A graph-theoretic approach to descriptive set theory and structural dichotomy theorems

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Lecture 1
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Part I

Introduction
I. Introduction
Basic definitions

Definition
A set $X$ is **countable** if there is an injection of $X$ into $\mathbb{N}$.  

Definition
Let $c$ denote the cardinality of $\mathbb{R}$.  

I. Introduction
Basic definitions

Definition
A set is **dense** with respect to a linear order if it intersects every non-empty open interval.

Definition
A linear order is **separable** if it has a countable dense set.
I. Introduction
Basic definitions

**Definition**
An antichain in a linear order is a pairwise disjoint family of non-empty open intervals.

**Definition**
A linear order is ccc if every antichain is countable.
I. Introduction
Two conjectures

Cantor’s continuum hypothesis
Every uncountable subset of the reals has cardinality $\mathfrak{c}$.

Souslin’s hypothesis
Every ccc linear order is separable.
I. Introduction

Two conjectures

Such questions figured prominently in the development of set theory, motivating the discovery of the constructible universe and forcing.

Both were eventually shown independent of the standard axioms.
I. Introduction

Two conjectures

These questions also played an important role in the development of descriptive set theory, where one focuses on suitably definable sets.

In this context, one can often avoid the sort of independence phenomena which appear in the study of more abstract sets.

We will give classical proofs of descriptive set-theoretic analogs of both conjectures.
Part II
Sequences
II. Sequences
Basic definitions

Definition
We use $X^I$ to denote the set of functions $s: I \rightarrow X$.

Definition
We refer to elements of $X^I$ as sequences.
II. Sequences
Basic definitions

Definition
We identify each natural number $n$ with $\{m \in \mathbb{N} \mid m < n\}$.

Definition
We refer to the elements of $X^n$ as $n$-sequences.
II. Sequences
Basic definitions

**Definition**
We use $\emptyset$ to denote the function with empty domain.

**Definition**
We use $(x)$ to denote the 1-sequence $t$ given by $t(0) = x$. 
II. Sequences
Basic definitions

<table>
<thead>
<tr>
<th>Definition</th>
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<tr>
<td>We use $X^{&lt;\mathbb{N}}$ to denote the set given by $X^{&lt;\mathbb{N}} = \bigcup_{n \in \mathbb{N}} X^n$.</td>
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<td>We refer to the elements of $X^{&lt;\mathbb{N}}$ as <strong>finite sequences</strong>.</td>
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II. Sequences
Basic definitions

Definition
The length of a finite sequence \( t \), or \(|t|\), is its domain.

Definition
The concatenation of finite sequences \( s \) and \( t \), or \( s \circ t \), is given by

\[
r(n) = \begin{cases} 
  s(n) & \text{if } n < |s| \text{ and } \\
  t(n - |s|) & \text{if } |s| \leq n < |s| + |t|. 
\end{cases}
\]
II. Sequences
Basic definitions

**Definition**
Given an $n$-sequence $s$ and a sequence $t$, we say that $s$ is an initial segment of $t$, $t$ is an extension of $s$, or $s ⊑ t$, if $\forall i < n \ s(i) = t(i)$.

**Definition**
If $t$ is an $(n + 1)$-sequence, we say $t$ is a one-step extension of $s$. 
II. Sequences

Trees

**Definition**

We say that a set $T$ of finite sequences is **closed under initial segments** if every initial segment of every element of $T$ is in $T$.

**Definition**

A **tree** on $X$ is a set $T \subseteq X^{<\mathbb{N}}$ closed under initial segments.
II. Sequences

Trees

Definition

A branch is an element of $X^\mathbb{N}$ whose initial segments are in $T$.

Definition

We use $[T]$ to denote the set of all such branches.
II. Sequences
Basic open sets

Definition
For each finite sequence $s$, we use $\mathcal{N}_s$ to denote the set given by

$$\mathcal{N}_s = \{x \in X^\mathbb{N} \mid s \sqsubseteq x\}.$$
II. Sequences
Closed sets and trees

Proposition 1
Suppose that $T$ is a tree on a discrete space. Then $[T]$ is closed.

Proof
Suppose that $x \in [T]$.

Then for each $n \in \mathbb{N}$, there exists $y \in \mathcal{N}_{x|n} \cap [T]$.

So $x|n = y|n$ and $y|n \in T$, thus $x|n \in T$.

It follows that $x \in [T]$. 

\[\Box\]
Proposition 2

Suppose that $X$ is a discrete space and $F \subseteq X^\mathbb{N}$ is closed. Then there is a tree $T$ on $X$ with the property that $F = [T]$.

Proof

Set $T = \{ t \in X^{<\mathbb{N}} \mid F \cap \mathcal{N}_t \neq \emptyset \}$.
II. Sequences
Closed sets and trees

Proof of Proposition 2 (continued)

Suppose first that $x \in F$.

If $n \in \mathbb{N}$, then $x \in F \cap \mathcal{N}_{x\mid n}$, so $x \upharpoonright n \in T$.

It follows that $x \in [T]$. 
Proof of Proposition 2 (continued)

Suppose now that $x \in [T]$.

