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On Bayesian asymptotic information inequality for unbounded parametric set

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Abstract

An asymptotic version of the Borovkov–Sakhanenko's integral information inequality is established for unbounded parametric sets under general Lebesgue's integrability assumptions on the prior. No unbiasedness nor "weak unbiasedness" is required.

1 Introduction

We study Cramér–Rao type integral inequalities due to Borovkov and Sakhanenko [1, 2], which may be also interpreted as inequalities for the Bayesian risk. The first integral inequality of this sort in a rather general form, even though under a bit vague set of assumptions, was established in [9, 10]. The former of these references can be found in [7], however, [9] is just a one page seminar abstract without any proof. But the paper [10] does contain a rigorous proof, quite naturally, based on some version of Cauchy–Bouniakovsky–Schwarz inequality. There is also a useful link http://www-igm.univ-mlv.fr/~herstel/Mps on the publication list

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by M. P. Schützenberger. Strangely, the paper [10] was not noticed by theoretical statisticians: in particular, there is no reference on it even in [7]. To discover the paper [10], the second author ought to ask Maurice Schützenberger's disciples and his daughter which were so kind as to provide the required text. In any case, in about a decade after the publication of [10], the idea of integral version of Cramér–Rao inequality, – at the time frequently called by three names, including Fréchet (see [5]), – was re-discovered in [11], and since that time inequalities of this kind are called van Trees inequalities. The meaning of integral inequalities is that they *do not require unbiasedness* of estimators. Instead, a dramatically weaker condition called "weak unbiasedness" is used, but it has a questionable connection to unbiasedness. For example, one of sufficient conditions for the former is that the prior density vanishes on the boundary of the parametric set; by all means, this is far away from unbiasedness. A technical idea is, of course, integration by parts with respect to the parameter. So far, we did not discuss smoothness. As usual in this area, the most standard assumption on derivatives is C^1 (cf. below the Proposition 1), which, of course, may be slightly weakened, e.g., to existence of derivative in L_2. The next step was made by Borovkov and Sakhanenko [1, 2]. Firstly, they introduced a new functional for the lower bound of integral type CR inequality, more precise than earlier functional by Schützenberger and van Trees in the *asymptotical sense* (see below). Secondly, they showed that there is also an *asymptotic version* of the Bayesian Cramér-Rao inequality which remains valid for even more general class of prior densities, under a *Riemann's integrability* assumption. In the other words, for an asymptotic inequality no smoothness is needed, nor vanishing of the prior on the boundary of the parametric set. By the way, this makes the idea of "weak unbiasedness" disappear completely.

Notice that, of course, any limiting assertion becomes really useful where there is some rate of convergence established. In this sense, Borovkov–Sakhanenko's results required some complementary bounds of remainder terms. Under certain additional smoothness, such bounds have been established in [1], [2], [8]. Nevertheless, even without such estimates of convergence rate, a limiting assertion may be helpful, since it may show, e.g., asymptotical effectiveness of some estimators, while complementary bounds on convergence rate may help, e.g., verify effectiveness of a second order or some others. Another argument is that rate of convergence always requires certain smoothness assumptions, while a theoretical statistician in some cases may be interested whether in the worst situation, – i.e., without any smoothness,

– asymptotical effectiveness still holds true. In such situation just a limiting result like the Theorems 1 or 2 below may be of some help.

In the present paper we show that in the case $\Theta = R^1$, Riemann's integrability condition is not necessary either and may be replaced by more logical Lebesgue's one, thus, extending results from [12] on this unbounded case. Another result is yet another slightly corrected version of Borovkov and Sakhanenko's asymptotic inequality under the *Riemann's integrability (R.i.)* type condition, in addition to filling some minor gaps in the original proof similar to those discussed in [12, Remark 4]. Notice that, of course, more general unbounded parametric sets can be considered in this way, however, we leave it till further studies. In [1, 2], the cases of bounded and unbounded Θ are considered simultaneously, which, in fact, brings some difficulties with Riemann's integrability in the unbounded Θ case. It looks like original proof of Borovkov and Sakhanenko of the asymptotical inequality (see, e.g., [1]) shall require *direct Riemann's integrability (d.R.i.)*, not mentioned in the cited works. We do not pursue this goal here. Our methods are similar to those applied in [12], but the calculus is not a full analogue, due to technical difficulties related to unboundedness. In particular, it is clearly impossible in the case $\Omega = R^1$ to require $\inf q > 0$, the property used in [12] for bounded Ω. We also suggest some new version of weak integrability condition in auxiliary results for smooth functions, suitable for the unbounded case, cf. the Proposition 1 below.

