

Hardy spaces, inner and outer functions, Blaschke products

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1 Hardy spaces

For $1 \leq p < \infty$ the Hardy space H^p is defined as the space of all analytic functions f in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which the norm

$$\|f\|_p = \sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p}$$

is finite. The space H^∞ consists of all bounded analytic functions f in \mathbb{D} and the norm is now

$$\|f\|_\infty = \sup_{|z| < 1} |f(z)|.$$

For functions in $H^p(\mathbb{D})$, $1 \leq p \leq \infty$, the radial limit

$$\tilde{f}(e^{it}) = \lim_{r \rightarrow 1} f(re^{it})$$

exists almost everywhere in t (Fatou's theorem), and indeed $\tilde{f} \in L^p(\mathbb{T})$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Moreover

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |\tilde{f}(e^{it})|^p dt \right)^{1/p} =: \|\tilde{f}\|_{L^p(\mathbb{T})}.$$

We normally identify f with \tilde{f} , and can just regard H^p as the subspace of those $L^p(\mathbb{T})$ functions for which the negative Fourier coefficients vanish, that is:

$$\frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(e^{it}) e^{-int} dt = 0$$

for all $n < 0$. Then a function $\tilde{f} \sim \sum_{n=0}^{\infty} a_n z^n$ can be naturally identified with the power series $f(z) = \sum_{n \geq 0} a_n z^n$, defining an analytic function f in \mathbb{D} .

One can also obtain the extension from \tilde{f} to f by convolving with the Poisson kernel K_r , namely

$$f(re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} K_{r,t}(\theta - t) \tilde{f}(e^{i\theta}) d\theta,$$

where

$$K_r(\theta - t) = \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - t)} = \Re \left(\frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} \right).$$

The case $p = 2$ is simpler, since for a function $f : z \mapsto \sum_{n=0}^{\infty} a_n z^n$ we have

$$\|f\|_2 = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}.$$

We have the following inclusions:

$$H^\infty \subset H^p \subset H^q \subset H^1,$$

for $1 \leq q \leq p \leq \infty$.

2 Inner and outer functions

2.1 The canonical factorization

In this section we are concerned with the multiplicative structure of the Hardy spaces, in that we want to factorize a general Hardy class function as the product of two somewhat simpler functions, an inner factor and an outer factor. Here are their definitions.

Definition 2.1. An *inner function* is an H^∞ function that has unit modulus almost everywhere on \mathbb{T} .

An *outer function* is a function $f \in H^1$ which can be written in the form

$$f(re^{i\theta}) = \alpha \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} k(e^{it}) dt \right), \quad (1)$$

for $re^{i\theta} \in \mathbb{D}$, where k is a real-valued integrable function and $|\alpha| = 1$.

Proposition 2.1. Let f be an *outer function*, satisfying (1). Then $\log |f(e^{i\theta})| = k(e^{i\theta})$ almost everywhere.

Proof : By taking logarithms we can obtain an expression using the Poisson kernel, namely

$$\log |f(re^{i\theta})| = \Re \log f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} K_r(\theta - t) k(e^{it}) dt.$$

Now the result follows, since $\lim_{r \rightarrow 1} \log |f(re^{i\theta})|$ equals (a.e.) both $k(e^{i\theta})$, by properties of the Poisson kernel, and $\log |f(e^{i\theta})|$, by Fatou's theorem. \square

Clearly, an outer function cannot have any zero in the disc, since it is the exponential of something. Any function that is invertible in H^∞ is outer (e.g. $z - a$ where $|a| > 1$); in fact $z - a$ is also outer when $|a| = 1$.

Examples of inner functions include Blaschke products (see below) which have zeroes, but also some functions without zeroes, such as $\exp((z-1)/(z+1))$.

Example 2.1. A *finite Blaschke product* is a function of the form

$$B(z) = e^{i\varphi} \prod_{j=1}^n \frac{z - z_j}{1 - \bar{z}_j z},$$

where $\varphi \in \mathbb{R}$ and $|z_j| < 1$ for $j = 1, \dots, n$. It is easy to verify that B has the following properties.

1. B is analytic in \mathbb{D} and continuous in $\overline{\mathbb{D}}$.
2. B is inner.
3. B has zeroes at z_1, \dots, z_n only, and poles at $1/\overline{z_1}, \dots, 1/\overline{z_n}$ only.

