1. For the following systems of Lotka–Volterra equations, determine the positions and stability properties of the fixed points. Sketch the phase portraits.

   (a) $\dot{x} = x(4 - x - 3y)\,, \quad \dot{y} = y(2 - x - y)\ ,$

   (b) $\dot{x} = x(3 - x - y)\,, \quad \dot{y} = y(2 - x - y).$

   Use LOCBIF to verify your results.

2. Show that, for the following system, the origin is a saddle point:

$$\dot{x} = -2x - 3y - x^2\,, \quad \dot{y} = x + 2y + xy - 3y^2.$$

   Change to variables which diagonalises the linear part of the system and calculate third order expansions for the stable and unstable manifolds of the origin. Use LOCBIF to draw the phase portrait.

3. For each of the following systems, find the value of a such that the given function is an appropriate Liapunov function and use these to determine the stability properties of the fixed points at the origin.

   (a) $\dot{x} = y - x^3\,, \quad \dot{y} = -x^3, \qquad V = x^4 + ay^2,$

   (b) $\dot{x} = -y^3 - xy^2\,, \quad \dot{y} = x - x^2 y, \qquad V = ax^2 + y^4.$

   Use the Bendixson's negative test to show that neither system possesses periodic solutions.

4. Find the limit cycles of the following systems and determine their stability types:

   (a) $\dot{r} = r(r-1)(r-2)\,, \quad \dot{\theta} = -1\ ,$

   (b) $\dot{r} = r(r-1)^2\,, \quad \dot{\theta} = -1.$

5. Show that the systems:

$$\dot{x} = x\,, \quad \dot{y} = -y \quad \text{and} \quad \dot{x} = x(1-y)\,, \quad \dot{y} = -y(1-y),$$

   have the same first integrals and sketch the respective phase portraits.

6. A chemical system has the following irreversible reactions

$\mathbf{P} \rightharpoonup \mathbf{A}$	{	Pool of chemical in excess produces A,
$\mathbf{A} \rightharpoonup \mathbf{B}$	{	Molecule of chemical A produces B,
$\mathbf{A} + 2\mathbf{B} \rightharpoonup 3\mathbf{B}$	{	B in presence of A generates more B,
$\mathbf{B} \rightharpoonup \mathbf{C}$	{	B breaks down to product C

Assuming the "pool" is present in large amounts we can think of it as being constant in amount and we get mass action equations:

$$\frac{dA}{dt} = k_0 P_0 - k_1 A - k_2 A B^2 \,, \quad \frac{dB}{dt} = k_1 A + k_2 A B^2 - k_3 B$$

with five constants k_0, P_0, k_1, k_2, k_3.

Transform these equations using new dimensionless state variables α, β and a new time variable τ with

$$A = \bar{A}\alpha, \quad B = \bar{B}\beta, \qquad t = \bar{T}\tau.$$

Show that if the scaling factors are

$$\bar{T} = \frac{1}{k_3}, \quad \bar{A} = \bar{B} = \sqrt{\frac{k_3}{k_2}},$$

then the equations take the form

$$\frac{d\alpha}{d\tau} = \mu - \kappa\alpha - \alpha\beta^2, \quad \frac{d\beta}{d\tau} = \kappa\alpha + \alpha\beta^2 - \beta,$$

which have just two independent dimensionless parameters:

$$\mu = \sqrt{\frac{k_0^2 k_2}{k_3^3}} P_0, \quad \kappa = \frac{k_1}{k_3}.$$