The paper consists of the Introduction (above), the Section 2 with a setting and main results, the Section 3 with auxiliary results, and the Sections 4 and 5 with proofs. The last Section 6 provides simple arguments that Borovkov–Sakhanenko's functional for the lower bound ("J" below) is optimal *in the asymptotical setting*, unlike Schützenberger–van Trees' one. Notice that optimal choice of functional in the pre-limiting inequalities is, of course, more involved (see some examples in [1], about various choices of functionals see [6]); however, optimality in [6] is not discussed. In such a situation, asymptotically optimal Borovkov–Sakhanenko's functional could be a reasonable approach for large samples.

2 Bayesian setting: assumptions and main results

Let us consider a family of probability densities $f(x|\theta)$, $x \in R^d$, with respect to Lebesgue measure, with a parameter $\theta \in \Theta \subset R^1$. In the present paper we tackle the case $\Theta = R^1$; other unbounded cases could be treated similarly. We assume that there is a *prior density* $q(\theta)$, $\theta \in \Theta$, and denote $f(x,\theta) := f(x|\theta)q(\theta)$. Let $\theta^*(X)$ denote any estimator of θ. The quality of the estimator is assessed by the integral or Bayesian functional

$$\int (E_\theta(\theta^*(X) - \theta)^2) q(\theta)\, d\theta \equiv E(\theta^* - \theta)^2,$$

where E_θ means expectation with respect to the density $f(\cdot,\theta)$, integration $\int \ldots d\theta$ is performed over the domain of q, – i.e., in our case over the whole line R^1, – $X = (X_1, \ldots, X_n)$ is a sample of i.i.d. random variables from the distribution $f(\cdot,\theta)$. The notation E is used for the "complete expectation", i.e., with respect both to X and θ. In fact, the estimator depends also on the sample size n, and in the asymptotical sense we may be interested in the effectiveness in the sense of the functional

$$\liminf_{n\to\infty} n \int (E_\theta(\theta^*(X) - \theta)^2) q(\theta)\, d\theta, \qquad (1)$$

which is, of course, oriented on "smooth" case in the sense of smoothness of the densities with respect to x.

We always assume that the Fisher information function is well-defined,

$$I(\theta) = E_\theta \left(\frac{\partial L(X,\theta)}{\partial \theta} \right)^2,$$

where $L(X,\theta)$ is a conditional likelihood function for the sample X given θ. The latter, as well as the existence of the densities $f(x|t)$, is a part of the setting and will not be repeated in the main assumptions below. The problem under consideration is lower bounds for the functional (1).
We assume the following.

(A1)

$$0 < J := \int_{-\infty}^{\infty} \frac{q(t)}{I(t)} dt < \infty, \qquad \text{and} \qquad \int_{-m}^{m} \sqrt{I(t)} dt < \infty, \qquad \forall\, m > 0.$$

(A2) For every $m > 0$ there exists $C_m > 0$ such that

$$C_m^{-1} \le \frac{q(t)}{I(t)} \le C_m, \quad -m < t < m,$$

and

$$\inf_{t\in[-m,m]} I(t) > 0, \qquad \forall\, m > 0.$$

As a consequence of (A2), for every $m > 0$, $\inf_{t\in[-m,m]} q(t) > 0$.

Remark 1. In general, continuity of I is not required in (A2), unlike in [1, 2] where the latter condition is implicitly assumed.

Remark 2. In some papers on the subject, formally a more general situation is considered, without any assumption which prevents the density q from vanishing. However, some auxiliary constructions may become more cumbersome. We wished to avoid this, and assume explicitly that q does not vanish. Clearly, *some* generalisation is possible, but we do not pursue this goal here.