Theorem 2.2. (Inner–outer factorization). *Let f be a nonzero function in H^1 . Then f has a factorization $f = \theta \cdot u$, where θ is inner and u is outer. This factorization is unique up to a constant of modulus one.*

Proof : We define u by

$$u(z) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |f(e^{it})| dt\right),$$

which is an outer function as in (1). Now $\theta := f/u$ is analytic in the disc and $|\theta(z)| = 1$ a.e. for $|z| = 1$, and thus θ is inner. The factorization is unique, as if we have two outer functions u, u_1 with $|u| = |u_1| = |f|$ a.e. on \mathbb{T} , then u/u_1 and u_1/u are both inner. By the maximum modulus principle, $|u/u_1| \leq 1$ and $|u_1/u| \leq 1$ everywhere in the disc, which implies that $u = \alpha u_1$ for some constant α of modulus 1. □

The next thing we want to do is to break the inner part into two factors, an inner function with zeroes (which will be an infinite Blaschke product) and an inner function without zeroes (a so-called *singular inner function*). To do this we need to understand the properties of the zero set of a function in H^p .

Theorem 2.3. (G. Szegő) *Let $f \in H^1$ be such that f is not identically zero. Then the zeroes (z_n) of f are countable in number and satisfy the Blaschke condition*

$$\sum_1^\infty (1 - |z_n|) < \infty.$$

Proof : Without loss of generality, we may suppose that $f(0) \neq 0$, since otherwise we can consider $f(z)/z^k$ for a suitable $k \geq 1$. Now take $r < 1$, and consider the zeroes z_1, \dots, z_m in $\{|z| < r\}$, repeated according to multiplicity, supposing that none satisfy $|z_k| = r$; there can only be finitely many, since they are isolated.

The function $f_r(z) = f(rz)$ is analytic in $\{|z| \leq 1\}$, and has zeroes at the points $z_1/r, \dots, z_m/r$. Thus we can write

$$f(rz) = g(z) \prod_1^m \frac{z - z_k/r}{1 - \overline{z_k}z/r},$$

where g is analytic and non-zero in an open set containing $\overline{\mathbb{D}}$. Thus

$$\log g(0) = \frac{1}{2\pi} \int_0^{2\pi} \log g(e^{i\theta}) d\theta.$$

Taking real parts, we obtain

$$\log |g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |g(e^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta.$$

That is,

$$\log |f(0)| + \sum_{|z_k| < r} \log \left(\frac{r}{|z_k|} \right) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta. \quad (2)$$

Now Jensen's inequality (see, for example, [10], Chapter 1) asserts that

$$\varphi \left(\int_E h(x) d\mu(x) \right) \leq \int_E \varphi(h(x)) d\mu(x)$$

whenever $\varphi : [a, b] \rightarrow \mathbb{R}$ is convex, $h : E \rightarrow [a, b]$ is measurable, μ is a probability measure on E , and both integrals exist. Hence, in our case,

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \leq \log \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| d\theta$$

since $(-\log x)$ is a convex function. Thus

$$\log |f(0)| + \sum_{|z_k| < r} \log \left(\frac{r}{|z_k|} \right) \leq \log \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| d\theta \leq \log \|f\|_{H^1}. \quad (3)$$

Finally, letting $r \rightarrow 1$, we see that $\sum_1^\infty \log 1/|z_k| < \infty$. This is equivalent to the assertion that $\sum_1^\infty (1 - |z_k|) < \infty$, since

$$1 - |z| \leq \log 1/|z| \leq 2(1 - |z|)$$

for $1/2 \leq |z| \leq 1$. □

Theorem 2.4. *Let $f \in H^1$. Then the infinite Blaschke product*

$$B(z) = z^m \prod_{|z_n| \neq 0} \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1 - \bar{z}_n z},$$

where (z_n) are the zeroes of f , m of them being at 0, converges uniformly on compact sets to an H^∞ function the only zeroes of which are the (z_n) , with the correct multiplicities. Moreover, $|B(z)| \leq 1$ and $|B(e^{i\theta})| = 1$ almost everywhere.

Proof : It will be sufficient to prove the result for $f(z)/z^m$. Write

$$b_n(z) = \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1 - \bar{z}_n z},$$

where the first term is a factor chosen to make $b_n(0) > 0$.

