P_1$, and let $P =$ orthogonal projection onto $L_1 \cap L_2 \cap \cdots \cap L_K$. Then $|T^n z - Pz| \to 0$ for every $z \in H$.

Proof sketch:

$T$ is a contraction since each $P_i$ is a contraction.

Pythagoras $\Rightarrow$

$|x - Tx|^2 \leq c(|x|^2 - |Tx|^2)$ for all $x \in H$ and some $c > 0$

$\Rightarrow$ $T$ is asymptotically regular

(that is, $T^n x - T^{n+1} x \to 0$ for each $x \in H$)

$\Rightarrow$ $T^n z \to Pz$ for every $z \in H$

More generally: if $T : H \to H$ is an asymptotically regular contraction, then $T^n z \to Pz$ for every $z \in H$, where $P$ is the projection onto the set of all fixpoints of $T$. 
Rate of Convergence, cyclic case

$L_1, L_2, \ldots, L_K$ closed subspaces of a Hilbert space $H$

$z_n = T^n z = (P_K \ldots P_1)^n z$, $Pz = \text{orthoprojection of } z \text{ onto } \bigcap L_i$

Then $z_n \to Pz$ in norm, but DICHOTOMY!!

Either the convergence is exponentially fast everywhere on $H$, or it is arbitrarily slow.

Either there exist $r \in [0, 1)$ and $c_z > 0$ so that $|T^n z - Pz| \leq c_z r^n$ for every $z \in H$ (and hence $\|T_n - P\| \leq Cr^n$),
or for any sequence $r_n \to 0$ there is $z \in H$ so that $|T^n z - Pz| \geq r_n$.

[Badea, Deutsch, Grivaux, Hundal, Müller, 2010]
\[ L_1^\perp + \cdots + L_K^\perp \text{ closed } \Rightarrow \text{ exponentially fast convergence} \]

Let \( L_1, L_2, \ldots, L_K \) be closed subspaces of a Hilbert space \( H \), let \( T = P_K \cdots P_1 \) and \( P \) be the projection onto \( L = \bigcap L_i \).
Then \( T^n \) converges to \( P \) exponentially fast iff \( L_1^\perp + \cdots + L_K^\perp \) is a closed subspace of \( H \).
[Deutsch, Hundal]

Question: Assume \( L_1^\perp + \cdots + L_K^\perp \) is not closed. Where do the initial points for the arbitrarily slow convergence lie?

Conjecture: in general in \( H \setminus (L \oplus (L_1^\perp + \cdots + L_K^\perp)) \).
[Deutsch, Hundal]
convergence slow ⇒ on a dense subspace it has to be relatively fast

Let $L_1, L_2, \ldots, L_K$ be closed subspaces of a Hilbert space $H$, let $T = P_K \ldots P_1$ and $P$ be the projection onto $L = \bigcap L_i$. Then $T^n$ converges to $P$ exponentially fast iff $L_1^\perp + \cdots + L_K^\perp$ is a closed subspace of $H$. [Bauschke, Deutsch, Hundal]

Assume $L_1^\perp + \cdots + L_K^\perp$ is not closed in $H$. Then there is a dense subspace $X$ of $H$ so that for all $\alpha > 0$

$$|T^n x - Px| = o(n^{-\alpha}).$$

[Badea, Seifert]
cyclic versus random

$L_1, L_2, \ldots, L_K$ closed subspaces of a Hilbert space $H$
$T_n z = z_n$ iterates of orthonormal projections of a point $z \in H$
$Pz =$ orthogonal projection of $z$ onto $L_1 \cap L_2 \cap \cdots \cap L_K$

**H FINITE DIMENSIONAL:**
- in both cyclic and random case $\{z_n\}$ converges.
- in the cyclic case $\{z_n\}$ converges exponentially fast.
- in the random case $\{z_n\}$ may converge arbitrarily slow.
- in both cyclic and random case $\|T_n - P\| \to 0$.

**H INFINITE DIMENSIONAL:**
- in the cyclic case $\{z_n\}$ converges, possibly arbitrarily slow.
- in the random case $\{z_n\}$ may diverge.

