

Decomposing departures from bilateral symmetry

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1 Introduction

Let M ($k \times m$) represent a configuration of k landmarks in m dimensions. Suppose M is bilaterally symmetric about an $(m - 1)$ -dimensional hyper-plane P in \mathbb{R}^m . In this case, the landmarks of M are of two types: pairs and solos. Paired landmarks are equispaced either side of the plane of symmetry P and the solos lie in P . In this paper, we mainly focus on $m = 2$ and without loss of generality let P be the horizontal axis, though the results extend easily to $m > 2$.

Let \mathcal{S} be the vector space of dimension $p = 2k - 4$ of $k \times m$ Procrustes tangent matrices representing deformations from M . It can be shown (Mardia *et al.*, 2000; Kent and Mardia, 2001) that \mathcal{S} splits into a direct sum

$$\mathcal{S} = \mathcal{S}^S \oplus \mathcal{S}^A \quad (1.1)$$

where \mathcal{S}^S , of dimension $p^S = k - 2$, contains bilaterally symmetric changes to M and \mathcal{S}^A , of dimension $p^A = k - 2$, is orthogonal to \mathcal{S}^S .

Further, whether or not M is bilaterally symmetric, \mathcal{S} can be decomposed as a direct sum of subspaces based on the partial warps;

$$\mathcal{S} = \bigoplus_{j=1}^{k-m} \mathcal{S}_j. \quad (1.2)$$

The first subspace \mathcal{S}_1 represents the span of uniform components and contains affine transformations. The remaining subspaces \mathcal{S}_j , $j = 2, \dots, k - m$ are spanned by the partial warps, which are determined by the eigenvectors of the $k \times k$ bending energy matrix B with non-zero eigenvalues $\lambda_2 < \dots < \lambda_{k-2}$ (Bookstein, 1989); note the non-standard indexing of eigenvalues. These subspaces represent the quadratic-like and other higher order (non-uniform) components describing large-scale to small-scale deformations from M .

In this paper, we demonstrate how the approaches (1.1) and (1.2) can be combined. The following theorem gives the theoretical result which is illustrated in the next two sections.

Theorem 1. *Each $\mathcal{S}_j \in \mathcal{S}$, $j = 1, \dots, k - 2$ can be split into a direct sum of subspaces*

$$\mathcal{S}_j = \mathcal{S}_j^S \oplus \mathcal{S}_j^A.$$

In $m = 2$ dimensions, \mathcal{S}_j^S and \mathcal{S}_j^A each have dimension one.

2 An artificial example: Isosceles triangle

Consider a simple example of an isosceles triangle in two dimensions with M (centred and scaled) given by

$$M^T = \xi \begin{bmatrix} 1 & 1 & -2 \\ \beta & -\beta & 0 \end{bmatrix}$$

where $\xi = (6 + 2\beta^2)^{-1/2}$, $\beta \in \mathbb{R}$. Note M is bilaterally symmetric about the horizontal axis.

In this case, where $k = 3$ and $m = 2$, the Procrustes tangent space is spanned by affine transformations of M only, i.e.

$$\mathcal{S} = \mathcal{S}_1 = \mathcal{S}_1^S \oplus \mathcal{S}_1^A$$

where

$$\mathcal{S}_1^S = \text{sp} \left\{ M \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\} = \text{sp} \left\{ \begin{bmatrix} 1 & -\beta \\ 1 & \beta \\ -2 & 0 \end{bmatrix} \right\}$$

$$\mathcal{S}_1^A = \text{sp} \left\{ M \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} = \text{sp} \left\{ \begin{bmatrix} \beta & 1 \\ -\beta & 1 \\ 0 & -2 \end{bmatrix} \right\}.$$

Figure 1 shows the corresponding departures from M with $\beta = 4$. The direction of the change is represented by drawing an arrow from each landmark.