Then $\prod b_n$ converges to an analytic function with the correct zeroes if and only if $\sum \log |b_n|$ converges locally uniformly; this happens if and only if

$$\sum |1 - b_n|$$

converges locally uniformly.

However,

$$\begin{aligned}
|1 - b_n(z)| &= \left| 1 + \frac{\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z} \right| \\
&= \frac{(1 - |z_n|)(\bar{z}_n z + |z_n|)}{|z_n| \cdot |1 - \bar{z}_n z|} \\
&\leq \frac{(1 - |z_n|)(1 + |z|)}{1 - |z|},
\end{aligned}$$

which gives convergence, by Szegő's theorem.

Thus $B(z) \in H^\infty$, and $\|B\|_{H^\infty} \leq 1$, so that the boundary function satisfies $|B(e^{i\theta})| \leq 1$ almost everywhere. But, writing $B_n = \prod_1^n b_k$, we see that B/B_n is another Blaschke product, and so

$$\left| \frac{B(0)}{B_n(0)} \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{B(e^{i\theta})}{B_n(e^{i\theta})} \right| d\theta = \frac{1}{2\pi} \int_0^{2\pi} |B(e^{i\theta})| d\theta.$$

Letting $n \rightarrow \infty$, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} |B(e^{i\theta})| d\theta = 1,$$

and so $|B(e^{i\theta})| = 1$ almost everywhere. □

For any H^p function we can remove a Blaschke factor which accounts for the zeroes, as follows.

Lemma 2.5. (F. Riesz) *Let $f \in H^p$, $f \neq 0$, and let $B(z)$ be the (possibly infinite) Blaschke product formed using the zeroes (z_n) of f . Then $f(z) = g(z)B(z)$ for some $g \in H^p$ with $\|f\|_p = \|g\|_p$.*

Proof : Let $g(z) = f(z)/B(z)$ and $g_n(z) = f(z)/B_n(z)$, where B_n is the Blaschke product corresponding to the first n zeroes of f . If $r < 1$ and $1 \leq p < \infty$, then

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} |g_n(re^{i\theta})|^p d\theta &\leq \lim_{R \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(Re^{i\theta})|^p}{|B_n(Re^{i\theta})|^p} d\theta \\
&= \lim_{R \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} |f(Re^{i\theta})|^p d\theta
\end{aligned}$$

since $|B_n(Re^{i\theta})| \rightarrow 1$ uniformly as $R \rightarrow 1$. Hence

$$\frac{1}{2\pi} \int_0^{2\pi} |g_n(re^{i\theta})|^p d\theta \leq \|f\|_{H^p}^p.$$

But $|g_n|$ increases to $|g|$ as $n \rightarrow \infty$, and so, by the monotone convergence theorem,

$$\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p d\theta \leq \|f\|_{H^p}^p.$$

However, $|g(z)| \geq |f(z)|$ for all $z \in \mathbb{D}$, so we have equality.

A similar (easier) argument holds for the case $p = \infty$. □

Corollary 2.6. Any nonzero function $f \in H^1$ can be written as $f = B \cdot S \cdot u$, where B is a Blaschke product, S a singular inner function, and u an outer function. This factorization is unique up to constants of modulus 1.

Proof : This follows from Theorem 2.2 and Lemma 2.5. □

In order to study singular inner functions, we recall the following result. It has two equivalent formulations, since any harmonic function in the disc is the real part of an analytic function.

Theorem 2.7. (G. Herglotz). A complex-valued harmonic function u in the disc is the Poisson integral of a finite positive measure μ on the circle, that is,

$$u(re^{i\theta}) = \int_{\mathbb{T}} K_r(t - \theta) d\mu(t),$$

if and only if it is non-negative. If h is an analytic function in the unit disc with values in the right-hand half-plane, such that $h(0) > 0$, then

$$h(re^{i\theta}) = \int_{\mathbb{T}} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} d\mu(t)$$

for some positive measure μ defined on \mathbb{T} .

The next result explains why an inner function without zeroes is called a singular inner function.

Corollary 2.8. Let g be an inner function without zeroes. Then there is a unique positive measure μ , singular with respect to Lebesgue measure, and a constant α of modulus 1, such that

$$g(re^{i\theta}) = \alpha \exp \left(-\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} d\mu(t) \right). \quad (4)$$

Formula (4) is very similar to (1), except that the integral is now taken with respect to a singular measure, rather than $k(t) dt$.

