Wavelet transform for Voronoi organized meshes

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1 Introduction

The meshing of 3D objects, when the surface of the object is a cloud of points, is realized in general by triangles using the algorithm of Delaunay. Another possibility exists *via* the dual algorithm, the Voronoi diagram. An alternative to the algorithm of Voronoi is proposed by the utilization of a *Voronoi organized meshes*. Vertices of valence 3, forming in general "hexagons" and some "polygons" compose by 5,7,8... edges to insure the curvature. An example is the fullerenes. Fullerenes are symmetrical closed convex graphite shells, consisting of 12 pentagons and various numbers of hexagons, for which the Gaussian curvature is positive. We present in this paper the different algorithms of subdivision and compression applied on this type of surfaces.

The algorithm of subdivision of surfaces has been introduced by Catmull and Clark (1978) and by Doo and Sabin (1978). They generalize the bicubic and biquadratic B -splines. Loop (1989) generalizes the quartic triangular B-splines. Rectangular patches have also been studied by Ball and Storry (1988), who use the Fourier Transform to find the tangent plane in singular points. Halstead, Kass and DeRose (1993) use the analysis of the eigenvalues in the determination of the sufficiently conditions for the convergence of the algorithm of subsection and the possibility to calculate the normal from the clean value structure.

2 Subdivision of the organized Voronoi surfaces

Proposed algorithms are developed for surfaces having the vertices of valence 3. In this case algorithms of subdivision can be considered as methods of interpolation of gravity centers. This advantage is used for the representation of objects 3D by metaballs to the different levels of resolution. Two methods of subdivision are used.

1. The first method uses filters associated to a orthogonal basis of wavelets (Doncescu and Gourret, 1998).

2. The second allows to avoid the problem, that is not again solved, the construction of the wavelets in 3D of any angle using the techniques of finite length filters. Algorithms are recursive and convergent.

Another possibility is the method of dual patches. Three centers of gravity of the faces having the same vertex in common are connected. Under this vertex, an equilateral triangle is built, then the each edge is subdividing in 3 segments determined a hexagon. The disadvantage of this method is that it does not preserve the centers of gravity of each face.

2.1 Subdivision by rotational wavelets

Let consider the next three wavelets ψ_1,ψ_2,ψ_3 having the properties:

1. compactly supported
2. regular
3. $\psi_3 = R(\psi_2)$ and $\psi_2 = R(\psi_1)$ where R is the matrix of rotation of angle $2\pi/3$.

The basis $2^j\psi(2^jx - k), k \in Z^2, j \in Z, \psi \in \psi_1, \psi_2, \psi_3$ is an orthogonal basis and is unconditional of $L^2(R^2)$

The construction of a wavelet basis in $2D$ or $3D$ uses the tensorial product. Starting with $1D$ basis,the scaling function ϕ is invariant to the rotation of $2\pi/3$. The wavelets ψ_1,ψ_2,ψ_3 are defined:

$$\psi_1(x,y) = \phi(x)\psi(y)$$
$$\psi_2(x,y) = \psi(x)\phi(y)$$
$$\psi_3(x,y) = \psi(x)\psi(y)$$

To avoid this difficult problem a possibility is to start with the matrix formula of the wavelet transform. In matrix form the formula of exact reconstruction is written:

$$H^TH + G^TG = I$$

So, a matrix R composed of several R_θ is introduced. R and R_θ are orthogonal matrices with $R_\theta^T R_\theta = I$,$R^TR = I$, so we can write

$$H^TR^TRH + G^TR^TRG = I$$

Using this idea we could applied it to the subdivision.

The subdivision process starts at level i and behind each vertex a new hexagon is constructed (level $i+1$). The solution is to take the center of gravity of each facet and to calculate :

$$P_{1m} = tC_g + (1-t)P_1$$

where P_1 corresponds to a C_{g_i}, P_2 corresponds to one vertex point obtained from the first step and P_3 is obtained from this second step.

With the Haar basis $t = 0.5$, P_{1m} is the middle of the segment and we can write

$$P_{1m} = h_0C_g + h_1P_1$$

where $h_0 = 0.5$ and $h_1 = 0.5$ are the coefficients of the Haar filter.

The matrix obtained after the wavelet transform, which we denote by H̲ex, is

$$0.5\begin{pmatrix} (P_{1x}+C_{gx}) & (P_{2x}+C_{gx}) & (P_{3x}+C_{gx}) & (P_{4x}+C_{gx}) & (P_{5x}+C_{gx}) & (P_{6x}+C_{gx}) \\ (P_{1y}+C_{gy}) & (P_{2y}+C_{gy}) & (P_{3y}+C_{gy}) & (P_{4y}+C_{gy}) & (P_{5y}+C_{gy}) & (P_{6y}+C_{gy}) \\ (P_{1z}+C_{gz}) & (P_{2z}+C_{gz}) & (P_{3z}+C_{gz}) & (P_{4z}+C_{gz}) & (P_{5z}+C_{gz}) & (P_{6z}+C_{gz}) \end{pmatrix}.$$

