Simulating non-stationary Gaussian processes using the wavelet transform

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1 Introduction

Consider an $N \times 1$ random vector, $\boldsymbol{X} = (X_1, \ldots, X_N)^T$, of observations from a multivariate Gaussian distribution. Suppose that the random variable X_i has mean μ_i and variance σ_i^2. In the most general case $\boldsymbol{X}$ is non-stationary with Cov$(X_i, X_j) = \gamma_{i,j}$ for $i \neq j$. Let the variance matrix of $\boldsymbol{X}$ be $\boldsymbol{\Sigma}$.

In this paper we discuss a wavelet-based technique for simulating $\boldsymbol{X}$, and present an application to simulating rainfall.

2 Cholesky decomposition

If the structure of $\boldsymbol{\Sigma}$ can be simplified by assuming stationarity then techniques such as that in Davies and Harte (1987) can be used. If no simplifications can be made then Cholesky decomposition is the standard method of simulation (Hörmann *et al.*, 2004).

Provided there are no linear dependencies between the $\{X_i\}$, $\boldsymbol{\Sigma}$ will be positive definite and can be written as $\boldsymbol{\Sigma} = \boldsymbol{\Gamma}^T\boldsymbol{\Gamma}$, where $\boldsymbol{\Gamma}$ is an upper triangular matrix. Given a sequence of independent observations from a Gaussian distribution with zero mean and unit variance, $\boldsymbol{Z} = (Z_1, \ldots, Z_n)^T$, we define $\boldsymbol{Y} = \boldsymbol{\Gamma}^T\boldsymbol{Z}$.

The variance matrix of $\boldsymbol{Z}$ is the $N \times N$ identity matrix $\boldsymbol{I}_N$. By definition, $\boldsymbol{Y}$ will have variance matrix $\boldsymbol{\Gamma}^T\boldsymbol{I}_N\boldsymbol{\Gamma} = \boldsymbol{\Sigma}$. The transformation:

$$X_i = \mu_i + Y_i, \quad i = 1, \ldots, N, \tag{1}$$

generates a vector $\boldsymbol{X}$ with the desired mean and variance structure.

3 Discrete Wavelet Transform (DWT) method

The Cholesky decomposition method can be used to simulate observations with exactly the desired variance structure. However, for large N, the algorithm becomes computationally expensive. We describe an alternative approximate method based on a wavelet decomposition.

We begin with a brief review of the DWT (see Percival and Walden, 2000, for further details). Let $N = 2^J$. The wavelet transform, $\boldsymbol{W}$, of the vector, $\boldsymbol{X}$, may be written as:

$$\boldsymbol{W} = \mathcal{W}\boldsymbol{X} = \left(w_{J-1,1}, \ldots, w_{J-1,2^{J-1}}, w_{J-2,1}, \ldots, w_{J-2,2^{J-2}}, \ldots, w_{1,1}, w_{1,2}, w_{0,1}, v_{0,1}\right)^T,$$

where $\mathcal{W}$ is the $N \times N$ wavelet transform matrix, and $\boldsymbol{W}$ is the resulting $N \times 1$ vector of transformed values. The wavelet transform matrix is orthonormal, i.e. $\mathcal{W}^{-1} = \mathcal{W}^T$.

The matrix $\mathcal{W}$ is sparse and highly structured. Its non-zero elements can be written in terms of the wavelet filter coefficients $\{h_l : l = 0, 1, \ldots, L-1\}$ where L (an even integer) is the length of the wavelet filter. For simplicity, we restrict attention to the so-called Haar wavelet filter (Haar, 1910), defined by $\{h_l\} = \left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$. The Haar filter gives rise to a $N \times N$ transform matrix:

$$\mathcal{W} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & 0 & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{\sqrt{N}} & -\frac{1}{\sqrt{N}} & -\frac{1}{\sqrt{N}} & -\frac{1}{\sqrt{N}} & \cdots & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} \\ \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \cdots & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} \end{pmatrix}.$$

The wavelet coefficient $w_{i,j}$ can thus be related to the elements of the vector $\boldsymbol{X}$ by a linear equation containing 2^{J-j} terms.