If the “angles” between all $L_i$’s are positive, $\{z_n\}$ converges exponentially fast.
Areas of application

...choose ingeniously subspaces $L_1, L_2, \ldots, L_K$ and a point $z_0$. The sequence $z_n = P_{k_n}z_{n-1}$ of projections converges to the projection of $z_0$ onto $\bigcap L_i$; this is the object you sought...

- solving systems of linear equations
- constructing conformal mappings
- computed tomography

Sir Godfrey Hounsfield, the inventor of the first commercially viable CT scanner, shared the 1979 Nobel Prize in Medicine.
Convex sets

closed and convex $C_1, C_2, \ldots, C_K \subset H$
\[ \bigcap C_i \neq \emptyset \]

$z_n = P_{k_n}z_{n-1}$ iterates of the nearest point projections of a point $z$

If $H = \mathbb{R}^d$ then $\{z_n\}$ converges. [Dye, Kuczumow, Lin, Reich ’96]

If $K = 3$ and $H = \ell_2$ then $\{z_n\}$ converges weakly.
[Bruck ’82], [Dye, Reich ’92]
For $K = 4$ this is not known!

If $H = \ell_2$, then every sequence $\{z_n\}$ of periodic iterates converges weakly. [Bregman ’65]

There exist $C, D \subset \ell_2$ closed and convex with $0 \in C \cap D$, and a sequence $\{z_n\}$ of iterates of nearest point projections on these sets which converges weakly but not in norm. [Hundal ’04]
No norm-convergence in $\ell_2$ already for 2 convex sets

In $\ell_2$ there exist a closed convex set $C$, a hyperplane $D$, with $0 \in C \cap D$, and a point $z$ so that the iterates $(P_C P_D)^n z$ do not converge in norm. The iterates approximately contain an ON sequence $\{e_n\}$. 
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In $\ell_2$ there exist a closed convex set $C$, a hyperplane $D$, with $0 \in C \cap D$, and a point $z$ so that the iterates $(P_C P_D)^n z$ do not converge in norm. The iterates approximately contain an ON sequence $\{e_n\}$. 
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\[ (1 - \varepsilon_n) e_{n+1} \]
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From a convex to a linear counterexample

Let $u$ and $v$ be orthonormal, $E$ be the span of $u$ and $v$, $\varepsilon > 0$.

[Hundal] There exists a convex cone $C \subset \mathbb{R}^3$ and a product $\varphi$ of nearest point projections onto $C$ and $E$ so that

$$|\varphi(C, E)(u) - v| < \varepsilon.$$ 

[Paszkiewicz] There exist subspaces $X$ and $Y$ of $\mathbb{R}^d$ and a product $\varphi$ of projections onto $X$, $Y$ and $E$ so that

$$|\varphi(X, Y, E)(u) - v| < \varepsilon.$$
From $u$ to $v$ via 3 linear subspaces of $\mathbb{R}^d$

Let $u$ and $v$ be orthonormal, $E$ be the span of $u$ and $v$, $\varepsilon > 0$.

There exist subspaces $Z_1 \subset Z_2 \subset \cdots \subset Z_k$, $\dim Z_j = j + 1$ and a product $\varphi$ of projections on these spaces and on $E$ so that

$$|\varphi(Z_1, \ldots, Z_k, E)(u) - v| < \varepsilon.$$

Suppose $Z_1 \subset Z_2 \subset \cdots \subset Z_k = X$ are subspaces of $\mathbb{R}^d$. There is a subspace $Y \approx X$ and $m_1 > m_2 > \cdots > m_k$ so that for all $j = 1, \ldots, k$

$$\|(P_X P_Y P_X)^{m_j} - P_{Z_j}\| < \varepsilon.$$

Corollary: There exist subspaces $X$ and $Y$ and a product $\varphi$ of projections onto $X$, $Y$ and $E$ so that

$$|\varphi(X, Y, E)(u) - v| < \varepsilon.$$
Construction of $Z_1 \subset Z_2 \subset \cdots \subset Z_k$

Let $u$ and $v$ be orthonormal, $E$ be the span of $u$ and $v$, $\varepsilon > 0$.

