The Dimple in Gneiting’s Spatial-temporal Covariance Model

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Abstract

Gneiting (2002) proposed a nonseparable covariance model for spatial-temporal data. In present paper we show that in certain circumstances his model possesses a counterintuitive “dimple”, which detracts from its modelling appeal.

Keywords: Spatial-temporal covariance function, Nonseparable, Asymmetry, Generalized Gamma Convolution, Laplace transform.

1 Introduction

Consider a stationary spatial-temporal process \( \{Z(s, t), s \in \mathbb{R}^d, t \in \mathbb{R}\} \), where \( s \) represents a site in \( d \)-dimensional space and \( t \) represents time. Denote the covariance function by

\[
C(h, u) = \text{Cov}(Z(s, t), Z(s + h, t + u)),
\]

where \( h \in \mathbb{R}^d \) and \( u \in \mathbb{R} \) are the space and time lags, respectively. It is nontrivial to develop tractable nonseparable models in this setting. One class of spatially isotropic
models proposed by Gneiting (2002) and Gneiting et al. (2007), and building on Cressie and Huang (1999), takes the form

\[ C(h, u) = \frac{\sigma^2}{\psi(u^2)^{d/2}} \varphi \left( \frac{|h|^2}{\psi(u^2)} \right), \quad (h, u) \in \mathbb{R}^d \times \mathbb{R}. \]  

(1.1)

Here \( \sigma^2 > 0 \) is a scale parameter, and it is assumed that

(a) \( \varphi(z) \) is a completely monotone function of \( z \in (0, \infty) \) with \( \lim_{z \to 0} \varphi(z) = 1 \) and \( \lim_{z \to \infty} \varphi(z) = 0 \).

(b) \( \psi(w) \) is a positive function of \( w \in (0, \infty) \) with a completely monotone derivative such that \( \lim_{w \to 0} \psi(w) = 1 \) and \( \lim_{w \to \infty} \psi(w) = \infty \).

Recall that a completely monotone function \( \varphi(z) \) has the property that its derivatives alternate in sign, \((-1)^n \varphi^{(n)}(z) > 0\) for \( z \in (0, \infty) \) and integer \( n \geq 0 \). The limiting properties in (a) and (b) are not usually stated explicitly but have been included here to ensure that \( C(h, u) \to 0 \) as \( |h| \to \infty \) and/or \( |u| \to \infty \).

Some examples of choices for \( \varphi(\cdot) \) and \( \psi(\cdot) \) are given in Tables 1 and 2, taken largely from Gneiting (2002). In particular, choice 4 in Table 1 with \( \nu = -1/2 \), for which the Bessel function simplifies to \( I_{-1/2}(x) = (2\pi x)^{-1/2}(e^x + e^{-x}) \), corresponds to one of his choices. Here \( I_\nu \) and \( K_\nu \) denote modified Bessel functions of order \( \nu \) (see e.g., Abramowitz and Stegun, 1972).

Unfortunately for many choices of \( \varphi(\cdot) \), Gneiting’s model possesses a counterintuitive “dimple” in the time lag \( u \). Intuitively, the dimple property can be described as follows. Let \( s \) denote “here” and \( t \) denote “now”, and consider the correlation of \( Z(s, t) \) with the value of the process at a nearby site “there” at two possible times, “now” and “yesterday”. If a dimple is present, then the correlation with “yesterday”, or equivalently with “tomorrow”, can be higher than the correlation with “now”. These concepts are made

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**Table 1:** Some completely monotone functions.

<table>
<thead>
<tr>
<th>Choice</th>
<th>Function</th>
<th>Parameter</th>
<th>( Q(\infty) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \varphi(z) = \exp(-z) )</td>
<td>–</td>
<td>( \infty )</td>
</tr>
<tr>
<td>2</td>
<td>( \varphi(z) = (1 + z)^{-\nu} )</td>
<td>( \nu &gt; 0 )</td>
<td>( \nu )</td>
</tr>
<tr>
<td>3</td>
<td>( \varphi(z) = [2^{\nu-1} \Gamma(\nu)]^{-1/2} z^{\nu/2} K_\nu(z^{1/2}) )</td>
<td>( \nu &gt; 0 )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>4</td>
<td>( \varphi(z) = [2^{\nu} \Gamma(\nu+1)]^{-1/2} z^{\nu/2} I_\nu(z^{1/2}) )</td>
<td>( \nu &gt; -1 )</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