If $n \in \mathbb{N}$, then $x \upharpoonright n \in T$, so there exists $y \in F \cap \mathcal{N}_{x \upharpoonright n}$.

Thus $x \in \overline{F}$, and it follows that $x \in F$.  \qed
Part III

Borel sets
III. Borel sets
Basic definitions

Definition
A family of sets is a $\sigma$-algebra if it is closed under complements and countable unions.

Definition
A subset of a topological space is Borel if it is in the $\sigma$-algebra generated by the family of open sets.
III. Borel sets
An alternate characterization

**Proposition 3**
Every Borel set is in the closure $\mathcal{C}$ of the family of closed sets and open sets under countable intersections and countable unions.

**Proof**
Define $\mathcal{B} = \{ C \in \mathcal{C} \mid \sim C \in \mathcal{C} \}$.  
As open sets are in $\mathcal{B}$, it is enough to show that $\mathcal{B}$ is a $\sigma$-algebra.

Clearly $\mathcal{B}$ is closed under complements.
III. Borel sets
An alternate characterization

Proof of Proposition 3 (continued)

Suppose that \( B = \bigcup_{n \in \mathbb{N}} B_n \), where \( B_n \in \mathcal{B} \) for all \( n \in \mathbb{N} \).

Then both \( B_n \) and \( \sim B_n \) are in \( \mathcal{C} \) for all \( n \in \mathbb{N} \).

So both \( \bigcup_{n \in \mathbb{N}} B_n \) and \( \bigcap_{n \in \mathbb{N}} \sim B_n \) are in \( \mathcal{C} \).

Thus both \( B \) and \( \sim B \) are in \( \mathcal{C} \), hence \( B \in \mathcal{B} \).
Ill. Borel sets
An alternate characterization

**Definition**
A set is $F_\sigma$ if it is the union of countably many closed sets.

**Definition**
A set is $G_\delta$ if it is the intersection of countably many open sets.

**Definition**
A function is Borel if the preimage of every open set is Borel.
Part IV

Analytic sets
IV. Analytic sets
Basic definitions

Definition
A set $A \subseteq X$ is **analytic** if for some closed set $F \subseteq \mathbb{N}^\mathbb{N}$ there is a continuous surjection $\varphi : F \to A$. 

IV. Analytic sets
An alternate representation

**Definition**
We identify \((X \times Y)'\) with \(X' \times Y'\).

**Definition**
The projection of a tree \(T\) on \(X \times Y\) is the set \(p[T]\) given by

\[
p[T] = \text{proj}_{X^\mathbb{N}}(T).
\]
IV. Analytic sets
An alternate representation

Proposition 4
Suppose that $X$ is a discrete space and $A \subseteq X^\mathbb{N}$ is analytic. Then there is a tree $T$ on $X \times \mathbb{N}$ such that $A = p[T]$.

Proof
Fix a closed set $F \subseteq \mathbb{N}^\mathbb{N}$ and a continuous surjection $\varphi : F \rightarrow A$. 
IV. Analytic sets
An alternate representation

Proof of Proposition 4 (continued)

Then the set $G = \{(x, y) \in X^\mathbb{N} \times F \mid x = \varphi(y)\}$ is closed.

Note that $A = \text{proj}_{X^\mathbb{N}}(G)$.

Fix a tree $T$ on $X \times \mathbb{N}$ such that $G = [T]$.

It follows that $A = p[T]$. 

\square
IV. Analytic sets
Closure properties

Proposition 5
Suppose that $X$ is a Hausdorff space and $A_n \subseteq X$ is analytic for all $n \in \mathbb{N}$. Then $\bigcup_{n \in \mathbb{N}} A_n$, $\prod_{n \in \mathbb{N}} A_n$, and $\bigcap_{n \in \mathbb{N}} A_n$ are analytic.

Proof
We will only show that $\bigcup_{n \in \mathbb{N}} A_n$ is analytic.

The proofs that $\prod_{n \in \mathbb{N}} A_n$ and $\bigcap_{n \in \mathbb{N}} A_n$ are analytic are similar.
Proof of Proposition 5 (continued)

Fix closed sets $F_n \subseteq \mathbb{N}^{\mathbb{N}}$ and continuous surjections $\varphi_n : F_n \to A_n$.

Then the set $F = \{ (n, x) \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \mid x \in F_n \}$ is closed.

Define $\varphi : F \to \bigcup_{n \in \mathbb{N}} A_n$ by $\varphi(n, x) = \varphi_n(x)$.

As $\bigcup_{n \in \mathbb{N}} A_n = \varphi(F)$, it is therefore analytic.
IV. Analytic sets
Closure properties

Proposition 6
Suppose that $X$ and $Y$ are Hausdorff and $\varphi : X \to Y$ is continuous.

1. If $X$ is analytic, then so too is $\varphi(X)$.
2. If both $X$ and $A \subseteq Y$ are analytic, then so too is $\varphi^{-1}(A)$.

Proof
Fix a closed set $F \subseteq \mathbb{N}^\mathbb{N}$ and a continuous surjection $\psi : F \to X$.