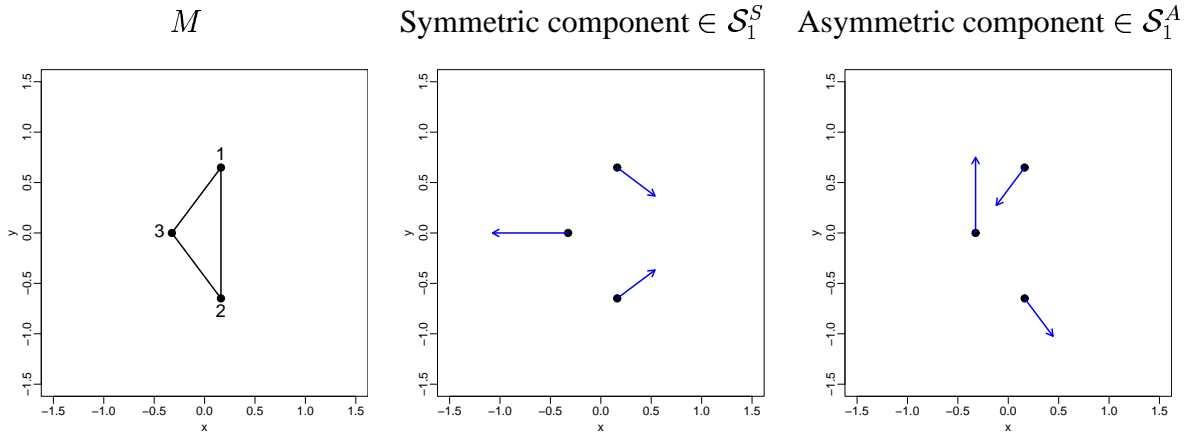


Figure 1: Decomposition of uniform departures from M

The symmetric component represents compression along the axes which makes the isosceles triangle narrower and longer, or vice versa. The asymmetric component represents stretch in 45° angle direction which breaks the isosceles nature of the triangle.

3 A real example: T2 mouse vertebrae

Consider the large and small groups of T2 mouse vertebrae data (Dryden and Mardia, 1998). Double the pooled dataset of the two groups and compute the Procrustes mean shape M . Since the doubled dataset is invariant under reflection, M is automatically bilaterally symmetric. In this example, M has $k = 6$ landmarks in $m = 2$ dimensions. Among them, two are paired and two are solo landmarks, and

$$M^T = \begin{bmatrix} -0.13 & -0.13 & 0.15 & 0.15 & 0.44 & -0.47 \\ 0.49 & -0.49 & 0.08 & -0.08 & 0.00 & 0.00 \end{bmatrix}.$$

The uniform departures from M can be defined as the same way as in Example 1.

