MCMC implementation of rock fracture modelling

A.N. Walder,*[1], C. Xu,[2], P.A. Dowd,[3], K.V. Mardia,[1], R.J. Fowell,[2]
& J.T. Kent[1]

[1] Department of Statistics, University of Leeds
[2] School of Process, Environmental & Materials Engineering, Univ. of Leeds
[3] Engineering, Computer and Mathematical Sciences, Univ. of Adelaide

1 Introduction

In this paper we consider the simplest rock fracture modelling problem of a plane section containing line segment fractures, see Xu *et al.* (2003). We start with a number of parallel, vertical boreholes. On each of these we have recorded the depths where it is intersected by fractures; the unknowns are the number of fractures, and for each fracture its intercept, slope and extent. Having established a plausible model for the likelihood of such a data set, we estimate the parameters for the fractures using an implementation of Markov chain Monte Carlo.

2 Implementation

1. Label the boreholes by $i = 1, ..., I$ and the intersection points on each borehole by $j = 1, ..., n_i$, where n_i is the number of points on the ith borehole. Let x_i be the horizontal location of the i^{th} borehole.

2. Let y_{ij} denote the vertical position of the j^{th} intersection point on the i^{th} borehole, listed in increasing order in j, for each i, and let $Y = \{y_{ij}\}$ denote this dataset.

3. Let $k = 1, ..., K$ denote the (unknown) fracture lines. When lines are fitted to the data then the k^{th} line will intersect a consecutive set of boreholes near $\alpha_k + \beta_k x_i$.

4. A key parameter is a "matching function" $\pi(i, j) \in 1, ..., K$ that associates each intersection point with one of the fracture lines. Define the orthogonal squared distances between the intersection points and the k^{th} fracture line by $d^2_{ij;k} = \frac{(y_{ij} - \alpha_k + \beta_k x_i)^2}{1+\beta_k^2}$. If d is normally distributed, with zero mean, then the likelihood for Y, π, K and the $\{\alpha_k, \beta_k\}$ is given by

$$L(Y; \pi, K, \{\alpha_k, \beta_k\}) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \prod_{i=1}^{I} \prod_{j=1}^{n_i} \exp\left(-\frac{d^2_{ij;\pi(i,j)}}{2\sigma^2}\right),$$

   where $n = \sum n_i$.

5. Combining the likelihood above with prior densities for $\pi, K, \{\alpha_k, \beta_k\}$ gives us the posterior density $p(\pi, K, \{\alpha_k, \beta_k\}|Y)$. One way to calculate the posterior mean is by a simulation: a popular technique is Markov chain Monte Carlo. This procedure generates a Markov chain whose equilibrium distribution is the posterior density of $\pi, K, \{\alpha_k, \beta_k\}|Y$. For simplicity write $\boldsymbol{\Theta} = (\pi, K, \{\alpha_k, \beta_k\})$.

For the MCMC implementation, at each iteration, generate Θ_{new} and calculate the Hastings' ratio

$$p = \frac{p[\Theta_{new}|Y]g(\Theta_{old}|\Theta_{new})}{p[\Theta_{old}|Y]g(\Theta_{new}|\Theta_{old})}.$$

In some cases the proposal density is symmetric, $g(\Theta_{old}|\Theta_{new}) = g(\Theta_{new}|\Theta_{old})$, in which cases $p = p[\Theta_{new}|Y]/p[\Theta_{old}|Y]$. We accept $\Theta = \Theta_{new}$ with probability $\min(1, p)$, otherwise we keep $\Theta = \Theta_{old}$, *i.e.* if $p \geq 1$, we take $\Theta = \Theta_{new}$, whereas if $p < 1$, we perform a further randomisation by drawing a random sample from uniform (0,1), and accept $\Theta = \Theta_{new}$ with probability p. Typically, a burn in period is allowed in initial simulations and an average is taken of a subset of the remaining simulations.

6. At each iteration we first propose a type of change, chosen at random, from the following proposals:

   - Slope - choose a line at random, and vary the slope.
   - Intercept - similarly, choose a line at random, and change the intercept.
   - Endpoints - choose at random two lines which have their endpoints at adjacent boreholes. Move the last point of the left hand line to be the new first point of the right hand line (or vice versa).
   - Labels - choose a borehole at random, and from it choose two adjacent intersection points. Swap over the labelling, so each now belongs to a different fracture line.
   - Split - choose a line at random, and split it into two.
   - Join - choose at random two lines where endpoints are at adjacent boreholes, and join them together.

   Note that the last two proposals either increases or decreases by one the number of fractures we are fitting, and so require the reversible jump MCMC algorithm; see Green (1995).

3 Conclusions

In our work so far we have obtained satisfactory results from both simulated and actual 2D data. Work is now proceeding to implement a 3D version, with line fractures replaced by planes. Specific problems this causes for our proposals include (i) extending the idea of adjacency for swapping "edgepoints" (formerly endpoints), and (ii) splitting a plane, instead of a line.

References

Green, P.J. (1995). Reversible jump Markov chain Monte Carlo computation and Bayesian model determination. *Biometrika*, **82**, 711-732.

Xu, C., Dowd, P.A., Mardia, K.V. & Fowell, R.J. (2003). Stochastic approaches to fracture modelling. *International Association for Mathematical Geology conference*, Sept. 2003, Portsmouth.