Extreme amenability and ultrahomogeneous posets

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A continuous action of a topological group $G$ on a compact Hausdorff space is called $G$-flow.
A continuous action of a topological group $G$ on a compact Hausdorff space is called $G$-flow.

Topological group $G$ is *extremely amenable* if every $G$-flow $K$ has a fixed point:

$$gx = x \text{ for all } g \in G, x \in K.$$
Dynamics-flows

$G$-flow $X$ is *minimal* if every its orbit is dense:

\[ G \cdot x = X \text{ for all } x \in X. \]

**Theorem (folklore)**

*Given a topological group $G$, there is minimal $G$-flow $M(G)$ such that for any other minimal $G$-flow $Y$ there exists surjective homomorphism from $M(G)$ onto $Y$. Moreover such $G$-flow $M(G)$ is uniquely determined up to isomorphism.*

Homomorphism of $G$-flows:

\[ \varphi : X \to Y \]

\[ \varphi(gx) = g \varphi(x). \]
G-flow $X$ is *minimal* if every its orbit is dense:

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**Theorem (folklore)**

Given a topological group $G$, there is minimal $G$-flow $M(G)$ such that for any other minimal $G$-flow $Y$ there exists surjective homomorphism from $M(G)$ onto $Y$. Moreover such $G$-flow $M(G)$ is uniquely determined up to isomorphism.

Homomorphism of $G$-flows:

$$\varphi : X \to Y$$

$$\varphi(gx) = g\varphi(x).$$

The minimal $G$-flow $M(G)$ established in the previous theorem is called the *universal minimal $G$-flow.*
Theorem

\textit{(Schmerl, 1979)} A countable partially ordered set \((P, \leq)\) is ultrahomogeneous iff it is isomorphic to the one of the following:

- \((A_n)_{n=1}^{\aleph_0}\) - antichains.
- \((B_n)_{n=1}^{\aleph_0}\) - incomparable chains.
- \((C_n)_{n=2}^{\aleph_0}\) - linearly ordered antichains.
- \(D\) - universal countable ultrahomogeneous poset.
Kečrís-Pestov-Todorčević (KPT theory) 2005

- Fraïssé limits,
- Ramsey theory,
- and topological dynamics of automorphism groups.

\[ RP \leftrightarrow \text{extreme amenability,} \]
\[ (RP + OP) \leftrightarrow \text{universal minimal flow.} \]

\( \mathcal{K} \) satisfies ordering property (OP) if for every \( A \in \mathcal{K}_0 \) there is \( B \in \mathcal{K}_0 \) such that for every linear ordering \( \prec_A \) with \( (A, \prec_A) \in \mathcal{K} \) and every linear ordering \( \prec_B \) with \( (B, \prec_B) \in \mathcal{K} \) there is an embedding \( \phi : A \to B \) of structure \( (A, \prec_A) \) into structure \( (B, \prec_B) \).

- graphs and linearly ordered graphs
- equivalence relations and linearly ordered equivalence relations
\( \mathcal{K}_0 \) -Fraïssé class, \( \mathbf{K}_0 = \text{F} \lim(\mathcal{K}_0) \),
\( \mathcal{K} \) be its reasonable order expansion of \( \mathcal{K}_0 \), \( \mathbf{K} = \text{F} \lim(\mathcal{K}) = (\mathbf{K}_0, \prec_0) \).
We assume that domain of \( \mathbf{K}_0 \) is the set of natural numbers.
Linear ordering \( \prec_0 \) can be treated as a member of compact space
\( \mathcal{L} \mathcal{O} \subset 2^{\mathbb{N}^2} \), where \( \mathbb{N} \) is the set of natural numbers and \( \mathcal{L} \mathcal{O} \) is the set of all linear orderings of natural numbers.
Group of automorphism \( G_0 = \text{Aut}(\text{F} \lim(\mathcal{K}_0)) \) of \( \mathbf{K}_0 \) acts continuously on \( \mathcal{L} \mathcal{O} \), and in particular on:
\[
X_{\mathcal{K}} = \overline{G_0 \cdot \prec_0} \subseteq \mathcal{L} \mathcal{O}.
\]
Ramsey property

Let $\mathcal{K}$ be class of structures in signature $L$, then the collection of all substructures of $B$ isomorphic with $A$ is denoted by:

$$(B_A) = \{C \leq B : C \cong A\}.$$

We say that class $\mathcal{K}$ satisfies Ramsey property (RP) if for any two structures $B, A \in \mathcal{K}$ and any natural number $r$ there is a structure $C \in \mathcal{K}$ such that for any coloring:

$$c : (C_A) \rightarrow \{1, \ldots, r\},$$

there exist a structure $B' \in (C_B)$ and natural number $l \in \{1, \ldots, r\}$ such that:

$$c \upharpoonright (B_A) = l.$$

We write this in the form:

$$C \rightarrow (B)_r^A.$$
\[ A_n : \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \]

\[ B_n : \]