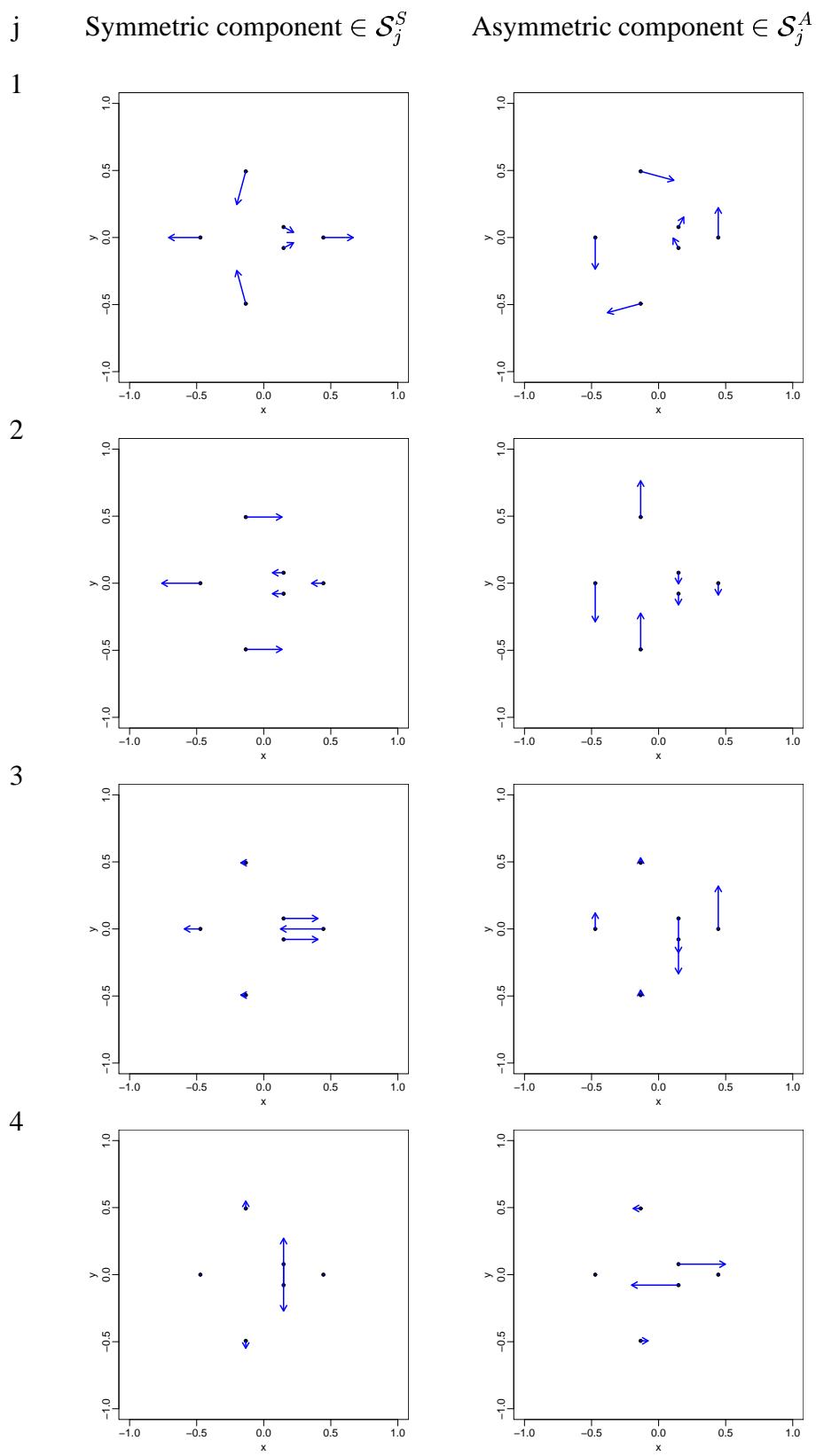


Figure 2: Decomposition of departures from M

To represent non-uniform departures from M , we compute the eigenvectors of the bending energy matrix B of M . The rank of B is 3, and hence there are three non-trivial eigenvectors:

$$\gamma_1 = \begin{bmatrix} 0.54 \\ 0.54 \\ -0.17 \\ -0.17 \\ -0.18 \\ -0.57 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0.07 \\ 0.07 \\ -0.51 \\ -0.51 \\ 0.64 \\ 0.24 \end{bmatrix}, \quad \gamma_3 = \begin{bmatrix} -0.11 \\ 0.11 \\ 0.70 \\ -0.70 \\ 0.00 \\ 0.00 \end{bmatrix}$$

Eigenvectors γ_1 and γ_2 can be described as “symmetric vectors” as they have equal entries for paired landmarks. The third eigenvector γ_3 can be described as an “asymmetric vector” as it has equal but opposite entries for paired landmarks and zero entries for solo landmarks.

The non-uniform departures from M are spanned by partial warps which are defined as “pairs” in two dimensions, i.e.

$$\mathcal{S}_j = \text{sp} \left\{ \left[\gamma_{j-1} \quad \vdots \quad \mathbf{0}_k \right], \left[\mathbf{0}_k \quad \vdots \quad \gamma_{j-1} \right] \right\}, \quad j = 2, 3, 4.$$

If γ_{j-1} is symmetric,

$$\mathcal{S}_j^S = \text{sp} \left\{ \left[\gamma_{j-1} \quad \vdots \quad \mathbf{0}_k \right] \right\} \quad \text{and} \quad \mathcal{S}_j^A = \text{sp} \left\{ \left[\mathbf{0}_k \quad \vdots \quad \gamma_{j-1} \right] \right\}.$$

The labelling is reversed if γ_{j-1} is asymmetric.

Figure 2 shows the deformations from M of the large T2 mouse vertebra.

References

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