There exist subspaces $Z_1 \subset Z_2 \subset \cdots \subset Z_k$, $\dim Z_j = j + 1$ and a product $\varphi$ of projections on these spaces and on $E$ so that

$$|\varphi(Z_1, \ldots, Z_k, E)(u) - v| < \varepsilon.$$ 

Choose orthonormal vectors $x_0, x_1, \ldots, x_{k-1} \in E^\perp$.

$$Z_j = \text{span}\{h_0 + \alpha_0 z_0, h_1 + \alpha_1 z_1, \ldots, h_j + \alpha_j z_j\}.$$ 

For some rapidly decreasing sequence of nonnegative numbers $\alpha_0 > \alpha_1 > \cdots > \alpha_{k-1} > \alpha_k = 0$. 
Replacing $Z_1 \subset Z_2 \subset \cdots \subset Z_k$ by $X$ and $Y$

Suppose $Z_1 \subset Z_2 \subset \cdots \subset Z_k = X$ are subspaces of $\mathbb{R}^d$. There is a subspace $Y \approx X$ and $m_1 > m_2 > \cdots > m_k$ so that for all $j = 1, \ldots, k$, \[\| (P_X P_Y P_X)^{m_j} - P_{Z_j} \| < \varepsilon.\]

Let \(\{e_i\}_{i=0}^k\) be an orthonormal basis of $Z_k = X$ such that \(\{e_i\}_{i=0}^j\) is an orthonormal basis of $X_j$ for $j \in \{1, \ldots, k\}$.

Choose orthonormal vectors $w_0, w_1, \ldots, w_k \in X^\perp$.

Define $Y = \text{span}\{e_i + \beta_i w_i\}_{i=0}^k$.

The coefficients $\beta_i$ and exponents $m_i$ are chosen by induction starting at $k$ and going ”backwards” so that $\beta_k > \beta_{k-1} > \cdots > \beta_0 > 0$ are rapidly decreasing, and $m_k < m_{k-1} < \cdots < m_1$ are rapidly increasing.
From $u$ to $v$ via 3 linear subspaces of $\mathbb{R}^d$

Let $u$ and $v$ be orthonormal, $E$ be the span of $u$ and $v$, $\varepsilon > 0$.

There exist subspaces $Z_1 \subset Z_2 \subset \cdots \subset Z_k$, $\dim Z_j = j + 1$ and a product $\varphi$ of projections on these spaces and on $E$ so that

$$|\varphi(Z_1, \ldots, Z_k, E)(u) - v| < \varepsilon.$$

Suppose $Z_1 \subset Z_2 \subset \cdots \subset Z_k = X$ are subspaces of $\mathbb{R}^d$. There is a subspace $Y \approx X$ and $m_1 > m_2 > \cdots > m_k$ so that for all $j = 1, \ldots, k$

$$\|(P_X P_Y P_X)^{m_j} - P_{Z_j}\| < \varepsilon.$$

Corollary: There exist subspaces $X$ and $Y$ and a product $\varphi$ of projections onto $X$, $Y$ and $E$ so that

$$|\varphi(X, Y, E)(u) - v| < \varepsilon.$$
Iterates of 1 point (and of points near to it) diverge

Let $H$ be an infinite dimensional Hilbert space.
There exist 2 closed and convex sets $C, D \subset \ell_2$ with $0 \in C \cap D$, and a sequence of iterates of nearest point projections of a point $z$ on these sets which does not converge in norm, since it approximately contains an ON sequence.

[Hein Hundal 2004]

There exist 5 closed subspaces $E, X_{\text{even}}, X_{\text{odd}}, Y_{\text{even}}, Y_{\text{odd}}$ of $H$ and a sequence of iterates of orthonormal projections of a point $z$ on these spaces which does not converge in norm, since it approximately contains an ON sequence.

[Adam Paszkiewicz, 2012]

There exist 3 closed subspaces $E, X, Y$ of $H$ and a sequence of iterates of orthonormal projections of a point $z$ on these spaces which does not converge in norm, since it approximately contains an ON sequence.

[Eva Kopecká & Vladimír Müller, 2013]
Iterates of ALL points diverge

Let $H$ be an infinite dimensional Hilbert space. There exist 3 closed subspaces $X_1, X_2, X_3$ with the following property. For every $0 \neq w_0 \in H$ there is a sequence $k_1, k_2, \cdots \in \{1, 2, 3\}$ so that the sequence of iterates defined by $w_n = P_{X_{k_n}}w_{n-1}$ does not converge in norm.