As $\varphi(X) = (\varphi \circ \psi)(F)$, it is therefore analytic.
Proof of Proposition 6 (continued)

Fix a closed set $G \subseteq \mathbb{N}^\mathbb{N}$ and a continuous surjection $\pi : G \to A$.

Then the set $H = \{(x, y) \in F \times G \mid \varphi \circ \psi(x) = \pi(y)\}$ is closed.

As $\varphi^{-1}(A) = (\psi \circ \text{proj}_F)(H)$, it is therefore analytic. \qed
IV. Analytic sets
Analytic representations of Borel sets

Proposition 7
Suppose that $X$ is an analytic Hausdorff space and $F \subseteq X$ is closed. Then $F$ is analytic.

Proof
Fix a closed set $G \subseteq \mathbb{N}^\mathbb{N}$ and a continuous surjection $\varphi: G \to X$.

Then the set $H = \varphi^{-1}(F)$ is closed.

As $F = \varphi(H)$, it is therefore analytic.
Proposition 8
Suppose that $X$ is an analytic Hausdorff space and $U \subseteq X$ is open. Then $U$ is analytic.

Proof
Fix a closed set $F \subseteq \mathbb{N}^\mathbb{N}$ and a continuous surjection $\varphi : F \rightarrow X$.

Then the set $V = \varphi^{-1}(U)$ is relatively open in $F$. 
IV. Analytic sets
Analytic representations of Borel sets

Proof of Proposition 8 (continued)

Fix sequences \( s_n \in \mathbb{N}^<\mathbb{N} \) such that \( V = F \cap \bigcup_{n \in \mathbb{N}} \mathcal{N}_{s_n} \).

Then each of the sets \( A_n = \varphi(F \cap \mathcal{N}_{s_n}) \) is analytic.

As \( U = \bigcup_{n \in \mathbb{N}} A_n \), it is therefore analytic.
Proposition 9

Every Borel subset of an analytic Hausdorff space is analytic.

Proof
We have already seen that closed sets and open sets are analytic.

We have also seen that the family of analytic sets is closed under countable intersections and countable unions.

It follows that every Borel set is analytic.
IV. Analytic sets

Separation

**Definition**

We say that \( C \) separates \( A \) from \( B \) if \( A \subseteq C \) and \( B \subseteq \sim C \).
IV. Analytic sets

Separation

**Theorem 10 (Lusin)**

Suppose that $X$ is Hausdorff and $A, B \subseteq X$ are disjoint analytic sets. Then there is a Borel set $C \subseteq X$ separating $A$ from $B$.

**Proof**

Fix closed subsets $F$ and $G$ of $\mathbb{N}^\infty$ as well as continuous surjections $\varphi: F \to A$ and $\psi: G \to B$.

For each $t \in \mathbb{N}^{<\infty}$, set $A_t = \varphi(F \cap \mathcal{N}_t)$ and $B_t = \psi(G \cap \mathcal{N}_t)$. 
### Lemma 11

Suppose that \( s, t \in \mathbb{N}^\prec \mathbb{N} \) and \( C_{m,n} \) separates \( A_{s^\prec (m)} \) from \( B_{t^\prec (n)} \) for all \( m, n \in \mathbb{N} \). Then \( C = \bigcup_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} C_{m,n} \) separates \( A_s \) from \( B_t \).

### Proof

If \( a \in A_s \), then there exists \( m \in \mathbb{N} \) such that \( a \in A_{s^\prec (m)} \), in which case \( a \in C_{m,n} \) for all \( n \in \mathbb{N} \), thus \( a \in C \).

If \( b \in B_t \), then there exists \( n \in \mathbb{N} \) such that \( b \in B_{t^\prec (n)} \), in which case \( b \notin C_{m,n} \) for all \( m \in \mathbb{N} \), thus \( b \notin C \).
Proof of Theorem 10 (continued)

Suppose that no Borel set separates $A$ from $B$.

Recursively construct $x, y \in \mathbb{N}^\mathbb{N}$ such that for no $n \in \mathbb{N}$ is there a Borel set separating $A_{x\upharpoonright n}$ from $B_{y\upharpoonright n}$.

Set $a = \varphi(x)$ and $b = \psi(y)$. 
Proof of Theorem 10 (continued)

Fix disjoint open neighborhoods $U$ and $V$ of $a$ and $b$.

Fix also $n \in \mathbb{N}$ sufficiently large that $A_x|_n \subseteq U$ and $B_y|_n \subseteq V$.

Set $C = \overline{A_x|_n}$.

Then $C$ separates $A_x|_n$ from $B_y|_n$, a contradiction.
IV. Analytic sets

Separation

Definition

A set is **co-analytic** if its complement is analytic.

Definition

A set is **bi-analytic** if it is analytic and co-analytic.
IV. Analytic sets
Separation

**Theorem 12 (Souslin)**
Every bi-analytic subset of a Hausdorff space is Borel.

**Proof**
Suppose that $A$ is bi-analytic.

Set $B = \sim A$.

Fix a Borel set $C$ separating $A$ from $B$.

Then $A = C$, so $A$ is Borel.
Theorem 13 (Souslin)

A subset of an analytic Hausdorff space is Borel iff it is bi-analytic.