Analysis of projective shapes of curves using projective frames

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1 Introduction

In classical projective shape analysis one works with a finite number of landmarks. In many practical problems, however, planar curves are really of interest. Examples include identification of shapes of large areas of land, shape changes in contours of seas, forests, deserts, etc, as well as in the detection of silhouettes in human surveillance, medical imaging, etc.
Our example is a "toy model" of analysis involving "footprint" like images. To compare the projective shape of two configurations of curves plus a projective frame, one uses a statistical testing method for functional data. One approach to Projective Shape Analysis (Patrangenaru, 1999), is based on the idea of a projective frame selected from the points of a finite generic k-ad in m dimensions. The resulting space of projective shapes of k-ads of is a product of k-m-2 copies of axial spaces $\mathbb{R}P^m$ (Mardia and Patrangenaru, 2005) . Such a representation has the advantage that it associates to the full set of projective invariants (Goodall and Mardia, 1999) of such a configuration one point on a projective shape manifold. The projective frame approach can be used to identify the projective shape of a planar curve in the context of a scene that contains four points in general position, that are not necessarily on the curve. The projective frame determined by such control points is used for registration. Ideally two registered images of the same scene should be identical; nevertheless due to registration errors and departure from a planar scene, they are different, and this raises questions about identification of the mean projective shape of a curve, testing for equality of such mean projective shapes of curves, etc. There are a few problems that arise in such a testing problem from curves in digital images. One is of image processing and of automatic detection of the actual curve, even from less noisy images. Secondly one has to address the curve registration problem. Thirdly, the classical simple null hypothesis of equality of two mean curves, even when they arise form the same scene, is very likely to be rejected because of inherent registration errors; therefore a neighborhood null hypothesis for functional data is preferred (Dette and Munk, 1998). Finally once the test is established one has to dwell with intensive computational algorithms involved in functional data analysis. In this preliminary report, we discuss the first three steps in such a testing problem, for the "Bigfoot" data set, a toy example of pattern recognition of contours from aerial images.

2 Image processing

We define our planar curve to be the edge of the footprint shown in the figures below. This footprint was cut from a piece of black paper as were the four small accompanying squares. These squares are landmarks which will be used to register the foot print image. Image processing was performed in the Image Processing Toolbox (IPT), MATLAB 7.1. The end result of processing the footprint shown in Figure 1 is the registered curve shown in Figure 5. The first step in processing our image is edge detection. This was performed using IPT function "edge". Here we found that overall, the Sobel's method worked best for finding edges. It was however necessary to determine the proper thresholding value for this function via trial-and-error. Notice that near the "toes" of the foot there appears to be two edges. This feature is an artifact of the paper footprint not being as flat as we would have liked. Next, we cleaned up Figure 2 using Microsoft Paint to yield the planar curve shown in Figure 3. This curve was then registered with respect the rectangular registration frame shown in Figure 4 to yield the registered planar curve in Figure 5. It was necessary to register with respect to this standard frame for the application of our proposed statistical techniques. Here, IPT functions "cpselect", "cp2tform" and "imtransfrom" were used to determine the projective transformation which maps the landmarks in Figure 3 to the dots in Figure 4. This projective transformation was used to map Figure 3 into Figure 5.

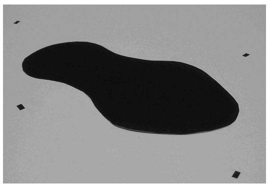

Figure 1: Cropped Footprint Image.

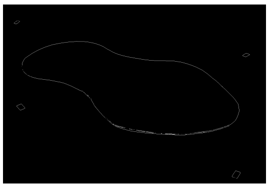

Figure 2: Edge of the Footprint Image.

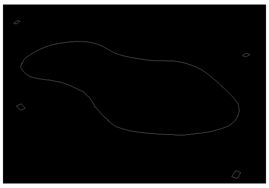

Figure 3: Planar Curve with Landmarks.

3 Projective alignment

Side view images of the same planar scene can be aligned along projective frames given by four landmarks that are present in all these images. A *projective frame* in $\mathbb{R}P^2$ is an ordered system of 4 points in general position.
Let (e_1, e_2, e_3) be the standard basis of $\mathbb{R}^3$. The *standard* projective frame is $([e_1], [e_2], [e_3], [e_1 + e_2 + e_3])$. The last point of the frame is referred to as the *unit* point. Since $\mathrm{PGL}(2)$ acts simply transitively on the set of projective frames in $\mathbb{R}P^3$, given a projective frame $\pi = (p_1, p_2, p_3, p_4)$, there is a unique $\alpha \in PGL(2),$ with

$$\alpha([e_j]) = p_j, j = 1, 2, 3 \text{ and } \alpha([[e_1 + e_2 + e_3]) = p_4. \quad (1)$$

The *projective coordinate(s)* of a point $p \in \mathbb{R}P^2$ w.r.t. a projective frame $\pi = (p_1, p_2, p_3, p_4)$ is (are) defined as $p^\pi = \alpha^{-1}(p)$, where $\alpha \in PGL(2)$ is given by (1).