**Table 2:** Some functions with a completely monotone derivative.

<table>
<thead>
<tr>
<th>Choice</th>
<th>Function</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \psi(w) = (w + 1)^\beta )</td>
<td>( 0 \leq \beta \leq 1 )</td>
</tr>
<tr>
<td>2</td>
<td>( \psi(w) = \log(w + b)/\log(b) )</td>
<td>( b &gt; 1 )</td>
</tr>
</tbody>
</table>
more precise in Section 2. The dimple property contradicts a natural monotonicity requirement that might be desired of a covariance function in many applications; namely, the covariance should decrease in $|h|$ for each $u$ and should decrease in $|u|$ for each $h$. Whether the dimple property is severe enough to limit the usefulness of Gneiting’s model in practice needs further study.

Using the first choices in Tables 1 and 2 provides the example

$$C(h, u) = \frac{1}{(u^2 + 1)^{d/2}} \exp \left(-\frac{|h|^2}{u^2 + 1}\right), \quad (h, u) \in \mathbb{R}^d \times \mathbb{R}. \quad (1.2)$$

Figure 1(a) gives a perspective plot of (1.2) in dimension $d = 1$. Figure 1(b) gives some cross-sectional plots of $C(h, u)$ versus $u$, for various values of $h$. As can be seen in this figure, the covariance function $C(h, u)$ is a unimodal function in $u$ for small spatial lags $|h|$, while it is bimodal for large spatial lags. This model was also proposed in Cressie and Huang (1999).

This paper is organized as follows: Section 2 proves the existence of the dimple in Gneiting’s model under certain assumptions. This model is fully symmetric. An extension to an asymmetric version and links to related models of Ma (2003) and Stein (2005) are given in Section 3.

## 2 The dimple

Consider a spatially isotropic fully symmetric covariance function $C(h, u)$; that is, it depends on $h$ and $u$ only through $|h|$ and $|u|$. Say that $C$ has a “dimple” in the time lag $u$ if there exists a real value $z^* > 0$ such that the following properties hold.
(a) For fixed $|h|^2 < z^*$, $C(h, u)$ is decreasing in $u \geq 0$.

(b) For fixed $|h|^2 > z^*$, $C(h, u)$ is increasing for $u \in (0, u^*)$ for some $u^* = u^*(|h|^2) > 0$ depending on $|h|^2$, and decreasing for $u \in (u^*, \infty)$.

Clearly if a covariance function possesses a dimple, then it cannot have the natural monotonicity property described in the last section. The following theorem investigates the circumstances under which Gneiting’s model will possess a dimple.

This theorem will make use of the function $Q(z) = -z\varphi'(z)/\varphi(z)$, $z > 0$, where $\varphi(z) = \int_0^\infty e^{-zs}F(ds)$ is the Laplace Transform of a probability measure $F(ds)$ on $(0, \infty)$. One key property of $Q(\cdot)$ is $Q(0) = \lim_{z \to 0} Q(z) = 0$. To verify this property, recall that $xe^{-x} \leq 1$ for all $x > 0$, so that for any $c > 0$,

$$-z\varphi'(z) = \int_0^\infty zse^{-zs}F(ds) \leq c \int_0^{c/z} F(ds) + \int_{c/z}^\infty F(ds). \quad (2.1)$$

The first and second terms in right hand side of (2.1) tend to $c$ and zero, respectively as $z \to 0$. Therefore $0 \leq -\lim_{z \to 0} z\varphi'(z) \leq c$. Since $c$ can be arbitrarily small, it follows that $-\lim_{z \to 0} z\varphi'(z) = 0$. Further, since $\varphi(0) = 1$, the desired property for $Q(z)$ holds.

**Theorem 1** Consider Gneiting’s model (1.1) and let $Q(z) = -z\varphi'(z)/\varphi(z)$, $z > 0$. If $Q$ is increasing in $z > 0$ and if $d/2 < \lim_{t \to \infty} Q(z) \leq \infty$, then Gneiting’s covariance function (1.1) has a dimple in $u$.