Theorem 1 *Let the assumptions (A1) and (A2) be satisfied. Then,*

$$\liminf_{n\to\infty} nE(\theta^* - \theta)^2 = \liminf_{n\to\infty} n \int_{-\infty}^{\infty} E_t(\theta^* - t)^2 q(t)dt \ge J. \tag{2}$$

Another version of assumption (A2) will be used in the next result. Clearly, under continuity of I, Riemann's integrability of q/I and of q are equivalent; the latter was used in [1, 2].

(A2') Function q/I is *Riemann integrable* on every bounded interval in R^1 and $\inf q = q^m > 0$ for every $m > 0$. In additions, the function I is required to be continuous.

Theorem 2 *Let the assumptions (A1) and (A2') be satisfied. Then, the inequality (7) holds true.*

3 Auxiliary results

Let us state two useful technical results which will be applied in the proof of the main theorems below.

Proposition 1 *Let $h_\epsilon(t)$ be a C^1–smooth function satisfying for any x*

$$\lim_{t\to\pm\infty} t\, h_\epsilon(t)\frac{f(x,t)}{q(t)} \equiv \lim_{t\to\pm\infty} t\, h_\epsilon(t) f(x \mid t) = 0, \tag{3}$$

and let the second part of (A1) be satisfied, – that is $\int_{-m}^{m}\sqrt{I(t)}dt < \infty$, for all $m > 0$, – be satisfied. Then,

$$n\int_{-\infty}^{\infty} E_t(\theta^* - t)^2 q(t)dt \ge \frac{\left(\int_{-\infty}^{\infty} h_\epsilon(t)dt\right)^2}{\int_{-\infty}^{\infty} I(t)\frac{h_\epsilon^2(t)}{q(t)}dt + \frac{1}{n}\int_{-\infty}^{\infty}\frac{(h_\epsilon'(t))^2}{q(t)}dt}. \tag{4}$$

Practically all papers on the subject contain one or another version of this inequality, see, e.g., See [1, Theorem 30.1], [12, Proposition 1]. However, the authors did not manage to find he assumption (3) in earlier works. We skip the details of the proof; however, for the convenience of the reader we show the basic identity on which the latter is based,

$$E\left((\theta^*(X) - \theta)\,\frac{(f(X \mid \theta)h_\epsilon(\theta))'_\theta}{f(X,\theta)}\right) = \int h_\epsilon(t)\, dt = E\,\frac{h_\epsilon(\theta)}{q(\theta)}\,. \tag{5}$$

In turn, (5) follows from

$$E\left(\theta^*(X)\,\frac{(f(X \mid \theta)h_\epsilon(\theta))'_\theta}{f(X,\theta)}\right) = \int \theta^*(x)\left(\int (f(x \mid t)h_\epsilon(t))'_t\, dt\right) dx = 0,$$

and

$$-E\left(\theta\,\frac{(f(X \mid \theta)h_\epsilon(\theta))'_\theta}{f(X,\theta)}\right) = -\int\left(\int t\,(f(x \mid t)h_\epsilon(t))'_t\, dt\right) dx = E\,\frac{h_\epsilon(\theta)}{q(\theta)}\,,$$

both due to (3) used in the integration by parts. Now Cauchy–Bouniakovsky–Schwarz inequality applied to (5) gives (4).

Remark. Suppose $\int h_\epsilon < \infty$ and $0 \le h_\epsilon \le C < \infty$. Then $I/q \le C$ would suffice for convergence of the integral $\int I h_\epsilon^2/q$; however, this condition is equivalent to $q/I \ge C^{-1} > 0$ which contradicts to the convergence of the integral $\int q/I$ over R^1, the latter convergence being a standing assumption.

