Some mathematical problems in flood modelling

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A flood

Some issues in flood modelling

State-of-the-art flood models (e.g. LISFLOOD-FP, UIM, Hi-PIMS) have certain features:

- *"... simplified models may reproduce numerical results comparable to a full model [however] they are unable to simulate supercritical flows accurately."* [Liang and Smith, 2015, J. Hydroinfomatics].
- *"For an explicit code this [time-step restriction] means that the computational cost will increase as* $(1/\Delta x)^4$*."* [Bates et al., 2010, J. Hydrology].
- *"The optimum time step is determined ... to avoid the 'chequer board' oscillations."* [Chen et al., 2012, J. Hydrology].
- *"... grid resolutions below the length scales of building size and street width [are] required to provide consistent and accurate estimates of urban flooding."* [Neal et al. 2007, J. Flood Risk. Man.].

Some issues in flood modelling

These raise certain questions...

- When are approximate ('diffusion wave') models acceptable? When are the full shallow water equations required?
- How can onerous time-step restrictions be circumvented?
- How are numerical instabilities best suppressed?
- How can coarse resolution models be constructed which take account of small-scale structure?

Can mathematics help...?

Shallow water equations

Consider the *nondimensional* 1D SWE with drag:

$$\begin{aligned} u_t + uu_x + h_x + b_x &= -Ch^{-4/3}|u|u \\ h_t + (uh)_x &= 0. \end{aligned} \qquad \boxed{\text{SWE}}$$

Here $u(x,t)$ velocity, $h(x,t)$ layer thickness, $b(x)$ topography.

- Units: $h, b \sim H$, $u \sim (gH)^{1/2}$.

Leaves

$$C = \frac{n^2 g}{H^{1/3}} \frac{L}{H}, \qquad n: \text{Manning coefficient } (\mathrm{sm}^{-1/3}).$$

(L: typical horizontal scale).

Friction dominated flows

Alternatively

$$C = \frac{L}{L_F}, \quad L_F = \frac{H^{4/3}}{n^2 g}, \quad \boxed{\text{Friction length}}$$

- Flows with $L \gtrsim L_F$ will be *friction dominated*.
- Flows with $L \lesssim L_F$ will exhibit SWE phenomenology (shocks, hydraulic control etc.).

Typical values: $n \approx 0.02$ (road surface) to 0.1 (floodplain with heavy brush) $\mathrm{sm}^{-1/3}$, $g \approx 10\mathrm{ms}^{-2}$, $H \approx 1\mathrm{m}$, gives

$$L_F \approx 10 - 250\mathrm{m}.$$

Friction dominated flows: Asymptotics

Write $C = \varepsilon^{-1}$ where $\varepsilon \ll 1$. Then expand

$$u(x,t,\varepsilon) = \varepsilon^{1/2}\left(u_0(x,t) + \varepsilon u_1(x,t) + \ldots\right).$$

$\Longrightarrow$ hierarchy of models (!?).

Leading order: Find $u_0 = U(h,s)$ where $s = h_x + b_x$ (interface slope), and

$$U(h,s) = -\mathrm{sgn}(s)|s|^{1/2}h^{2/3}.$$

Leads to the *diffusion wave model* (with $b = 0$ here)

$$h_\tau = (\mathrm{sgn}(h_x)|h_x|^{1/2}h^{5/3})_x, \quad \boxed{\text{DIFFWAVE}}$$

where $\tau = \varepsilon^{1/2}t$ is a rescaled time.

Friction dominated flows: Asymptotics

First order: Obtain

$$h_\tau = -\left(U(h,s)h + \varepsilon u_1 h\right)_x ,$$

where

$$u_1 = -\frac{|s|^{1/2} h^2}{2}\left(\frac{s_x}{6s} - \frac{4h_x}{9h} - \frac{h}{2s}\left(\frac{5h_{xx}}{3h} - \frac{5h_x^2}{9h^2} + \frac{5h_x s_x}{3hs} + \frac{s_{xx}}{2s} - \frac{s_x^2}{4s^2}\right)\right).$$

- Well-posed (sign of hyperdiffusion term always negative).
- Ill-conditioned as $s \to 0$.
- Needs to be regularised to be useful!

