Characterising the structure of the shape tangent space for objects with complex symmetry

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1 Introduction

The method for analysing object symmetry (Mardia *et al.* 2000; Kent and Mardia 2001) has been developed for the studies of shape variation in structures that are bilaterally symmetric (e.g. human faces). Further work related object symmetry to the structure of the shape tangent space which was decomposed into one subspace of symmetric shape variation and one subspace of asymmetric shape variation (Kolamunnage and Kent 2003; Kolamunnage and Kent 2005). For instance, principal component analysis (PCA) yields separate principal components (PCs) containing either symmetric or asymmetric shape variation (Kolamunnage and Kent 2003).

This approach has been extended for the shape analysis of any type of symmetry (Savriama and Klingenberg 2006). Every type of symmetry is associated with a set of symmetry transformations, which forms a symmetry group. An original configuration of landmarks is used and copies of it to which all transformations in the symmetry group of the object have been applied. Thereafter, a Procrustes fit superimposes all configurations and the resulting Procrustes mean (consensus) is symmetric. To separate the components of symmetric and asymmetric shape variation with respect to the symmetry transformations in the symmetry group of the object, we used PCA (Savriama and Klingenberg 2007).

Here, we suggest an approach to decompose the shape tangent space for object symmetry involving rotations into appropriate subspaces of symmetric and asymmetric shape variation. Also, we propose a method to determine the shape dimension for the respective subspaces.

2 Subspaces of symmetric and asymmetric shape variation

For object symmetry involving reflection, the shape tangent space is decomposed into two types of subspaces that are orthogonal and complementary to each other: one subspace of shape changes that are symmetric with respect to reflection and one subspace of shape changes that are asymmetric with respect to reflection (Mardia *et al.* 2000; Kent and Mardia 2001). Similarly, if the symmetry group contains rotations, the shape tangent space is decomposed into two kinds of orthogonal subspaces: one subspace of symmetric shape changes relative to the rotations and one subspace of asymmetric shape changes considering rotations. If the symmetry group contains both reflection and rotations, the shape tangent space is decomposed into various subspaces as a consequence of the combinations between the reflection and rotations.

3 Case study: Rotational symmetry

Let us consider a two dimensional object that is symmetric under rotations of order o. It is equally divided into o sectors arranged around the centre of rotation (like the slices of a pie). We describe the geometry of the object with a configuration of landmarks that consists of c landmarks ($c = 0, 1$) for the centre of rotation and k landmarks per sector. The k landmarks are identical for every sector and they are off the centre of rotation. The total number of landmarks is $ok + c$. The shape dimension for the shape tangent space is $2(ok + c) - 4$.

In the symmetric shapes, the centre is fixed and there are not any free parameters associated to it. By contrast, the k landmarks are free to move in any direction so that there are $2k$ free parameters associated to them. However, the Procrustes fit imposes two constraints due to the size and orientation of the whole configuration which remove one degree of freedom each. Therefore, the shape dimension for the subspace of symmetric shape variation is $2k - 2$.

To obtain the shape dimension for the asymmetry, we subtract the shape dimension for the subspace of symmetric variation from the shape dimension for the shape tangent space since it has been demonstrated that the shape dimension for the subspaces of symmetric and asymmetric variation sum up to the shape dimension for the shape tangent space (Mardia *et al.* 2000; Kent and Mardia 2001). Thus, the shape dimension for the asymmetric variation is $2(o - 1)k + 2c - 2$. Note that the Procrustes fit adds a constraint (two dimensions) due to location for these parameters.

In addition, if o is not a prime number (e.g. 4 or 6) there is further structuring within the subspace of asymmetry. The subspace of asymmetry can be decomposed into subspaces of shape variation symmetric with respect to rotations which order corresponds to each prime factor of o and subspaces of shape variation asymmetric relative to any rotations. Besides, if o has only two prime factors the shape dimension for each subspace of shape variation can be obtained with $(2q_ik - 2) - (2k - 2)$, with $q_i = \frac{o}{o_i}$ for $i = 1, ..., o$ and o_i represents the prime factors of o. Nevertheless, this decomposition is not viable with a higher number of prime factors.

To illustrate the decomposition of the total shape tangent space for object symmetry involving rotations, we use Venn diagrams (Figure 1). We consider two examples of configurations with object symmetry regarding rotations, with o as a prime number and as a composite number with two prime factors.