Proof : Since g has no zeroes, and is inner, we can write it as $g = \alpha \exp(-h)$, where h is analytic, takes values in \mathbb{C}_+ , and satisfies $h(0) > 0$ (thus α is chosen to make $g(0)/\alpha$ real and positive). By Theorem 2.7 we have expression (4), except that we need to show that μ is singular with respect to Lebesgue measure. This follows since the nontangential limits of $h(z)$ are a.e. purely imaginary as $|z| \rightarrow 1$. But

$$\operatorname{Re} h(re^{i\theta}) = \int_{\mathbb{T}} K_r(t - \theta) d\mu(t),$$

and its nontangential limit is $\frac{1}{2\pi} \frac{d\mu}{dt}$, which must therefore vanish a.e. Hence μ is a singular measure. □

Example 2.2. Let $a \in [0, 2\pi)$ and consider the singular measure $\mu_a := 2\pi\delta_a$ where δ_a is the Dirac measure at a . Then the singular inner function associated with μ_a is

$$z \mapsto \exp \left(\frac{z + e^{ia}}{z - e^{ia}} \right). \quad (5)$$

In particular $z \mapsto \exp\left(\frac{z+1}{z-1}\right)$ and $z \mapsto \exp\left(\frac{z-1}{z+1}\right)$ are singular inner functions (take $a = 0$ and $a = \pi$) and a finite products of functions of the form (5) is also a singular inner function.

2.2 Consequences

It is easy to check (using the Cauchy–Schwarz inequality) that the product of two H^2 functions is always in H^1 . The converse, which is harder, is also true; namely, that any H^1 function can be written as the product of two H^2 functions.

Theorem 2.9. (The Riesz factorization theorem) *A function f is in H^1 if and only if there exist $g, h \in H^2$ with $f = g \cdot h$ and $\|f\|_1 = \|g\|_2 \|h\|_2$.*

Proof : Note that if g and h are in H^2 then $g \cdot h \in H^1$ and $\|g \cdot h\|_1 \leq \|g\|_2 \|h\|_2$, by the Cauchy–Schwarz inequality.

Conversely, given $f \in H^1$, write $f(z) = f_1(z)B(z)$, where B is as in Theorem 2.5, $\|f_1\|_{H^1} = \|f\|_{H^1}$, and f_1 has no zeroes in \mathbb{D} .

Since f_1 is never zero it has an analytic square root g (see, for example, [9]); that is, we can write

$$f_1(z) = g(z)^2.$$

Now $f(z) = g(z)g(z)B(z)$ and $\|f\|_{H^1} = \|g\|_{H^2}^2$, so $\|f\|_1 = \|g\|_2 \|gB\|_2$ since $\|gB\|_2 = \|g\|_2$. □

We are now ready to look at the boundary behaviour of H^p functions.

Theorem 2.10. *Suppose that $f \in H^1$ and that f is not identically zero. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta > -\infty,$$

and hence $f(e^{i\theta}) \neq 0$ almost everywhere.

Proof : It is sufficient to prove the result for $f \in H^2$, and then invoke Theorem 2.9. Without loss of generality, we may suppose that $f(0) \neq 0$, as otherwise we may consider $f(z)/z^n$ for some suitable n . Writing $f_r(z) = f(rz)$ for $r < 1$, we know from Fatou’s theorem that $f_r(e^{i\theta}) \rightarrow f(e^{i\theta})$ almost everywhere as $r \rightarrow 1$. We recall from the proof of Theorem 2.3 that, if (z_k) are the zeroes of f , then we have

$$\log |f(0)| + \sum_{|z_k| < r} \log(r/|z_k|) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta,$$

and so

$$\log |f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta.$$

Let us write $\log(x) = \log^+(x) - \log^-(x)$ for $x \geq 0$, where

$$\log^+(x) = \max(0, \log x) \quad \text{and} \quad \log^-(x) = \max(0, -\log x).$$

Then, since $\log^+(x) \leq x^2$, it follows that

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \\ &= \|f_r\|_2^2 \leq \|f\|_2^2. \end{aligned}$$

Hence

$$\frac{1}{2\pi} \int_0^{2\pi} \log^- |f(re^{i\theta})| d\theta \leq \|f\|_2^2 - \log |f(0)|$$

for each r . Thus, by Fatou's Lemma, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \log^- |f(e^{i\theta})| d\theta \leq \|f\|_2^2 - \log |f(0)|,$$

and hence

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta > -\infty,$$

as required. □

The above result underlines the fact that being in H^1 is far from being equivalent to the existence of a radial limit in $L^1(\mathbb{T})$. Indeed, N. Lusin gave in 1915 ([5]) a non-zero holomorphic function f with $\lim_{r \rightarrow 1} f(r\xi) = 0$ almost everywhere on \mathbb{T} .