![Diagram of posets \( A_n \) and \( B_n \).]
$C_n :$

```
  ⬤ ⬤ ⬤ ⬤
   ▼
    ⬤
   ▼
  ⬤ ⬤
```

$D :$

```
  ⬤
  ▼
  ⬤ ⬤ ⬤
  ▼
  ⬤
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Miodrag Sokić (California Institute of TechnoExtreme amenability and ultrahomogeneous p
Ages of structures from the Schmerl list

\[ \mathcal{K}^{A_n}, \]

\[ \mathcal{K}^{A_{\aleph_0}} = \bigcup_{n<\aleph_0} \mathcal{K}^{A_n}. \]

\[ \mathcal{K}^{B_n} \]

\[ \mathcal{K}^{B_{\aleph_0}} = \bigcup_{n<\aleph_0} \mathcal{K}^{B_n}. \]

\[ \mathcal{K}^{C_n} \]

\[ \mathcal{K}^{C_{\aleph_0}} = \bigcup_{n<\aleph_0} \mathcal{K}^{C_n}. \]

\[ \mathcal{K}^{D} = \text{Age}((D, \leq)). \]
Fraïssé classes of finite posets

\[ \mathcal{K}^{A_{\aleph_0}}, \]

\[ \mathcal{K}^{B_n} \text{ for } 1 \leq n \leq \aleph_0, \]

\[ \mathcal{K}^{C_n} \text{ for } 2 \leq n \leq \aleph_0, \]

\[ \mathcal{K}^{D}. \]

Adding linear ordering to posets:

- linear extensions
- arbitrary linear ordering
\[ \mathcal{K}^{e,A_n} = \{(A, \leq, \prec) : (A, \leq) \in \mathcal{K}^{A_n}, \prec \text{ is l. e. of } \leq\} \]

\[ = \mathcal{K}^{o,A_n} = \{(A, \leq, \prec) : (A, \leq) \in \mathcal{K}^{A_n}, \prec \text{ is a l. o. on } A\} \]
Posets with linear orderings

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\mathcal{K}^{e,A_n} = \{ (A, \leq, \prec) : (A, \leq) \in \mathcal{K}^{A_n}, \prec \text{ is l. e. of } \leq \} \\
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\]

\[
\mathcal{K}^{e,B_n} = \{ (B, \leq, \prec) : (B, \leq) \in \mathcal{K}^{B_n}, \prec \text{ is l. e. of } \leq \} \\
\subset \mathcal{K}^{o,B_n} = \{ (B, \leq, \prec) : (B, \leq) \in \mathcal{K}^{B_n}, \prec \text{ is a l. o. on } B \},
\]
\( \mathcal{K}^{e,C_n} = \{(C, \leq, \prec) : (C, \leq) \in \mathcal{K}^{C_n}, \prec \text{ is l. e. of } \leq\} \)
\( \subset \mathcal{K}^{o,C_n} = \{(C, \leq, \prec) : (C, \leq) \in \mathcal{K}^{C_n}, \prec \text{ is a l. o. on } C\}, \)
Posets with linear orderings

\[ \mathcal{K}^e, C_n = \{ (C, \leq, \prec) : (C, \leq) \in \mathcal{K}^{C_n}, \prec \text{ is l. e. of } \leq \} \]
\[ \subset \mathcal{K}^{o, C_n} = \{ (C, \leq, \prec) : (C, \leq) \in \mathcal{K}^{C_n}, \prec \text{ is a l. o. on } C \}, \]

\[ \mathcal{K}^e, D = \{ (E, \leq, \prec) : (E, \leq) \in \mathcal{K}^D, \prec \text{ is l. e. of } \leq \} \]
\[ \subset \mathcal{K}^{o, D} = \{ (E, \leq, \prec) : (E, \leq) \in \mathcal{K}^D, \prec \text{ is a l. o. on } E \}. \]
Class $\mathcal{K}$ of finite structures in signature $L$ that is countable, contains structures of arbitrary large cardinality and satisfies HP, JEP and AP is called *Fraïssé class*.

Structure in a countable signature $L$ that is infinite countable, locally finite and ultrahomogeneous is called *Fraïssé structure*.

Structure $F\lim(\mathcal{K})$ is called *Fraïssé limit* of the class $\mathcal{K}$.

$$\mathcal{K} \hookrightarrow F\lim(\mathcal{K}), \ A \hookrightarrow \text{Age}(A).$$
Theorem

The following are all Fraïssé classes of linear ordered posets appearing on our list:

\[ \mathcal{K}^{e, A_{\aleph_0}} = \mathcal{K}^{o, A_{\aleph_0}}, \]
