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On exponential bounds and convergence rate for Reciprocal Gamma diffusion processes

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Abstract

For a parametric class of "reciprocal gamma diffusion processes", certain exponential bounds for β-mixing and rate of convergence to stationary distribution are established.

1 Introduction

In our recent preprint [1] we considered a parametric class of 1D diffusion processes called Student diffusions. The reason for studying this class was a demand from stochastic financial applications; this study was initiated in [2]. The class of reciprocal gamma diffusions is another particular class which is interesting from the same point of view, [5]. Namely, in applied stochastic finance theory there is a need to have description of parametric classes of processes with certain special properties, in particular, with heavy or light tails, "short" or "long" memory, and exponential or some other rate of mixing and convergence towards stationary distributions. For applications it is also highly desirable that the latter are known exactly. Hence, in this paper we investigate certain mixing properties and convergence rate to equilibrium distribution for this new particular class suggested in [5]. It turns out that the processes from this class possess polynomial tails and exponential mixing.

The organisation of the paper is as follows. Section 2 relates to the definition of the gamma and reciprocal gamma density distribution. The section

3 is devoted to the presentation of the parametric class of ergodic stationary reciprocal gamma diffusions with marginal inverse gamma distribution. The section 4 presents main results, the section 5 contains auxiliary lemmae. Main results are proved in the section 6.

2 Gamma and Reciprocal gamma distribution

If random variable Y has gamma distribution with probability density function of the form,

$$\mathfrak{g}(x) = \begin{cases} \dfrac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x}, & x > 0, \\ 0 & x \leq 0. \end{cases} \tag{1}$$

where $\alpha > 0$ is scale parameter and $\beta > 1$ is shape parameter. Then random variable $X = \dfrac{1}{Y}$ has a reciprocal gamma distribution with probability density function,

$$\mathfrak{rg}(x) = \begin{cases} \dfrac{\alpha^\beta}{\Gamma(\beta)} x^{-\beta-1} e^{-\alpha/x}, & x > 0, \\ 0 & x \leq 0. \end{cases} \tag{2}$$

with the same parameters α and β. These distribution are denoted as follows, $Y \sim \mathfrak{G}(\alpha, \beta)$, and $X \sim \mathfrak{RG}(\alpha, \beta)$. The moments of k-th order of reciprocal gamma random variable is given by the following expression,

$$E[X^K] = \frac{\alpha^k}{\prod_{i=1}^{k}(\beta - i)} = \alpha^K \frac{\Gamma(\beta - k)}{\Gamma(\beta)}, \quad \beta > k. \tag{3}$$

In particular, expectation and variance of random variable $X \sim \mathfrak{RG}(\alpha, \beta)$ are:

$$E[X] = \frac{\alpha}{\beta - 1}, \qquad Var(X) = \frac{\alpha^2}{(\beta-1)^2(\beta-2)}. \tag{4}$$

Important property of reciprocal gamma distribution is scaling property, i.e., if $\mathfrak{G}(\alpha, \beta)$ and $\mathfrak{RG}(\alpha, \beta)$ are gamma and reciprocal gamma random variables, respectively, with the same parameters α and β, then it is known that

$$\mathfrak{RG} = \frac{1}{\mathfrak{G}(\alpha, \beta)} = \frac{\alpha}{\mathfrak{G}(\alpha, \beta)} = \alpha\, \mathfrak{RG}(1, \beta). \tag{5}$$

For references see [5].

3 Reciprocal gamma diffusion

Consider a stochastic differential diffusion equation

$$dX_t = -\theta(X_t - \frac{\alpha}{\beta - 1})\,dt + \sqrt{\frac{2\theta}{\beta - 1}X_t^2}\,dW_t, \quad t \geq 0, \tag{6}$$

with some initial data X_0. Here $\theta > 0$, $\alpha > 0$, $\beta > 1$, and W_t is a standard Brownian motion. Equivalently,

$$X_t = X_0 - \theta\int_0^t (X_s - \frac{\alpha}{\beta - 1})\,ds + \int_0^t \sqrt{\frac{2\theta}{\beta - 1}X_s^2}\,dW_s, \quad t \geq 0,$$

where X_0 is a random variable independent of Brownian motion W_t; in particular a non-random value is, of course, allowed. Due to global Lipschitz condition on both coefficients, the stochastic differential equation above has a unique strong solution which is a strong Markov process. Moreover, solution $X = \{X_t, t \geq 0\}$ is ergodic with invariant reciprocal gamma probability density function (2), see [5]. Notice that this invariant density, of course, does not depend on $\theta > 0$.