**Proof:**

Let $f(h, u) = \log C(h, u) = \log \sigma^2 - \frac{d}{2} \log \psi(u^2) + \log \varphi(|h|^2/\psi(u^2))$, with partial derivative

$$\frac{\partial f(h, u)}{\partial u} = -2u \left\{ \frac{d \psi(u^2)}{2 \psi(u^2)} + \frac{|h|^2 \psi'(u^2)}{\psi^2(u^2)} \frac{\varphi'(z)}{\varphi(z)} \right\} = -2u \frac{\psi'(u^2)}{\psi(u^2)} \left\{ Q(z) - \frac{d}{2} \right\}, \quad \text{say,} \quad (2.2)$$

where $z = |h|^2/\psi(u^2)$. Thus, as $u$ increases from 0 to $\infty$ for fixed $|h|^2$, $\psi(u^2)$ increases from 1 to $\infty$, $|h|^2/\psi(u^2)$ decreases from $|h|^2$ to 0, and $Q(z)$ decreases from $Q(|h|^2)$ to 0. Here we have used the properties that $\psi$ and $Q$ are increasing continuous functions. By assumption $Q(\infty) > d/2$, so it is possible to define unambiguously $z^* = Q^{-1}(d/2)$. Also the assumptions on $\psi$ in Gneiting’s model ensure that the factor $2u\psi'(u^2)/\psi(u^2)$ is always positive for $u > 0$ in (2.2). Hence

(a) if $|h|^2 < z^*$, then $\partial f(h, u)/\partial u$ is negative for all $u > 0$, and

(b) if $|h|^2 > z^*$, then $\partial f(h, u)/\partial u$ is positive for $u \in (0, u^*)$ and negative for $u \in (u^*, \infty)$, where $u^* = \{\psi^{-1}(|h|^2/z^*)\}^{1/2}$.
These are precisely the conditions for a dimple to exist.

The next question is when the function \( Q(\cdot) \) will have the required monotonicity properties. A sufficient condition is given by the following theorem. This result uses the Thorin class of infinitely divisible distributions, also known as the class of generalized gamma convolutions, which has been systematically studied by Bondesson (1992), Bondesson et al. (2008) and Steutel and van Harn (2004). Denote the class of Laplace transforms of Thorin distributions by \( T \). These Laplace transforms take the form

\[
\varphi(z) = \exp \left\{ -az + \int_0^\infty \log\left( \frac{s}{s+z} \right) U(ds) \right\}, \quad z \geq 0
\]  

(2.3)

where \( a \geq 0 \) and \( U(ds) \) is a nonnegative measure on \((0, \infty)\) satisfying

\[
\int_0^1 |\log(s)| U(ds) < \infty \quad \text{and} \quad \int_1^\infty \frac{1}{s} U(ds) < \infty.
\]  

(2.4)

For simplicity exclude the degenerate case \( a = 0 \) and \( U(ds) = 0 \). The simplest example occurs when \( U(ds) = \nu \delta_b(ds) \) is atomic measure at \( b > 0 \) with magnitude \( \nu > 0 \), for which \( \varphi(z) = (1 + z/b)^{-\nu} \) is Laplace transform of the gamma distribution \( \Gamma(\nu, b) \).

**Theorem 2** If \( \varphi(z) \in T \), where \( a > 0 \) and/or \( \int_0^\infty U(ds) > d/2 \) in (2.3), then \( Q(z) = -z\varphi'(z)/\varphi(z) \) is increasing in \( z > 0 \) with \( d/2 < \lim_{t \to \infty} Q(z) \leq \infty \).

**Proof:**

Differentiating \( \log \varphi(z) \) yields \( \varphi'(z)/\varphi(z) = -a - \int_0^\infty 1/(s+z)U(ds) \), and multiplying by \( -z \) yields

\[
Q(z) = az + \int_0^\infty \frac{z}{s+z} U(ds).
\]

Since \( z \) and the integrand \( z/(s+z) \) are increasing functions of \( z \), so is \( Q(z) \). Further, if \( a > 0 \), then \( \lim_{z \to \infty} Q(z) = \infty \), and if \( a = 0 \), then \( \lim_{z \to \infty} Q(z) = \int_0^\infty U(ds) \in (d/2, \infty] \).

The Thorin class includes all the distributions in Table 1. This conclusion is straightforward to check for choices 1 and 2. For choices 3 and 4, see, e.g., Kent (1987, 1980, 1982). For choices 1, 2 and 4, it can be shown that \( Q(\infty) = \infty \). Choice 2 is the Laplace transform of the gamma distribution \( \Gamma(\nu, 1) \), for which \( Q(\infty) = \nu \).