In fact: need to solve DIFFWAVE regularisation problem first.

DIFFWAVE versus SWE

Notice that:

- The SWE are *hyperbolic* (c.f. wave equation)
  $\implies$ information propagates along characteristics[1].
- DIFFWAVE is *parabolic* (c.f. diffusion equation)
  $\implies$ information transmitted instantaneously everywhere.

Important influence on numerics:

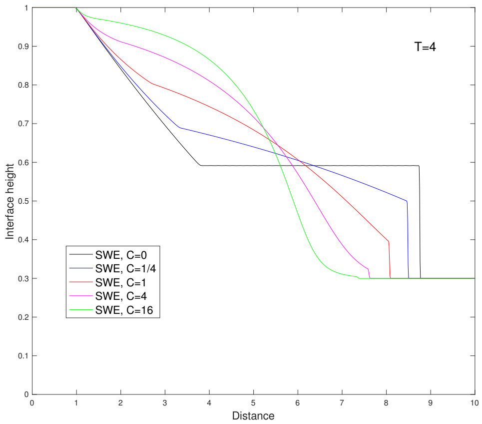
- In hyperbolic system CFL criterion $\Delta t \lesssim \Delta x / c_{\max}$.
- In parabolic system $\Delta t \lesssim (\Delta x)^2/\kappa \quad (\kappa$ diffusivity).

Worse news: In DIFFWAVE $\kappa \sim |s|^{-1/2} \implies \Delta t \to 0$ as the interface flattens ($s \to 0$).

[1] (with max. speed $c_{\max} = |u| + h^{1/2}$).

Friction in the SWE

SWE numerical solutions (CLAWPACK) with increasing C.



Friction: • damps shocks and • does not affect propagation of information in rarefactions.

Dam-breaks in DIFFWAVE

The diffwave (partial) dam-break ($h_l < 1$) problem:

$$h_\tau = (h_x^{1/2} h^{5/3})_x, \quad h(x,0) = \begin{cases} h_l & x < 0 \\ 1 & x \geq 0 \end{cases} .$$

has a *similarity solution*:

$$h(x,\tau) = F(s), \quad \text{where} \quad s = \frac{x}{\tau^{2/3}}.$$

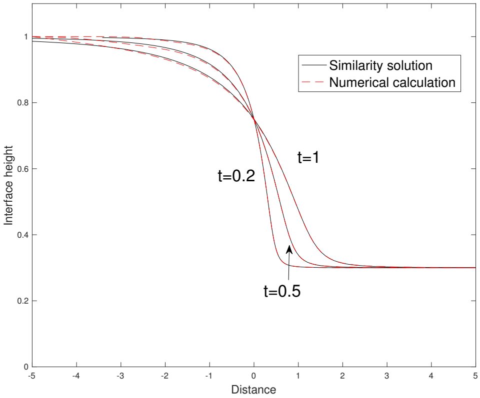
Here $F(s)$ satisfies the 2nd-order ODE boundary value problem

$$F''(s) + \frac{10F'(s)^2 F(s)^{2/3} + 4sF'(s)^{3/2}}{3F(s)^{5/3}} = 0, \qquad \begin{array}{c} F(-\infty) = h_l, \\ F(+\infty) = 1. \end{array}$$

Can be solved numerically using the shooting method.

Dam-breaks in DIFFWAVE

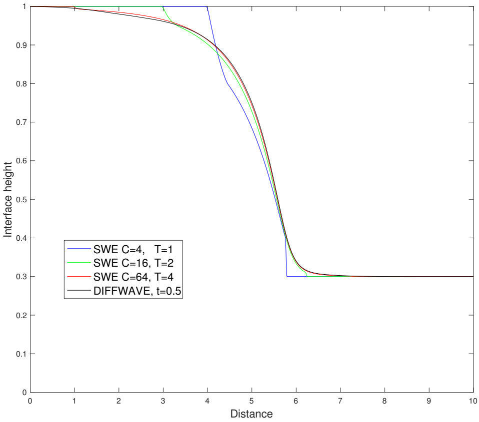
Dam-break solution has a *universal profile*:



- Notice that $h(0, t) = F(0)$ is constant.
- Flux of fluid across $x = 0$ is $F'(0)^{1/2} F(0)^{5/3} t^{-1/3}$.