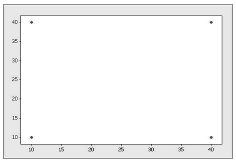

Figure 4: Rectangular Registration Frame.

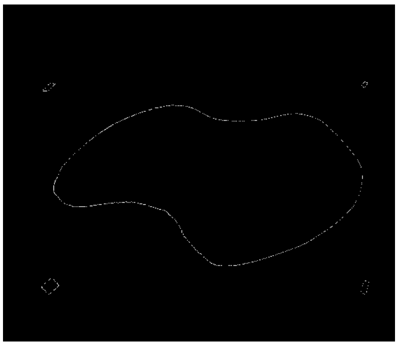

Figure 5: Registered Planar Curve with Landmarks.

Let us assume that x_1, x_2, x_3, x_4 are points in general position in plane and let (x, y) be an arbitrary point in $\mathbb{R}^2$. Explicit formulas for the affine representative (ξ, η) of the projective coordinates of (x, y) w.r.t. π are given in Mardia and Patrangenaru (2005). In particular if $x(t), y(t)$ is a planar curve, the affine representative $(\xi(t), \eta(t))$ of its projective coordinates yield another planar curve. If two curves are obtained from a planar curve viewed from different perspective, then the associated affine curves are the same. This projective representation of a (closed) curve will be used in this paper.

4 The one sample identification problem for a curve

Here we will be concerned with an instance of curve recognition. A curve

$$\gamma(t) = (\xi(t), \eta(t)), t \in [0, 0.5], \tag{2}$$

in the plane is observed with random errors

$$\Gamma(t) = (\xi(t), \eta(t)) + (\epsilon^X(t), \epsilon^Y(t)), t \in [0, 0.5], \tag{3}$$

where $\epsilon^X(t), \epsilon^Y(t)$ are stochastic independent error processes. We will identify Γ with

$$C(t) = \begin{cases} \xi(t) + \epsilon^X(t) & \text{if} \quad 0 \le t < 0.5 \\ \eta(t - 0.5) + \epsilon^Y(t - 0.5) & \text{if} \quad 0.5 \le t \le 1, \end{cases} \tag{4}$$

so that the observed curve can, for instance be considered as a random element in the Hilbert space $L^2[0, 1]$.
In practice two curves will not have exactly the same shape, even if they should agree according to some theory. In this case therefore, using the “neighborhood hypothesis” , stating the approximate equality of the shapes of the curves, seems appropriate.
We are led to consider the following one sample problem in a Hilbert space $\mathbb{H}$: given a sample of independent copies of a random element in $\mathbb{H}$, with finite fourth total moment, and mean $\mu \in \mathbb{H}$, and variance operator $\Sigma : \mathbb{H} \to \mathbb{H}$. To describe the null hypothesis suppose that L is

a linear subspace of $\mathbb{H}$, of finite dimension and let $\delta > 0$ be an arbitrary given number. Let us denote the orthoprojection onto L by Π, and also the projection on $L^{\perp}$ by $\Pi^{\perp}$. If we introduce the functional $\phi_L(x) = d^2(x, L) = \left\|\Pi^{\perp}x\right\|^2$, representing the squared distance from a point x in $\mathbb{H}$ to L(finite dimensional subspaces are closed), the neighborhood hypothesis to be tested is $H_\delta : \mu \in L_\delta \cup B_\delta$, for some $\delta > 0$.
Here $L_\delta = \{x \in \mathbb{H}, \phi_L(x) < \delta\}$ and $B_\delta = \{x \in \mathbb{H}, \phi_L(x) = \delta, \left\langle\Pi^{\perp}x, \Sigma\Pi^{\perp}x\right\rangle > 0\}$. The alternative hypothesis is $A_\delta : \mu \in L_\delta^c \cap B_\delta^c$. For testing such hypotheses see Dette and Munk (1998). More details for the present situation can be found in Munk *et al.* (2005). The test statistic is based on $\phi_L\left(\bar{X}\right) - \phi_L\left(\mu\right)$, where $\bar{X}$ is the sample mean. This statistic has approximately a simple, normal distribution with variance that can be consistently estimated. The implementation of the estimation techniques described in that paper, although straightforward, is computationally intensive, given the large number of pixels on a curve, and will be performed in subsequent work. Estimation and testing should be also analyzed in the context of similarity shapes of planar curves, that is on Hilbert manifolds of similarity shapes of curves. For more on such Hilbert manifolds see Joshi *et al.* (2005). The estimation technique in this section is applied to the "Bigfoot" data set.

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