Dynamical Systems and Small Divisors

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This book nominally contains expanded versions of three of the lectures that were given in a CIME meeting of the same title as well as transcriptions of an open problem session.

It should be remarked that some of the texts are not just pedagogical expositions of material that has been published elsewhere. For example, the lectures of Yoccoz contain material that appears in print for the first time. Nevertheless, it is true that the three expositions are very clear and have paid attention to being quite accessible.

The first lecture "Perturbation of linear quasi-periodic systems" by L. H. Eliasson is concerned with the study of discrete time quasi-periodic cocycles.

An illustrative example is the study of the quasi-periodic Schrödinger equation on $\ell^2(\mathbf{Z})$. Consider the operator H defined by:

$$(Hu)_n = \epsilon(u_{n+1} + u_{n-1} - 2u_n) + V(\theta + n\omega)u_n$$

where V is an smooth function and ω is a Diophantine vector. (The author suppresses the term $2u_n$, which can be clearly included in V.)

This operator has been considered very often in Solid State Physics so that there is an extensive literature on it. (See [1, 2] for references from this point of view not covered in the present paper.) Operators of this type also appear in dynamical systems in the linearization around quasiperiodic orbits of a dynamical systems.

Note that the eigenvalue equation $Hu = Eu$ can be transformed into

$$((\quad u_{n+1} \quad u_n\,)) = ((\quad -V(\theta + n\omega) + 2\epsilon - E \quad -\epsilon \quad 1 \quad 0\,))((\quad u_n \quad u_{n-1}\,))$$

This defines a dynamical system in $\Re^2$ which is quasiperiodic in time.

The existence of solutions of the dynamical systems which are in ℓ^2 corresponds to E being in the pure point spectrum. As it is well known in elementary quantum mechanics textbooks, existence of solutions that do not grow too fast implies that E is in the spectrum. Hence, the study of spectral properties leads quickly to the study of the asymptotic behavior of systems which depend on parameters.

Note that when $\epsilon = 0$, a function supported in the site n is an eigenfunction with eigenvalue $V(\theta + n\omega)$. One would like to try to study the addition of the term with ϵ by perturbation theory. Nevertheless, if ω is irrational, then the eigenvalues are dense in an interval.

The point of view taken by the author in this case is much closer to the quantum mechanics point of view.

Even if diagonalizing the operator H is very difficult, the author has the idea that it is much easier to reduce the infinite dimensional matrix representing H

to a normal form that consists only of finite dimensional blocks. Hence, the spectrum will remain pure point.

The proof is rather intricate, but the idea is that, given a matrix that decays far away from the diagonal and that is almost decoupled into finite dimensional blocks, then one can produce a better decoupling for the blocks that are not too resonant. Eventually, for a full measure set of θ, the process finishes and we obtain a block diagonalization.

The really remarkable part of the paper is that the results are obtained for a full measure set of θ and these are the trickiest estimates.

This paper follows the author's paper [3] but some of the estimates have been improved and there is more explanation of the arguments. Some more recent developments can be found in [4].

The second paper "KAM persistence of finite gap solutions" by S. Kuksin is a pedagogical version of the book [5]. It emphasizes the background and the main geometric ideas. It has a very careful discussion of the main structures present in the problem.

The main problem considered is the persistence of quasiperiodic solutions in perturbations of Lax integrable Hamiltonian PDE's.

Roughly speaking, Lax-integrable PDE's are PDE's for which one can find a representation in terms of a commuting pair. For the purposes of this paper, – indeed for any KAM theorem – one also has to make several regularity assumption about the pair that guarantee that finite dimensional approximations for the problem are reasonably good. It is worth to keep in mind that there is a big difference between formal integrability and some more precise meanings which involve precise definitions of spaces. "Integrability" is often used in many meanings and, for example, it is often said that Burgers equation is integrable even if it has finite time blow up.

Note that quasi-periodic solutions correspond to finite dimensional tori. Since we are in an infinite dimensional situation, this corresponds to a lower dimensional tori.

The main technical step is that the author reduces the problem to a problem of a KAM with parameters. This statement needs to make assumptions about how the normal eigenvalues move so that if there is a resonance, one can exclude parameters and proceed.

This theorem is not proved in the paper, but some hints and references to the literature are given. Indeed, there is a certain industry of producing theorems of this type. As pointed out in the presentation, at the end one could use several theorems in the literature.

The theorem used here and the methods used here should be very familiar to people familiar with lower dimensional elliptic tori in finite dimensional systems. Indeed the presentation is cleverly arranged so that all the usual finite dimensional structures have carefully chosen infinite dimensional counterparts. (Of course, some of the estimates are harder). So, most of the point of view is about evolution equations in scales of Hilbert spaces satisfying small divisors and non-degeneracy conditions.

There are other presentation of infinite dimensional KAM theory which have much more of a PDE flavor and use PDE techniques at different stages. A good exposition of these techniques and comparison between different approaches is [6].

The third paper "Analytic linearization of circle diffeomorphisms" by Y.C. Yoccoz contains a proof of a remarkable result that had been discussed for many years, even if, to the knowledge of the reviewer, this is the first time that appears in print.

The author introduces an arithmetic condition $\mathcal{H}$ on rotation numbers and shows that if analytic diffeomorphism has a $\mathcal{H}$ rotation number, it is analytically conjugate to a rotation. Moreover, he constructs examples that show that given a number which is not in $\mathcal{H}$ there is an analytic diffeomorphism which is not analytically conjugate to a rotation.

This should be contrasted with the case where one can assume that the system is not only analytic but also close to a rotation. There, one can show that a rotation R_α admits an analytic neighborhood in which all the maps with rotation number α are analytically conjugate to R_α if and only if α satisfies a Brjuno condition.

The paper also contains a brief description the the finite regularity theory of conjugacy of dynamical systems on the circle.

The final chapter is a series of problems on the subject of the lectures. I think that the problems have been formulated with care so that they are precise and interesting.

References

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