DIFFWAVE versus SWE

Comparison with DIFFWAVE - need to rescale time.



In fact: all SWE dam-breaks converge $\rightarrow$ the DIFFWAVE profile (timescale $L_F/\sqrt{gH}$, non-dimensional C^{-1}).

Multiscale methods and flooding

- Multiscale methods (MSM): techniques used to obtain averaged equations, accounting for small-scale structure in physical problems.
- For example: used to model flow through porous media, properties of metamaterials, crystallography.
- Small-scale structure can be regular (e.g. periodic) or random.
- How can this be applied to flooding?

Textbook example problem

(e.g. Holmes, Introduction to perturbation methods, Ch. 5)

- Consider the diffusion equation

$$u_t = (\kappa u_x)_x$$

  where $\kappa(x)$ is rapidly-varying with small-scale structure.
- Naive approach: Use a coarse-grain average

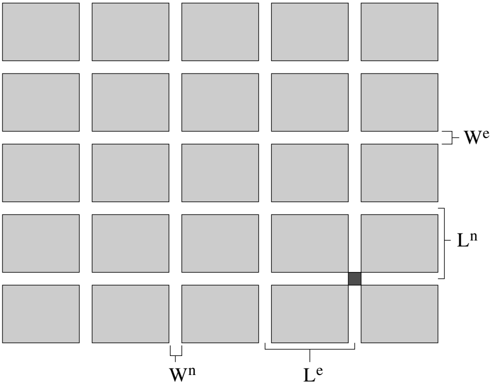
$$u_t = (\langle \kappa \rangle u_x)_x \,, \quad \langle f \rangle = \frac{1}{L} \int_{x-L/2}^{x+L/2} f(\bar{x}) \, \mathrm{d}\bar{x}$$

- MSM result:

$$u_t = (\kappa_{\text{eff}} u_x)_x \,, \quad \kappa_{\text{eff}} = \langle \kappa^{-1} \rangle^{-1} .$$

Multiscale methods and flooding: Example

As a (simple) model problem consider flooding through a network of streets.



Can we obtain a model for which we do not need to resolve every street and junction?

Model Equations

Flow in channels satisfies (generalised) DIFFWAVE equations

$$h^e_t + (F(h^e, h^e_x))_x = 0,$$
$$h^n_t + \left(G(h^n, h^n_y)\right)_y = 0.$$

Heights: $h^e(x,t)$ (east-west roads), $h^n(y,t)$ (north-south).

$F(h,s)$, $G(h,s)$ differentiable functions.

Junction boundary conditions (network limit):

$$h^e = h^n, \quad [F(h^e, h^e_x)]^+_- + [G(h^e, h^e_y)]^+_- = 0.$$

$[\cdot]^+_-$ - difference between quantity on positive x (or y) side of the junction and negative side.

Multiscale analysis

- Choose length $L \gg L^e, L^n$ (junction spacings) for (x, y).
- Small parameter $\varepsilon = L^e/L \ll 1$. Junction ratio $\beta = L^n/L^e$.
- Seek solution using multi-scale ansatz

$$h^e(x,y,t) = h_0(x,y,t) + \sum_{k=1}^{\infty} \varepsilon^k h_k^e(X,x,y,t)$$

$$h^n(x,y,t) = h_0(x,y,t) + \sum_{k=1}^{\infty} \varepsilon^k h_k^n(Y,x,y,t)$$

- Here $(X, Y) = (x/\varepsilon, y/\beta\varepsilon)$ are the ‘cell-scale’ variables.
- Cell-scale solutions are *periodic* on $X \in [0, 1)$.