4 Example: Rotational symmetry of order 6

Let us consider a two-dimensional symmetric object under rotations with $o = 6$; $k = 2$ and $c = 1$. Since 6 is a composite number and has two prime factors (2 and 3), the shape tangent space can be decomposed into four subspaces of shape variation: one subspace of shape variation symmetric with respect to rotations of order 6, two subspaces of shape variation asymmetric relative to rotations of order 6 but symmetric with respect to rotations which order is either 2 or 3 and one subspace of shape variation asymmetric relative to any rotations. Also, the shape dimension corresponding to each subspace can be obtained following the rationale developed in part 3:

Symmetric component relative to rotations of order 6: $2k - 2 = 2$ AS: Asymmetric component relative to rotations of order 6: $2(o - 1)k + 2c = 20$ AS3: Asymmetric component relative to rotations of order 6 but symmetric under rotations of order 3, with $q3 = 2$: $(2q3k - 2) - (2k - 2) = 4$ AS2: Asymmetric component relative to rotations of order 6 but symmetric under rotations of order 2, with $q2 = 3$: $(2q2k - 2) - (2k - 2) = 8$ Totally asymmetric component relative to any rotations: $AS - AS3 - AS2 = 8$ To represent the decomposition of the shape tangent space into complementary subspaces as well as their associated shape dimension, we use Venn diagrams (Figure 2).

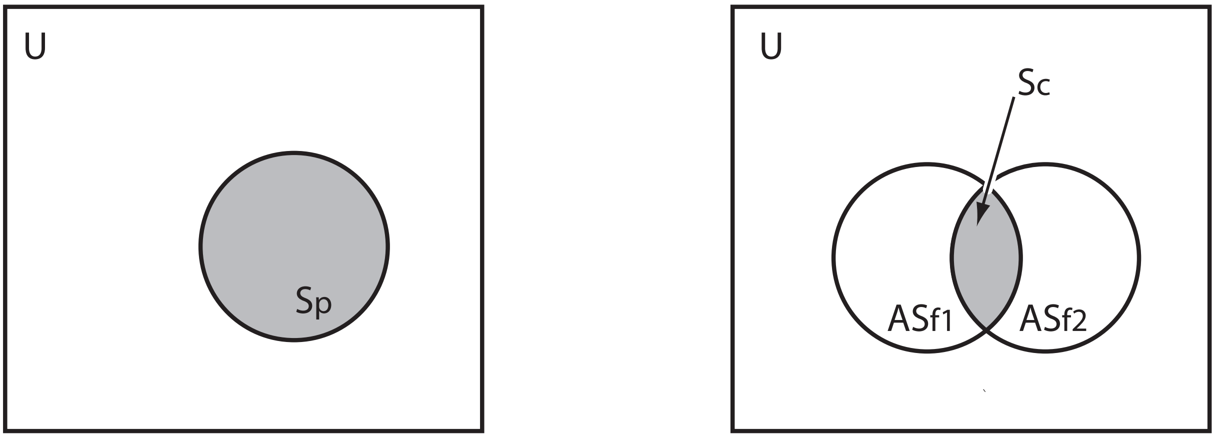

Figure 1: Venn diagrams showing the subspaces of shape variation for object symmetry considering rotations which order is a prime factor (left), and which order is a composite number (right). The shaded portion of the Venn diagrams represents the subspace of symmetric shape variation. Sp means symmetric under rotations which order is a prime number, Sc means symmetric under rotations which order is a composite number, ASf1 and ASf2 means asymmetric under rotations of order 6 but symmetric with respect to rotations which order is one of the two prime factors of the composite number.

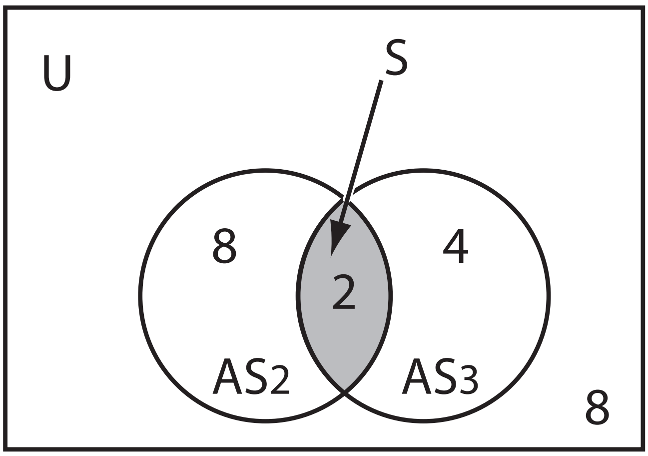

Figure 2: Venn diagram showing the subspaces of shape variation and their associated shape dimension for object symmetry involving rotations of order 6, with $k = 2$ and $c = 1$. The shaded portion of the Venn diagrams represents the subspace of symmetric shape variation. S represents the subspace associated with the symmetric component, AS2 stands for the subspace associated with the asymmetric component relative to rotations of order 6 but symmetric with respect to rotations of order 2 and AS3 stands for the subspace associated with the asymmetric component relative to rotations of order 6 but symmetric with respect to rotations of order 3.

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