After this contraction of 0.5 it is necessary to rotate the hexagon through an angle θ about a line passing through the origin with a direction u determinate by C_g.

$$\underline{\mathrm{R}}_\theta = \begin{pmatrix} u_1^2 + cos\theta(1-u_1^2) & u_1u_2(1-cos\theta) - u_3sin\theta & u_3u_1(1-cos\theta) + u_2sin\theta \\ u_1u_2(1-cos\theta) + u_3sin\theta & u_2^2 + cos\theta(1-u_2^2) & u_3u_2(1-cos\theta) - u_1sin\theta \\ u_1u_3(1-cos\theta) - u_2sin\theta & u_3u_2(1-cos\theta) + u_1sin\theta & u_3^2 + cos\theta(1-u_3^2) \end{pmatrix}.$$

where u_1, u_2, u_3 are the directive cosines for the axis origin, center of gravity.

This matrix multiplies a matrix [6x3] for hexagons and [5x3] for the pentagons. For the pentagons we rotate with a $\theta = \pi/5$ and for the hexagons with an angle $\theta = \pi/6$.

3 Implicit numbering

We have previously described the subdivision process. This process consists of adding vertices, edges and “polygonal facets”, in accordance with the following recursive relations:

Vertices :
$V_0 = 20$ *number of vertices for the dodecahedron (level 0)*
$V_1 = 60$ *number of vertices for the truncated icosahedron (level 1)*
$V_i = 6V_{i-2} + V_{i-1} = 3V_{i-1}$ $(level\ i \geq 2)$
$V_i = 20 \times 3^i$
Facets :
Pentagons
$P_i = 12$ *level* i $i = 0, 1, 2, \ldots$
Hexagons
$H_0 = 0$ *(level 0)*
$H_1 = 20$ *(level 1)*
$H_i = 60 + 6H_{i-2} + H_{i-1}$ $(level\ i \geq 2)$
$H_i = V_{i-1} + H_{i-1} = 20\ \ 3^{i-1} + H_{i-1}$
$H_i = 20(3^{i-1} + 3^{i-2} + ... + 1) = 20\frac{3^i - 1}{3-1} = 10(3^i - 1)$
Edges :
$generated\ by\ Euler\ relation$ *:*
$V_i - E_i + F_i = 2$
which gives $E_i = V_i + F_i - 2$ *with* $F_i = P_i + H_i$
$E_i = 20\ \ 3^i + 12 + 10(3^i - 1) - 2 = 10 \times 3^{i+1}$

It appears that the transformation of level i to level $i + 1$ keeps the number of pentagon $(P_i = 12)$ because each pentagon is reproduced, keeps the number of hexagons H_i because each hexagon is reproduced, add V_i hexagons because each vertex of level i create a new hexagon in level $i + 1$. It is important to remark that at each level of subdivision the number of vertices and the number of hexagons are known. The topological properties of the mesh generation lead to a system of numbering suitable for this kind of applications. In Fig. 1 we present this system. We remark that the old position of analyzed vertex $P1$ (level i) determine the vertices which participate at the construction of the new hexagon. For example the new position of $P1$ is $P1A$ and this new hexagon obtaining by contraction-rotation of FA involvement an edge $P1AP9A$ in the new hexagon behind $P1$. $P9A$ could be obtain taking the $P1A$ position k and decrease by 1, $k - 1$. Similarly for the 3 other hexagons, with condition to have the same counter clockwise inside of the hexagons and for scour around the vertex $P1$.

4 Conclusion

This new type of mesh allows :

1. a minimal representation of objects 3D
2. a method of implicit numbering
3. the fast subdivision-compression of 3D data in view of their transmission
4. the possibility to apply the wavelet transform to generate a database objects

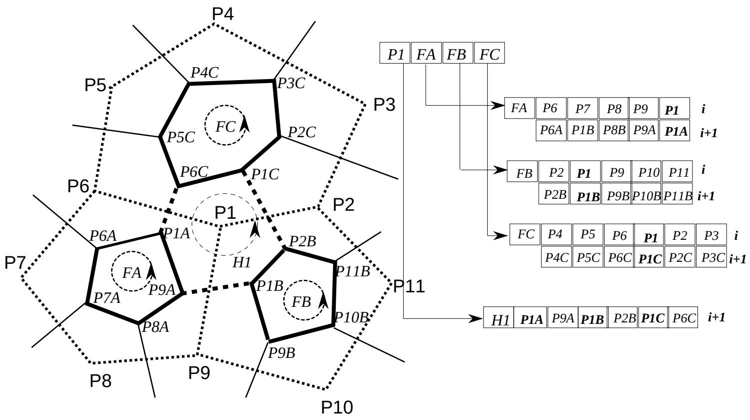
heightheight 

Figure 1: Numbering of vertices

5. a rapid deformation calculation

This search is significant to treat all the types of surfaces having the vertices of valence 3 and thus profiting from many properties of the Voronoi Organized Mesh highlighted in this article. All the algorithms could be generalized on the Voronoi Diagram.

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