Multiscale analysis

- Insert ansatz into DIFFWAVE equations. Use multi-scale formalism $\partial_x \to \varepsilon^{-1}\partial_X + \partial_x$.
- **Leading order**:

$$\partial_s F(h_0, h_{0x} + h^e_{1X})\, h^e_{1XX} = 0$$
$$\partial_s G(h_0, h_{0y} + h^n_{1Y})\, h^n_{1YY} = 0.$$

- Apply periodic boundary conditions:

$$h^e_1 = h^e_1(x, y, t), \quad h^n_1 = h^n_1(x, y, t).$$

Multiscale analysis

- **Next order** (cell problem):

$$h_{0t} + \partial_h F(h_0, h_{0x})\ h_{0x} + \partial_s F(h_0, h_{0x})\ (h_{0xx} + h^e_{2XX}) = 0$$
$$h_{0t} + \partial_h G(h_0, h_{0y})\ h_{0y} + \partial_s G(h_0, h_{0y})\ \left(h_{0yy} + h^n_{2YY}\right) = 0.$$

- Integrate over edges of ‘block’ to get

$$\partial_s F(h_0, h_{0x})[h^e_{2X}]^+_- = h_{0t} + \partial_x F(h_0, h_{0x})$$
$$\partial_s G(h_0, h_{0y})[h^n_{2Y}]^+_- = \beta\left(h_{0t} + \partial_y G(h_0, h_{0y})\right)$$

- Finally observe that

$$[F]^+_- = \varepsilon \partial_s F(h_0, h_{0x})[h^e_{2X}]^+_-, \quad [G]^+_- = \varepsilon \partial_s G(h_0, h_{0y})[h^n_{2Y}]^+_-,$$

which allows use of the flux b.c.

Multiscale equation

Using the flux boundary condition:

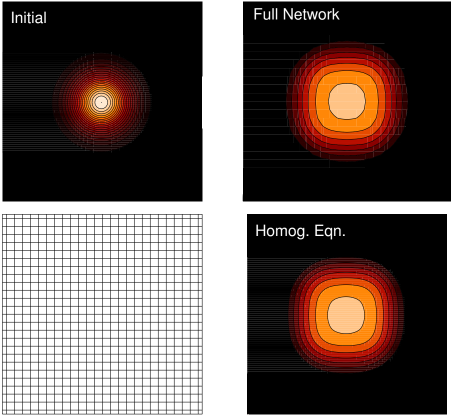
Homogenised eqn.

$$h_{0t} + \frac{1}{1+\beta}\left(F(h_0, h_{0x})\right)_x + \frac{\beta}{1+\beta}\left(G(h_0, h_{0y})\right)_y = 0.$$

- Describes large-scale (slow) evolution at leading order.
- Includes effect of the street network without resolving it.
- Can therefore be integrated on much coarser grid.

Model comparison

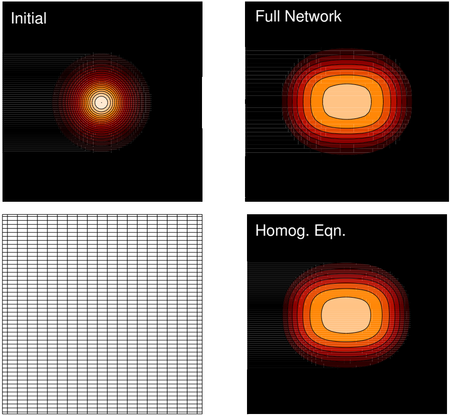
Comparison between full model (resolves network) and homogenised equations:



Square network: captures anisotropic diffusion.

Model comparison

Comparison between full model (resolves network) and homogenised equations:



Rectangular network: diffusion faster in x-direction.

Model comparison: computational savings

- **Full model**: needs to resolve junctions. 10 grid points between junctions $\implies$ 500×100 grid points. $\Delta t \propto (1/500)^2$.
- **Homogenised model**: Resolution 1 point per street $\implies$ 50×50 grid. $\Delta t \propto (1/50)^2$.

Saving factor (approx): 2000.

Conclusions and outlook

Mathematics will (should?) be useful for:

- Modifying DIFFWAVE models to:
  1. Track SWE solutions more accurately.
  2. Minimize expensive CFL restrictions.
  3. Optimize time-steps to suppress numerical instabilities.
- Providing (semi-)analytical results to calibrate models.
- Identifying conditions to switch between DIFFWAVE and SWE in regions where the latter are needed.
- Deriving coarse-grain homogenized models including the effects of small-scale structure.