\[ \mathcal{K}^{e, B_n}, \mathcal{K}^{o, B_n} \text{ for } 1 \leq n \leq \aleph_0, \]
\[ \mathcal{K}^{e, C_{\aleph_0}}, \mathcal{K}^{o, C_{\aleph_0}}, \]
\[ \mathcal{K}^{o, D}, \mathcal{K}^{e, D}. \]
OP-Ordering Property

Theorem

The following are Fraïssé classes of linearly ordered posets with the ordering property:

\[ \mathcal{K}^{e,A_{\mathbb{N}_0}} = \mathcal{K}^{o,A_{\mathbb{N}_0}}, \mathcal{K}^{e,B_1}, \mathcal{K}^{e,C_{\mathbb{N}_0}}, \mathcal{K}^{e,D}. \]
Theorem

(1) $K^{e, A_n}$ is without RP for $1 < n < \aleph_0$ and it has RP for $n = \aleph_0$ and $n = 1$.
(2) $K^{e, B_n}$ is without RP for $n \geq 2$ and has RP for $n = 1$.
(3) $K^{e, C_n}$ is without RP for $2 \leq n < \aleph_0$ and has RP for $n = \aleph_0$.
(4) $K^{e, D}$ has RP.
Theorem

(1) $\mathcal{K}^{o,B_n}$ is without RP for $n \geq 2$ and it has RP for $n = 1$.

(2) $\mathcal{K}^{o,C_n}$ is without RP for $2 \leq n < \aleph_0$ and has RP for $n = \aleph_0$.

(3) $\mathcal{K}^{o,D}$ is without RP.
Lemma

$\mathcal{K}^{o,D}$ is not a Ramsey class.

Recall that $\mathcal{K}^{o,D}$ is the class of finite posets with arbitrary linear orderings.
Partite construction

The following theorem requires new version of the partite construction.

Theorem

$\mathcal{K}^{o,B_1}$ has the RP.
Another theorem requires modification of the partite construction.

**Theorem**

The class $\mathcal{K}^0, C_{\mathbb{N}_0}$ has the RP.
Extreme amenability

Theorem

The following groups:

\[ \text{Aut}(F \lim(K^e, A_{\aleph_0})), \text{Aut}(F \lim(K^e, B_1)), \]

\[ \text{Aut}(F \lim(K^e, C_{\aleph_0})), \text{Aut}(F \lim(K^e, D)), \]

are extremely amenable, while groups \( \text{Aut}(F \lim(K^e, B_n)), 2 \leq n \leq \aleph_0 \), are not extremely amenable.
Theorem

The following groups

$$\text{Aut}(F \lim(K^o_{B_1})), \text{Aut}(F \lim(K^o_{C_{N_0}})),$$

are extremely amenable, while group

$$\text{Aut}(F \lim(K^o_{B_{N_0}}))$$

is not extremely amenable.
Theorem

The space $X_K$ is the universal minimal $G_0$-flow in the following cases:

1. $K = K^{e,A_{\infty}}$, $G_0 = \text{Aut}(A_{\infty})$,
2. $K = K^{e,B_{1}}$, $G_0 = \text{Aut}(B_{1})$,
3. $K = K^{e,C_{\infty}}$, $G_0 = \text{Aut}(C_{\infty})$,
4. $K = K^{e,D}$, $G_0 = \text{Aut}(D)$. 
Theorem

The space $X_K$ is not a minimal $G_0$-flow in the following cases:

1. $K = K^{o,B_n}, G_0 = \text{Aut}(B_n), 1 \leq n \leq \aleph_0$,
2. $K = K^{o,C_{\aleph_0}}, G_0 = \text{Aut}(C_{\aleph_0})$,
3. $K = K^{o,D}, G_0 = \text{Aut}(D)$.
alternative approach

add convex linear orderings: chains and antichains are intervals
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1. $G^{e,B_n}$ is the class of structures $(A, \leq, \prec) \in \mathcal{K}^{e,B_n}$ such that for all $x, y, z \in A$ it holds

$$x \leq y, x \prec z \prec y \Rightarrow x \leq z \leq y.$$
alternative approach

add convex linear orderings: chains and antichains are intervals

1. $G^{e,B_n}$ is the class of structures $(A, \leq, \prec) \in K^{e,B_n}$ such that for all $x, y, z \in A$ it holds

   $$x \leq y, x \prec z \prec y \Rightarrow x \leq z \leq y.$$ 

2. $G^{o,B_n}$ is the class of structures $(A, \leq, \prec) \in K^{o,B_n}$ such that for all $x, y, z \in A$ it holds