4 Main results

Let $F^X_{\leq s} = \sigma(X_u,\, u \leq s)$. Notations E_x and P_x (E_{st} and P_{st}) are used for the processes with initial data x (stationary distribution as initial data). We recollect the definition of two mixing coefficients, $\alpha(t)$ and $\beta(t)$:
Strong mixing coefficient or Rosenblatt's coefficient

$$\alpha^x(t) = \sup_{s\geq 0} \sup_{A\in F^X_{\leq s}, B\in F^X_{\geq t+s}} |P_x(AB) - P_x(A)P(B)|;$$

Complete regularity condition or Kolmogorov's coefficient

$$\beta^x(t) = \sup_{s\geq 0} E_x \mathrm{var}_{B\in F_{\geq t+s}X}(P(B|F_s) - P(B)).$$

Denote $\alpha^{st}(t)$ and respectively $\beta^{st}(t)$ the versions of both coefficients for stationary distributed initial data X_0. Denote by $\mu^x(t)$ the distribution of X_t with initial data x, and by μ^{st} the invariant measure for X.

Theorem 1 *For any $\beta > 1$ and $m < \beta$, there exist constants $C, c > 0$ such that*

$$\beta^x(t) \le K(x)e^{-ct}, \quad K(x) = C\left(\frac{1}{x} + |x|^m\right), \quad t \ge 0, \tag{7}$$

and for stationary regime,

$$\beta^{st}(t) \le Ce^{-ct}, \quad t \ge 0. \tag{8}$$

Theorem 2 *Under the assumptions of theorem 1,*

$$var(\mu^x(t) - \mu^{st}) \le K(x)e^{-ct}, \quad t \ge 0. \tag{9}$$

Notice that the bounds obtained above allow to apply Central Limit Theorem for functionals of the process, see, e.g., [4]. We do not pursue this goal here, cf., e.g., [1, Corollary 1].

5 Preliminary results

For any $R > 0$ denote $\tau_1 := \inf(t \ge 0, X_t \le R)$ and $\tau_2 := \inf(t \ge 0, X_t > R^{-1})$.

Lemma 1 *For any $\alpha, \theta > 0$, $\beta > 1$ and any $m < \beta$, there exists a constant $\alpha_1 > 0$ such that for any R large enough and $x > R$,*

$$E_x e^{\alpha_1 \tau_1} \le |x|^m.$$

Proof. Consider a Lyapunov function $f(t, x) = \exp(\alpha_1 t) x^m$. Then, the stochastic differentiation of this function along the trajectory X reads,

$$\begin{aligned}
df(X_t, t) &= \alpha_1 e^{\alpha_1 t} X_t^m dt + m X_t^{m-1} e^{\alpha_1 t}(dX_t) + \frac{1}{2} m(m-1) X_t^{m-2} e^{\alpha_1 t}(dX_t)^2 \\
&= \alpha_1 e^{\alpha_1 t} X_t^m dt + \frac{1}{2} e^{\alpha_1 t} \frac{2\theta}{\beta - 1} m(m-1) X_t^m dt - \theta m X_t^{m-1} e^{\alpha_1 t} X_t dt \\
&+ \frac{\alpha\theta m}{\beta - 1} X_t^{m-1} e^{\alpha_1 t} dt + \sqrt{\frac{2\theta}{\beta - 1} X_t^2} dW_t \\
&= e^{\alpha_1 t} X_t^m \left[\alpha_1 + \frac{\theta}{\beta - 1} m(m-1) + \frac{\alpha\theta m}{\beta - 1} X_t^{-1} - \theta m\right] dt \\
&+ \sqrt{\frac{2\theta}{\beta - 1} X_t^2} dW_t.
\end{aligned}$$

Due to the assumption,

$$m\theta - \frac{m(m-1)\theta}{\beta-1} > 0.$$

Let

$$0 < \alpha_1 < m\theta - \frac{m(m-1)\theta}{\beta-1}.$$

Then, for R large enough,

$$\left[\alpha_1 + \frac{\theta}{\beta-1} m\,(m-1) + \frac{\alpha\theta m}{\beta-1} X_t^{-1} - \theta m\right] < c \leq 0.$$

Hence, taking expectations, we obtain,

$$E_x f(t\wedge\tau_1, X_{t\wedge\tau_1}) - f(0,x) \leq c E_x \int_0^{t\wedge\tau_1} e^{\alpha_1 s} X_s^m ds \leq 0.$$

The Lemma 1 is proved.

Notice that to avoid any question about expectations of the stochastic integral above, as usual, we can apply a standard localization procedure.