3 Arithmetic and nontangential limits of Inner functions

Definition 3.1. Let Θ_1 and Θ_2 be two inner functions. We say that Θ_1 divides Θ_2 if $\Theta_2/\Theta_1 \in H^2$ (obviously this quotient is again inner).

We deduce the following elementary property.

Proposition 3.1. Let B_1 and B_2 be two Blaschke products. Then B_1 divides B_2 if and only if the set of zeroes of B_1 is contained in the set of zeroes of B_2 .

Proof : Suppose that $B_2/B_1 \in H^2$. Obviously B_2/B_1 is analytic in \mathbb{D} if and only if the set of zeroes of B_1 is contained in the set of zeroes of B_2 . Moreover if it is the case B_2/B_1 is a Blaschke product, and thus belongs to H^2 . □

Note that if $\mu \geq 0$ is a singular measure with respect to Lebesgue measure, then

$$\lim_{s \rightarrow 0} \frac{\mu([x-s, x+s])}{2s} = +\infty$$

almost everywhere with respect to the measure μ .

It follows from the properties of the Poisson kernel that:

Proposition 3.2. If u is a singular inner function associated with the positive singular measure μ , then its radial limit is equal to 0 almost everywhere with respect to the measure μ .

The above proposition is sometimes used in order to conclude that an inner function is a Blaschke product. Indeed, if an inner function u has its radial limits nowhere equal to 0, necessarily, u is a Blaschke product. Here is an example:

Consider the following atomic singular inner function $S(z) = \exp\left(\frac{z+1}{z-1}\right)$ and the Blaschke factor $b(z) = \frac{1/e-z}{1-z/e}$. The composite function

$$B(z) = b \circ S$$

satisfies the following properties: at the point 1, the function B has the radial limit $1/e$ and at all other points of \mathbb{T} , the function B has a radial limit of modulus 1. Therefore B is clearly an inner function and moreover, because its radial limit is nowhere 0, it is a Blaschke product. This result is not so surprising if one knows Frostman's theorem [6], p 45.

Theorem 3.2 (Frostman). *Let u be an inner function and $\xi \in \mathbb{T}$. Then the composition $b_{t\xi} \circ u$ are Blaschke products with simple zeroes for almost all $t \in (0, 1)$, where $b_\lambda(z) = \frac{\lambda-z}{1-\lambda z}$ for $\lambda \in \mathbb{D}$. In particular u is a uniform limit of Blaschke products with simple zeroes.*

The aim of the last part of this section is to give information about the cluster sets of infinite Blaschke product.

Definition 3.3. *Let f be an holomorphic function on \mathbb{D} and $\xi \in \mathbb{T}$. The radial cluster set $C(f, \xi)$ is defined as follows:*

$$C(f, \xi) = \{\lambda \in \overline{\mathbb{C}} : \lambda = \lim_{n \rightarrow \infty} f(z_n), \text{ where } z_n \rightarrow \xi \text{ nontangentially}\}.$$

In 1985, Belna, Colwell and Piranian proved the following result.

Theorem 3.4 ([1]). *Let $\{\xi_m\}_m$ and $\{K_m\}_m$ denote a countable set of distinct points on \mathbb{T} and a sequence of nonempty, closed, connected sets in $\mathbb{D} \cup \mathbb{T}$. Then some Blaschke product has the radial cluster set K_m at ξ_m , for $m = 1, 2, \dots$.*

In other words, there are some infinite Blaschke product for which the cluster sets can even be equal to $\overline{\mathbb{D}}$, whereas almost everywhere, the radial limits exists and are of modulus one.