By definition, the variance matrix of $\boldsymbol{W}$ is given by $\mathcal{V} = \mathcal{W}\Sigma\mathcal{W}^T$. The decorrelation property of the wavelet transform (see, for example, McCoy and Walden, 1996) implies that the off-diagonal elements of $\mathcal{V}$ will be relatively small. The diagonal elements are the variances of the wavelet coefficients $\{w_{i,j}\}$.

Define $\mathcal{S}$ to be a diagonal matrix with:

$$\text{diag}(\mathcal{S}) = \left(\sqrt{\text{Var}(w_{J-1,1})}, \ldots, \sqrt{\text{Var}(w_{J-1,2^{J-1}})}, \ldots, \sqrt{\text{Var}(w_{0,1})}, \sqrt{\text{Var}(v_{0,1})}\right).$$

We define $\boldsymbol{Y} = \mathcal{W}^T\mathcal{S}\boldsymbol{Z}$, where $\boldsymbol{Z}$ is an $N \times 1$ vector of independent observations from a Gaussian distribution with zero mean and unit variance.

Notice that $\mathcal{S}\boldsymbol{Z}$ will be a vector of observations from a Gaussian distribution with zero mean and variances given by the diagonal elements of $\mathcal{V}$. By the decorrelation property, the off-diagonal elements of $\mathcal{V}$ are relatively small and can be regarded as zero. We therefore assert that $\mathcal{S}\boldsymbol{Z}$ has variance matrix approximately equal to $\mathcal{V}$. The vector $\boldsymbol{Y}$ will be a sequence of Gaussian observation with zero mean and variance matrix approximately equal to $\boldsymbol{\Sigma}$. The transformation of Eqn. 1 generates a vector $\boldsymbol{X}$ with the desired means, and approximate variance structure.

4 Application to the simulation of rainfall

There is currently much interest in techniques for stochastically simulating and downscaling data on the spatial and temporal properties of rainfall (see, for example, www.rainmap.rl.ac.uk).

It has been suggested (see, for example, Kedem and Chiu, 1987) that, during rainfall, the natural logarithm of the rain rate measured at a point, L_i, follows a Gaussian distribution. A comprehensive study on the temporal properties of rain rate was carried out by Burgueño *et al.* (1990), who suggest modelling the autocovariance structure of log rain rate by:

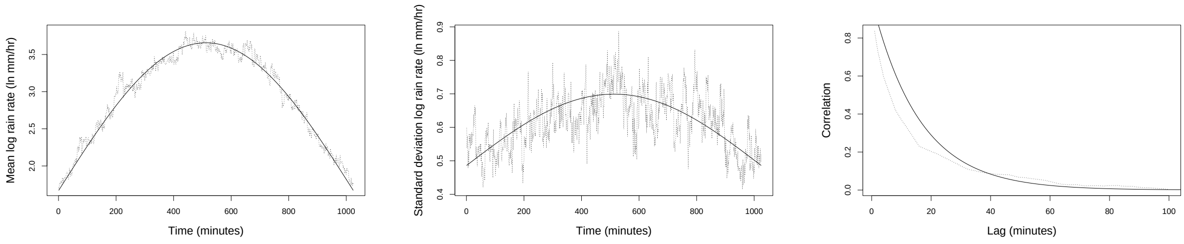
$$\gamma_{l_{i,j}} = \exp(-\beta|i-j|)\sigma_{l_i}\sigma_{l_j}. \tag{2}$$

Here β is an empirically chosen constant (taken to be 0.062 min^{-1} in their paper) and σ_{l_i} denotes the standard deviation of log rain rate at time i.