Lemma 2 *For any $\alpha, \theta > 0$, $\beta > 1$ and any $\lambda > 0$, there exists a constant $\alpha_2 > 0$, such that for any R large enough any and $x < \frac{1}{R}$,*

$$E_x e^{\alpha_2 \tau_2} \leq \frac{1}{x^\lambda}.$$

Proof. Here the initial condition is very near to the origin. Notice that our diffusion can never reach zero if it starts from positive values. Hence, it is convenient to transform the space and consider another process,

$$Y_t := \lambda \ln X_t \equiv \ln X_t^\lambda.$$

Its stochastic differential reads,

$$\begin{aligned}
\frac{1}{\lambda} dY_t = d(\ln X_t) &= \frac{dX_t}{X_t} - \frac{1}{2}\frac{dX_t^2}{X_t^2}, \\
&= \left(-(\theta + \frac{\theta}{\beta-1}) + \frac{\alpha\theta}{\beta-1}\exp(-Y_t)\right) dt + \sqrt{\frac{2\theta}{\beta-1}} dW_t, \\
&= \left[-\frac{\theta\beta}{\beta-1} + \frac{\alpha\theta}{\beta-1}\exp(-Y_t)\right] dt + \sqrt{\frac{2\theta}{\beta-1}} dW_t.
\end{aligned}$$

Let us consider a Lyapunov function $f(t, y) = \exp(-y + \alpha_2 t)$. We have,

$$
\begin{aligned}
df(t, Y_t) &= \alpha_2 f(t, Y_t)dt \, - \, f(t, Y_t)dY_t \, + \, \frac{1}{2} f(t, Y_t)(dY_t)^2 \\
&= \alpha_2 f(t, Y_t)dt \, - f(t, Y_t)\lambda \left[-\frac{\theta\beta}{\beta - 1} + \frac{\alpha\theta}{\beta - 1} \exp(-Y_t) \right] dt \\
&- f(t, Y_t)\lambda \left[\frac{2\theta}{\beta - 1} \right]^{1/2} dW_t + \frac{1}{2} f(t, Y_t) \, \frac{2\lambda^2\theta}{\beta - 1} dt \\
&= f(t, Y_t) \left[\alpha_2 + \frac{\lambda^2\theta}{\beta - 1} + \frac{\lambda\theta\beta}{\beta - 1} - \frac{\lambda\alpha\theta}{\beta - 1} \exp(-Y_t) \right] dt \\
&- \lambda f(t, Y_t) \sqrt{\frac{2\theta}{\beta - 1}} dW_t.
\end{aligned}
$$

For R large enough and $t < \tau$, we have,

$$
\left[\alpha_2 + \frac{\theta(\lambda\beta + \lambda^2)}{\beta - 1} - \frac{\lambda\alpha\theta}{\beta - 1} \exp(-Y_t) \right] \leq \kappa \leq 0.
$$

Hence, taking expectation, we get,

$$
Ef(t \wedge \tau_2, Y_{t\wedge\tau_2}) - f(Y_0, 0) \leq 0.
$$

So, using Fatou's Lemma as $t \to \infty$, we get,

$$
E_x \exp(\alpha_2 \tau_2) \leq \frac{1}{x^\lambda}.
$$

The Lemma 2 is proved.

Lemma 3 *For any $\alpha, \theta > 0$, any $\lambda > 0$ and any $m < \beta$,*

$$
E_{st} X^m + E_{st} \frac{1}{X^\lambda} < \infty.
$$

Proof. This is straightforward due to the convergence,

$$
\int_0^\infty x^m x^{-\beta-1} e^{-\frac{\alpha}{x}} dx + \int_0^\infty \frac{1}{x^\lambda} x^{-\beta-1} e^{-\frac{\alpha}{x}} dx < \infty.
$$

In what follows, we shall consider recurrence properties of a couple of independent Markov processes satisfying the same equation (). Consider direct product of two identical probability spaces where two independent copies of our Markov process are defined, say $(Z_t,\, t\geq 0)$ and $(Z'_t,\, t\geq 0)$, with the initial values $Z_0=z$, $Z'_0=z'$, respectively. Define a new function $\phi_R(z)$, as follows:

$$\phi_R(z)=\begin{cases} z^m, & \text{if } z>R,\\ \text{any} f\in C^2, & \text{if } \dfrac{1}{R}<z<R,\\ z^{-\lambda}, & \text{if } z<\dfrac{1}{R}. \end{cases}$$

Define a stopping time γ,

$$\gamma \;:=\; \inf(t\geq 0: Z_t\in[\frac{1}{R_1},R_1]\,\&\,Z'_t\in[\frac{1}{R_1},R_1]).$$