4 Interpolating Blaschke products

Definition 4.1. *A sequence $(z_k)_{k \geq 1}$ in \mathbb{D} is said to be an **interpolating sequence** if for every bounded sequence $(w_k)_{k \geq 1}$ there is a function $f \in H^\infty$ such that*

$$f(z_k) = w_k \text{ for all } k \geq 1.$$

Equivalently, $(z_k)_{k \geq 1}$ is an interpolating sequence whenever the evaluation operator

$$E : H^\infty \rightarrow \ell^\infty$$

defined by $E(f) = (f(z_k))_{k \geq 1}$ is surjective.

Note that not only do the points of an interpolating sequence have to be distinct, but Theorem 2.3 implies that they must also satisfy the Blaschke condition

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty, \quad (6)$$

since there is a function $f \in H^\infty$ such that $f(z_1) = 1$ and $f(z_k) = 0$ for $k \geq 2$. We can now deduce the following necessary condition, which will also turn out to be sufficient.

Lemma 4.2. *Any interpolating sequence $(z_k)_{k \geq 1}$ satisfies the following condition (C):*

$$\inf_{k \geq 1} \prod_{j \neq k} \left| \frac{z_j - z_k}{1 - \overline{z_k} z_j} \right| > 0$$

Proof : By Banach's open mapping theorem, the fact that $E : H^\infty \rightarrow \ell^\infty$ is surjective, implies that there is a constant M such that for every $(w_k)_k \in \ell^\infty$ there is a function $f \in H^\infty$ with $f(z_k) = w_k$ for each k and with

$$\|f\|_\infty \leq M \|(w_k)_k\|_\infty.$$

So, for each k we can find a function $f_k \in H^\infty$ such that $\|f_k\|_\infty \leq M$ and

$$f_k(z_j) = \delta_{j,k}.$$

Writing $f_k = B_k g_k$, where B_k is the Blaschke product formed on the points $(z_j)_{j \neq k}$, we see that $1 = B_k(z_k) g_k(z_k)$ and $\|g_k\| \leq M$. Thus

$$\prod_{j \neq k} \left| \frac{z_j - z_k}{1 - \overline{z_k} z_j} \right| = |B_k(z_k)| \geq \frac{1}{M} > 0,$$

as required. □

It turns out that Carleson's condition (C) is a necessary and sufficient condition for a sequence $(z_k)_k$ to be interpolating. If it is satisfied, the sequence must be sparse, in the sense that it has few points close together and drifts out to the boundary relatively quickly.

Example 4.1. *It can easily be verified that an example of a sequence satisfying (C) is the real sequence $z_k = 1 - \delta_k$, where $\delta_k / \delta_{k-1} \leq r$ for some fixed positive constant $r < 1$.*

There are some well-known conditions equivalent to the Carleson condition. We mention them without proof. This presentation is mainly based on [8, 7].

We write B for the infinite Blaschke product with zeroes $(z_k)_k$, namely

$$B(z) = \prod_{k=1}^{\infty} \frac{\overline{z_k}}{|z_k|} \frac{z_k - z}{1 - \overline{z_k} z},$$

and then $K = H^2 \ominus BH^2$, the orthogonal complement of BH^2 in H^2 . Note that the normalized Cauchy kernels e_k given by

$$e_k(z) = \frac{(1 - |z_k|^2)^{1/2}}{1 - \overline{z_k} z}$$

satisfy

$$\langle Bf, e_k \rangle B(z_k) f(z_k) (1 - |z_k|^2)^{1/2} = 0$$

for all $f \in H^2$. In other words we have $e_k \in K$. These functions $(e_k)_k$ will actually play a more significant role in K .

Definition 4.3. A sequence $(x_k)_k$ in a Hilbert space H is said to be a Riesz basis of H if there exist constants $A, B > 0$ such that

$$A \sum_{k=1}^{\infty} |c_k|^2 \leq \left\| \sum_{k=1}^{\infty} c_k x_k \right\|^2 \leq B \sum_{k=1}^{\infty} |c_k|^2$$

for all square summable sequences $(c_k)_k$.

We have the following characterization of interpolating sequences.

Theorem 4.4. Let $(z_k)_k \subset \mathbb{D}$ be a sequence satisfying the Blaschke condition (6). Then the following are equivalent.

1. The sequence $(z_k)_k$ is an interpolating sequence.
2. The functions $(e_k)_k$ defined by $e_k(z) = \frac{(1-|z_k|^2)^{1/2}}{1-\bar{z}_k z}$ form a Riesz basis of K .

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