When projecting on 5 closed subspaces this can be achieved using just 2 fixed sequences of indices:
There exists a sequence $k_1, k_2, \cdots \in \{1, 2, 3\}$ with the following property. Every infinite dimensional Hilbert space $H$ has closed subspaces $X, Y, X_1, X_2, X_3$ so that if $0 \neq z \in H$, and $u_0 = P_X z$, $v_0 = P_X P_Y z$, then at least one of the sequences of iterates $\{u_n\}_{n=1}^{\infty}$ or $\{v_n\}_{n=1}^{\infty}$ defined by $u_n = P_{X_{k_n}}u_{n-1}$, $v_n = P_{X_{k_n}}v_{n-1}$ does not converge in norm.

[Kopecká & Paszkiewicz, 2015]
Let $H$ be an infinite dimensional Hilbert space, and $X_1, X_2, X_3 \subset H$ be 3 of its closed subspaces. Let $Z$ be the set of good points in $H$, that is of all $z_0 \in H$ so that for every sequence $j_1, j_2, \cdots \in \{1, 2, 3\}$ the sequence defined by $z_n = P_{X_{j_n}}z_{n-1}$ does converge in norm. Then $Z$ is a closed subspace.

**ASSUME $Z \neq H$**

Then $L = Z\perp$ is an infinite dimensional subspace of $H$, and in $L$ all points are bad w.r.t. the spaces $\tilde{X}_i = L \cap X_i$, $i = 1, 2, 3$, that is:

For every $0 \neq w_0 \in L$ there is a sequence $k_1, k_2, \cdots \in \{1, 2, 3\}$ so that the sequence of iterates defined by $w_n = P_{\tilde{X}_{k_n}}w_{n-1}$ does not converge in norm.
Fix $k_1, k_2, \cdots \in \{1, 2, 3\}$ with the following property: Every infinite dimensional (separable) Hilbert space $H^m$ contains a point $z_0^m \in H$ and closed subspaces $X_1^m, X_2^m, X_3^m$ so that the sequence $z_n^m = P_{X_{kn}} z_{n-1}^m$ diverges.

**DEFINE:**

\[
H = H^1 \oplus_2 H^2 \oplus_2 H^3 \oplus \ldots \\
X_i = X_i^1 \oplus_2 X_i^2 \oplus_2 X_i^3 \oplus \ldots \quad i = 1, 2, 3 \\
X = \text{span} \{ z^1, z^2, z^3, \ldots \} \\
Y = (X + X^\perp)/2.
\]

**THEN:**

If $0 \neq z \in H$, and $u_0 = P_X z$, $v_0 = P_X P_Y z$, then at least one of the sequences of iterates $\{u_n\}_{n=1}^\infty$ or $\{v_n\}_{n=1}^\infty$ defined by $u_n = P_{X_{kn}} u_{n-1}$, $v_n = P_{X_{kn}} v_{n-1}$ diverges.
Let $H$ be an infinite dimensional Hilbert space. There exist 3 closed subspaces $X_1, X_2, X_3$ with the following property:

For every $0 \neq w_0 \in H$ there is a sequence $k_1, k_2, \ldots \in \{1, 2, 3\}$ so that the sequence of iterates defined by $w_n = P_{X_{k_n}} w_{n-1}$ converges weakly but does not converge in norm.
Open Problems in infinite dimensional Hilbert space $H$

Random projections onto linear subspaces $L_1, L_2, \ldots, L_K$ closed subspaces of $H$
$z_n = P_{k_n}z_{n-1}$ iterates of orthonormal projections of a point $z$
What is the probability that for $\{k_n\} \in \{1, \ldots, K\}^\mathbb{N}$ the sequence $\{z_n\}$ converges in norm?
Is it always positive? Is it always equal to 1?

Cyclic projections onto linear subspaces $L_1, L_2, \ldots, L_K$ closed subspaces of $H$ s.t. $0 = \bigcap L_i$. Assume $\bigoplus L_1 \perp + \cdots + \bigoplus L_K \perp$ is not closed in $H$. Where do the initial points for the arbitrarily slow convergence lie?

Random projections onto convex sets closed and convex $C_1, C_2, \ldots, C_K \subset H$, $4 \leq K$
$\bigcap C_i \neq \emptyset$
$z_n = P_{k_n}z_{n-1}$ iterates of the nearest point projections of a point $z$
Does $\{z_n\}$ converges weakly?