The Laplace transforms in the Thorin class are closed under several natural operations which allow the construction of a wider set of choices for \( \varphi \) from a simple initial list such as Table 1.

1. **Scaling:** If \( \varphi(z) \in T \), then \( \varphi_b(z) = \varphi(bz) \in T \), \( b > 0 \), with \( Q_b(\infty) = Q(\infty) \).

2. **Shifting:** If \( \varphi(z) \in T \), then \( \varphi_a(z) = \varphi(z + a) \in T \), \( a > 0 \), with \( Q_a(\infty) = Q(\infty) \).
3 Extensions and links to other work

The simplest extension of (1.1) is to exchange the roles of space and time, yielding a covariance function of the form

$$C(h, u) = \frac{\sigma^2}{\psi(|h|^2)^{1/2}} \varphi \left( \frac{u^2}{\psi(|h|^2)} \right), \quad (h, u) \in \mathbb{R}^d \times \mathbb{R}. \quad (3.1)$$

In this case, provided \( \lim_{t \to -\infty} Q(t) > 1/2 \), there exists a dimple in the spatial lag \( h \). That is for fixed \( |u|^2 < z^* = Q^{-1}(1/2) \), \( C(h, u) \) is decreasing in \( |h| \), whereas for \( |u|^2 > z^* \), \( C(h, u) \) is increasing and then decreasing in \( |h| \). Gneiting (2002) has also proposed versions of (1.1) and (3.1) in which time becomes multi-dimensional. These models can also have dimples.

Gneiting’s models (1.1) and (3.1) are fully symmetric, i.e. \( C(h, u) = C(-h, u) = C(h, -u) = C(-h, -u) \), \( (h, u) \in \mathbb{R}^d \times \mathbb{R} \). Starting from a fully symmetric model, one of the asymmetric extensions proposed by Ma (2003) takes the form

$$C_M(h, u; \eta) = C(h, u + \eta' h), \quad (h, u) \in \mathbb{R}^d \times \mathbb{R}, \quad (3.2)$$

where \( \eta \in \mathbb{R}^d \) is a drift parameter. In particular, starting from (3.1) Ma’s model (3.2) inherits the dimple properties of \( C(\cdot, \cdot) \) in any direction specified by a unit vector \( h_0 \) satisfying \( \eta' h_0 = 0 \). That is, for fixed \( |u|^2 < z^* \), \( C_M(\alpha h_0, u; \eta) \) is decreasing in \( \alpha > 0 \), whereas for \( |u|^2 > z^* \), \( C_M(\alpha h_0, u; \eta) \) is increasing and then decreasing in \( \alpha > 0 \).

Stein (2005) proposed a spatial-temporal covariance function motivated by a simple structure in the frequency domain for time. In its fully symmetric form, his model takes the
form
\[ C(h, u) = \int_{-\infty}^{\infty} k(\tau)C_0(|h|\gamma(\tau))e^{iu\tau}d\tau, \quad (h, u) \in \mathbb{R}^d \times \mathbb{R}, \tag{3.3} \]

where \( C_0(\cdot) \) is a purely spatial isotropic covariance function, and \( k(\tau) \) and \( \gamma(\tau) \) are non-negative even functions on \( \mathbb{R} \). Further, the simplest asymmetric version of Stein’s model takes the form (3.2). The reason for introducing Stein’s model here is to point out an overlap with Gneiting’s class of models. In particular, if \( k(\tau) = \exp(-\tau^2/4) \), \( \gamma(\tau) = |\tau| \) and \( C_0(|h|) = \exp(-|h|^2/4) \), then (3.3) reduces to (3.1), with \( \sigma^2 = 2\sqrt{\pi} \) and using the first choices in Tables 1 and 2. Thus in at least some cases, Stein’s model possesses a dimple in \( h \). Note that (3.1) in this case is the same as (1.2), but with the roles of time and space interchanged.

The dimple in Gneiting’s model does not seem to have been explicitly recognized before in the literature, though there are visual hints in some of the published figures, e.g. Cressie and Huang (1999, Fig. 2), and Gneiting (2002, Fig. 1), which are based on the model (1.2). More recently Gneiting’s covariance function has used as building block in more complicated models; see e.g. Kolovos et al. (2004) and Porcu et al. (2008).

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