   $$((x \leq y \text{ or } y \leq x) \text{ and } (x \prec z \prec y)) \Rightarrow (x \leq z \text{ or } z \leq z).$$
Lemma

For $2 \leq n < \aleph_0$, class $G^{e.B_n}$ does not satisfy RP and AP, while class $G^{e.B_{\aleph_0}}$ satisfies RP and AP.
Lemma

For $2 \leq n < \aleph_0$, class $\mathcal{G}^{e.B_n}$ does not satisfy RP and AP, while class $\mathcal{G}^{e.B_{\aleph_0}}$ satisfies RP and AP.

Lemma

Class $\mathcal{G}^{e.B_n}$ satisfies OP with respect to $\mathcal{K}^{B_n}$ for all $n \geq 2$. 
Lemma

For $2 \leq n < \aleph_0$, class $G^{e,B_n}$ does not satisfy RP and AP, while class $G^{e,B_{\aleph_0}}$ satisfies RP and AP.

Lemma

Class $G^{e,B_n}$ satisfies OP with respect to $K^{B_n}$ for all $n \geq 2$.

Lemma

Class $G^{o,B_n}$ does not satisfy AP and therefore it does not satisfy RP for all $2 \leq n < \aleph_0$, while it satisfies AP and RP for $n = \aleph_0$. Classes $G^{o,B_n}$ never satisfy OP for $n \geq 2$. 
Let \((\mathcal{K}_i)_{i=1}^{s}\) be a sequence of Ramsey classes of finite objects. Then for the two sequences \((B_i)_{i=1}^{s}\) and \((A_i)_{i=1}^{s}\) satisfying \(A_i \in \mathcal{K}_i\) and \(B_i \in \mathcal{K}_i\), we denote by \((B_i)_{i=1}^{s}\) the collection of all sequences \((A'_i)_{i=1}^{s}\) with \(A'_i \in \mathcal{K}_i\), \(A'_i \cong A_i\) and \(A'_i\) is a substructure of \(B_i\). In particular for \(s = 1\) we just write \((B_1)_{A_1}\) instead of \((A_1)_{i=1}^{1}\).
Let \((K_i)_{i=1}^s\) be a sequence of Ramsey classes of finite objects. Then for the two sequences \((B_i)_{i=1}^s\) and \((A_i)_{i=1}^s\) satisfying \(A_i \in K_i\) and \(B_i \in K_i\), we denote by \(\left(\frac{(B_i)_{i=1}^s}{(A_i)_{i=1}^s}\right)\) the collection of all sequences \((A'_i)_{i=1}^s\) with \(A'_i \in K_i\), \(A'_i \cong A_i\) and \(A'_i\) is a substructure of \(B_i\). In particular for \(s = 1\) we just write \(\frac{(B)}{A_1}\) instead of \(\left(\frac{(B_i)_{i=1}^1}{(A_i)_{i=1}^1}\right)\).

**Theorem**

*(Product Ramsey theorem for classes)*: Let \((K_i)_{i=1}^s\) be a sequence of Ramsey classes of finite objects. Fix the two sequences \((B_i)_{i=1}^s\) and \((A_i)_{i=1}^s\) such that for all \(i \in \{1, \ldots, s\}\) we have \(A_i \in K_i\), \(B_i \in K_i\). Then there is a sequence \((C_i)_{i=1}^s\) such that \(C_i \in K_i\) and for any coloring \(p: \left(\frac{(C_i)_{i=1}^s}{(A_i)_{i=1}^s}\right) \rightarrow \{1, \ldots, r\}\) there exists a sequence \((B'_i)_{i=1}^s\), \(B'_i \in K_i\) and \(B'_i \cong B_i\) for all \(i \in \{1, \ldots, s\}\), and number \(l \in \{1, \ldots, r\}\) such that \(p\left(\frac{(B'_i)_{i=1}^s}{(A_i)_{i=1}^s}\right) = l\).
Theorem

(i) \( \text{Aut}(G^e \cdot B_{\mathbb{N}_0}) \) is an extremely amenable group and \( X_{G^e \cdot B_{\mathbb{N}_0}} \) is a universal minimal \( \text{Aut}(B_{\mathbb{N}_0}) \)-flow.

(ii) \( \text{Aut}(G^o \cdot B_{\mathbb{N}_0}) \) is an extremely amenable group, while \( X_{G^o \cdot B_{\mathbb{N}_0}} \) is not a minimal \( \text{Aut}(B_{\mathbb{N}_0}) \)-flow.
$G^o,C_n$ is the class of structures $(C, \leq, \prec)$ from $K^o,C_n$ such that for all $x, y, z \in C$:

$$(x \perp y, x \prec z \prec y) \Rightarrow (x \perp z, y \perp z).$$
$G^{o, C_n}$ is the class of structures $(C, \leq, \prec)$ from $K^{o, C_n}$ such that for all $x, y, z \in C$:

$$(x \perp y, x \prec z \prec y) \Rightarrow (x \perp z, y \perp z).$$

Lemma

For $2 \leq n < \aleph_0$ class $G^{o, C_n}$ does not satisfy the RP and AP, while class $G^{o, C_{\aleph_0}}$ satisfies AP and RP.
\( \mathcal{G}^{o, C_n} \) is the class of structures \((C, \leq, \prec)\) from \( \mathcal{K}^{o, C_n} \) such that for all \( x, y, z \in C \):

\[(x \perp y, x \prec z \prec y) \Rightarrow (x \perp z, y \perp z).\]

**Lemma**

For \( 2 \leq n < \aleph_0 \) class \( \mathcal{G}^{o, C_n} \) does not satisfy the RP and AP, while class \( \mathcal{G}^{o, C_{\aleph_0}} \) satisfies AP and RP.

**Theorem**

The topological group \( \text{Aut}(\mathcal{G}^{o, C_{\aleph_0}}) \) is extremely amenable, while \( X_{\mathcal{G}^{o, C_{\aleph_0}}} \) is not a minimal \( \text{Aut}(C_{\aleph_0}) \)-flow.
\[ g^{e,A_2} \quad \ldots \quad g^{e,A_{\mathbb{N}_0}} \]
\[ g^{o,B_2} \quad \ldots \quad g^{o,B_{\mathbb{N}_0}} \]
\[ g^{o,C_2} \quad \ldots \quad g^{o,C_{\mathbb{N}_0}} \]
KPT theory does not calculate universal minimal flow for $\text{Aut}(\mathcal{B}_n)$ and $\text{Aut}(\mathcal{C}_n)$.
KPT theory does not calculate universal minimal flow for $\text{Aut}(\mathcal{B}_n)$ and $\text{Aut}(\mathcal{C}_n)$

Points are not Ramsey in any expansion with linear ordering
KPT theory does not calculate universal minimal flow for $Aut(B_n)$ and $Aut(C_n)$
Points are not Ramsey in any expansion with linear ordering
Modified version of KPT could work
Lemma

Suppose that \( G \) is a topological group such that \( G = B \rtimes A \). Groups \( A \) and \( B \) have subspace topology such that \( A \) is compact and \( B \) is extremely amenable. If the action of the group \( G \) on the subgroup \( A \) is continuous then this is a universal minimal flow for the group \( G \).
\[ \text{Aut}(B_n) = (\text{Aut}(\mathbb{Q}, <))^n \rtimes S_n \]

\[ \sigma : S_n \to \text{Aut}(\text{Aut}(\mathbb{Q}, <))^n \]

\[ (\sigma \cdot x)(r) = x(\sigma^{-1}r). \]
Lemma

Suppose that $G$ is a topological group such that $G = B \times A$. Groups $A$ and $B$ have subspace topology such that $A$ is extremely amenable and $B$ is compact. If the action of the group $G$ on the subgroup $B$ is continuous then this is a universal minimal flow for the group $G$. 
\[\text{Aut}(C_n) = (S_n)^Q \times \text{Aut}(Q, \prec)\]

\[\sigma : (\text{Aut}(Q, \prec) \to \text{Aut}((S_n)^n))\]

\[(\sigma \cdot x)(r) = x(\sigma^{-1} r).\]
Let $\mathcal{L}$ be the class of finite lattices with limit $\mathbb{L}$. Calculate universal minimal flow for $Aut(\mathbb{L})$ and get respective Ramsey statement.
Problems

Let $\mathcal{L}$ be the class of finite lattices with limit $\mathbb{L}$. Calculate universal minimal flow for $\text{Aut}(\mathbb{L})$ and get respective Ramsey statement. Let $\mathcal{OB}$ be the class of finite ordered boolean algebras $(B, \wedge, \vee, 0, 1, <)$ such that for $b \in B$ we have $0 < b < 1$. Is $\mathcal{OB}$ a Ramsey class?