Lemma 4 *For any $\alpha,\theta>0$, any $\lambda>0$ and any $m<\beta$, there exists $\alpha_3>0$ such that for $R_1\geq R$ large enough and for any $z,z'>0$ as initial starting points for Z and Z',*

$$E_{z,z'}\exp(\alpha_3\gamma)\leq \phi_{R_1}(z)+\phi_{R_1}(z').$$

Proof. Consider a Lyapunov function with $\alpha_3>0$,

$$f(t,\phi(z),\phi(z'))\;=\;\exp(\alpha_3\, t)(\phi(z)+\phi(z')).$$

Due to Itô's formula, we have,

$$df(t,\phi(Z_t),\phi(Z'_t))$$

$$=\exp{(\alpha_3 t)}\phi(Z_t)[\alpha_3+\frac{L\phi(Z_t)}{\phi(Z_t)}]dt+\exp{(\alpha_3 t)}\phi(Z'_t)[\alpha_3+\frac{L\phi(Z'_t)}{\phi(Z'_t)}]dt$$

$$+\exp{(\alpha_3 t)}\sqrt{\frac{2\theta}{\beta-1}Z_t^2}\phi'(Z_t)dW_t+\exp{(\alpha_3 t)}\sqrt{\frac{2\theta}{\beta-1}Z_t^2}\phi'(Z'_t)dW'_t.$$

Notice that

$$\sup_{\frac{1}{R}\leq z\leq R}\frac{L\phi_R(z)}{\phi_R(z)}\;\vee\;0:=C^*<\infty,$$

and also

$$\sup_{z\notin[\frac{1}{R},R]} \frac{L\phi_R(z)}{\phi_R(z)} \le 0.$$

Define $S_+ := \{z : L\phi_R(z) > 0\}$. Then in the integration

$$\int_0^{t\wedge\gamma} (\ldots)\, ds,$$

for every s, there are two main cases to be considered:

I: At time s, one process is either $(\ge R_1)$ or $(\le \dfrac{1}{R_1})$, while the other process is in $(R_1 \setminus S_+)$. Then contribution from both processes in the ds term are negative, and one of them provides a large negative value, $\le -K_{R_1}$ such that $-K_{R_1} + \alpha_3 < 0$ for any chosen $\alpha_3 > 0$, if R_1 is large enough.

II: At time s, one process is either $(\ge R_1)$, or $(\le \dfrac{1}{R_1})$, but the other process is in the domain (S_+). Then again the first process provide a large negative contribution $\le -K_{R_1}$, so that $-K_{R_1} + C^* + \alpha_3 < 0$ for any chosen $\alpha_3 > 0$, if R_1 is large enough.

In both cases the term with ds is negative. Hence, we can finish the proof similarly to the proofs of Lemmae 1 and 2. The Lemma 4 is proved.

Lemma 5 *For any $\alpha, \theta > 0$ and for every $m < \beta$, there exist $\lambda > 0$ and $C > 0$ such that for every $x > 0$,*

$$\sup_{t\ge 0} E_x \left(X_t^m + \frac{1}{X_t^\lambda} \right) \le C \left(1 + x^m + \frac{1}{x^\lambda} \right). \tag{10}$$

Notice that unit in the right hand side here has been added just for simplicity: clearly it may be dropped. This is the only place were $\lambda > 0$ is to be chosen. Perhaps, by some improvement of the method one can show (10) with any $\lambda > 0$. Nevertheless, for our purposes *some* lambda is sufficient.

Proof of

$$\sup_{t\ge 0} E_x X_t^m \le C(1 + x^m), \tag{11}$$

follows similarly to the proof of Lemma 2 from [1]. Hence, it suffices to show

$$\sup_{t\geq 0} E_x X_t^{-\lambda} \leq C\left(1 + x^{-\lambda}\right) \qquad (12)$$

for some $\lambda > 0$, for any $x < 1$. Consider transformation $Y_t = \ln X_t$ (without lambda). In the new scale, the equation has a constant diffusion coefficient and the drift, say, $\tilde{b}(y)$, satisfying

$$y\tilde{b}(y) \leq -c < 0, \quad |y| \geq c_1.$$

It was shown in [6] that under such condition, there exists $\lambda > 0$ such that

$$E_y \exp(\lambda |Y_t|) \leq C \exp(\lambda |y|), \quad t \geq 0, \qquad (13)$$

with some $C > 0$. Strictly speaking, only bounded coefficients were considered in [6]. However, our drift here is “better” in negative domain of y values in the sense that it is not just positive, but goes to $+\infty$ as $y \to -\infty$. Hence, e.g., a simple comparison theorem reduces our case to that considered in [6]. Finally, (13) implies (12). The Lemma 5 is proved.

Proof of Theorems 1 and 2

The proofs follow from the Lemmae 4 and 5 similarly to the calculus in the proof of [1, Theorem 1], via Harnack inequality. Notice that, of course, instead of using Harnack, it is possible to apply the intersection idea, similarly to [1, Proof of Theorem 1, step 3]. We leave such details till a journal publication. The Theorems 1 and 2 are proved.

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