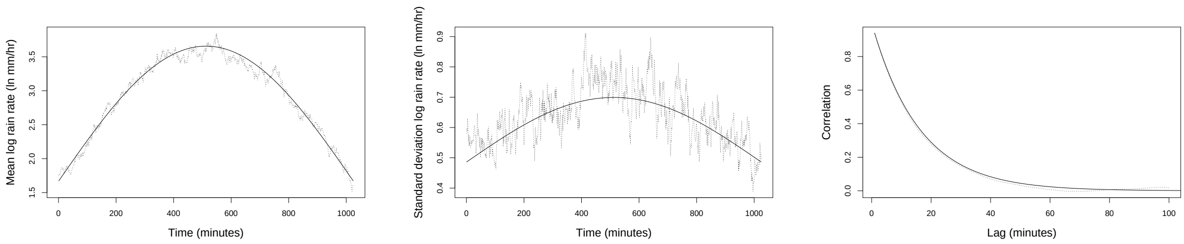
To objectively compare the methods of §2 and 3 we will simulate vectors, $\boldsymbol{L}$, of length $N = 2^{10}$. The mean and standard deviation of log rain rate will vary between values implied in

Burgueño *et al.* The vectors will be simulated to approximate the covariance structure of Eqn. 2 (to speed computation values of $\exp(-\beta|i-j|) < 10^{-6}$ will be regarded as zero).

The results of carrying out fifty simulations using the two methods are displayed in Fig. 1. The dotted lines represent calculations over the fifty simulations. The solid lines are the corresponding theoretical values that were used as inputs to the simulations. Figs. 1(a), (c) and (e) relate to wavelet simulations. Figs. 1(b), (d) and (f) correspond to Cholesky simulations.



(a) (c) (e)



(b) (d) (f)

Figure 1: Comparing wavelet and Cholesky simulation methods: (a) (b) mean of log rain rate; (c) (d) standard deviation of log rain rate; (e) (f) autocorrelation of log rain rate

Comparing Figs. 1(a) to (d), we note that both methods show similar variation in the mean and standard deviation of log rain rate about their theoretical values. Studying Figs. 1(e) and (f) we observe that the Cholesky simulation method more closely reproduces the required autocorrelation profile.

The Cholesky method is exact, whereas the wavelet approach is approximate. The advantage of the wavelet method is that it has a lower computational burden, and can be used to simulate long sequences of random observations. Table. 1 compares the time taken to simulate a vector $\boldsymbol{L}$ of length 2^J using the two methods. Simulations were run using the R environment for statistical computing (R Development Core Team, 2004) with the wavethresh library and LAPACK routines. No attempt was made to optimise the code. Notice that the Cholesky method fails for $J \geq 14$ and is around 3.5 times slower than the wavelet method for $J = 13$.

Table 1: Comparison of time taken (in seconds) to run wavelet and Cholesky simulations on a Pentium IV 2.4 GHz PC with 1 GB of RAM

J	7	8	9	10	11	12	13	14	15	16
wavelet	0.5	2	6	17	44	107	253	581	1376	3033
Cholesky	0.5	1.5	4	10	27	98	911	Failed	Failed	Failed

5 Further work

The wavelet approach presented in §3 used the Haar wavelet transform. A variety of alternative wavelet filters are available (for example, Daubechies, 1988). It would be interesting to see how changing the filter affects the performance of the simulation routine.

To evaluate the wavelet method, we simulated non-stationary Gaussian observations. However, the non-stationarity arose from a non-constant variance rather than a time-varying autocorrelation structure. It would be informative to run simulations in which the autocovariance structure evolved over time.

The simulation of rain rates discussed in §4 relied on the autocorrelation profile of Burgueño *et al.* It would be useful to study this model further, and investigate how it might be extended.

The model used in §4 allowed the mean and standard deviation of log rain rate to vary with time. Meteorological instruments with a coarse temporal resolution could be used to measure the mean and standard deviation with time. These could be used as inputs to a wavelet simulation in which the coarse temporal data was stochastically downscaled to have a specified covariance structure.

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