Lemma 1 *Let the assumption (A1) hold true, and let there be a sequence* $0 \le q_m(t) \uparrow q(t)$ *(a.e.) as* $m \to \infty$*, such that for any estimator* θ^**,*

$$\liminf_{n\to\infty} n \int_{-\infty}^{\infty} E_t(\theta_n^*(X) - t)^2 \widetilde{q}_m(t)dt \ge \int_{-\infty}^{\infty} \frac{\widetilde{q}_m(t)}{I(t)}dt, \qquad (6)$$

where

$$\widetilde{q}_m(t) = \frac{q_m(t)}{\kappa_m} \qquad \text{and} \qquad \kappa_m = \int_{-\infty}^{\infty} q_m(\theta)d\theta.$$

Then,

$$\liminf_{n\to\infty} nE(\theta^* - \theta)^2 = \liminf_{n\to\infty} n \int_{-\infty}^{\infty} E_t(\theta^* - t)^2 q(t)dt \ge J \qquad (7)$$

holds true with the prior $q(t)$*.*

For the proof in the case $\Theta = [a, b]$ see [12, Lemma 1]; in our case the proof is practically the same, so we skip it.

4 Proof of Theorem 1

1. We will approximate q by appropriate q_m and apply the Lemma 1. Let

$$q_m(t) := q(t)1(-m + 1 < t < m - 1), \quad m > 1.$$

Then, $\quad 0 \le q_m(t) \uparrow q(t), \quad m \to \infty.$
Denote

$$\kappa_m = \int_{-m}^{m} q_m(\theta)d\theta \qquad \text{and} \qquad \widetilde{q}_m(t) = \frac{q_m(t)}{\kappa_m}.$$

To prove the Theorem, it suffices to show that for every m,

$$\liminf_{n\to\infty} n \int_{-m}^{m} E_t(\theta_n^*(X) - t)^2 \widetilde{q}_m(t)dt \ge \int_{-m}^{m} \frac{\widetilde{q}_m(t)}{I(t)}dt. \qquad (8)$$

Denote

$$S_m := \mathrm{supp}(q_m) = [-m + 1, m - 1],$$

and

$$h_{0,m}(t) := \frac{\widetilde{q}_m(t)}{I(t)},$$

and consider the following continuous piece-wise linear function $\varphi = \varphi_{\epsilon,m}$, with $\epsilon \leq 1$ and $m > 1$,

$$\varphi(t) = \varphi_{\epsilon,m}(t) = \begin{cases} (\epsilon+1)t + \epsilon m, & -m \leq t \leq -m+1 \\ \left(1 - \frac{\epsilon}{m-1}\right) t, & -m+1 \leq t \leq m-1 \\ (\epsilon+1)t - \epsilon m, & m-1 \leq t \leq m \end{cases}$$

Notice that

$$\varphi(-m) = -m, \;\; \varphi(-m+1) = -m+1+\epsilon, \;\; \varphi(m-1) = m-1-\epsilon, \;\; \varphi(m) = m,$$

$$0 < C^{-1} \leq \varphi'_{\epsilon,m} \leq C < \infty, \quad \sup_{-m\leq t\leq m} |\varphi'_{\epsilon,m}(t) - 1| \to 0, \; as\ \epsilon \to 0,$$

$$\sup_{-m\leq t\leq m} |\varphi_{\epsilon,m}(t) - t| \to 0, \; as\ \epsilon \to 0,$$

and

$$\widetilde{q}_m(-m+1-) = \widetilde{q}_m(m-1+) = 0.$$

In particular, it follows,

$$\sup_v \left| 1 - \frac{1}{2\epsilon} \int_{\varphi^{-1}_{\epsilon,m}(v-\epsilon)}^{\varphi^{-1}_{\epsilon,m}(v+\epsilon)} dt \right| \to 0, \quad \epsilon \to 0. \tag{9}$$

Let

$$h_{\epsilon,m}(t) := \frac{1}{2\epsilon} \int_{\varphi_{\epsilon,m}(t)-\epsilon}^{\varphi_{\epsilon,m}(t)+\epsilon} h_{0,m}(v) dv. \tag{10}$$

Since $q_m \equiv 0$ outside $[-m+1, m-1]$, then

$$h_{\epsilon,m}(-m) = h_{\epsilon,m}(m) = 0 \quad \text{for} \quad \epsilon \leq 1.$$

Hence, the functions $h_{\epsilon,m}$ satisfy the assumption (3).

Moreover, the function $h_{\epsilon,m}(t)$ defined in (10) is absolutely continuous and differentiable almost everywhere, with a.e. (in)equalities,

$$\begin{aligned} h'_{\epsilon,m}(t) &= \frac{1}{2\epsilon}\left\{h_{0,m}(\varphi_{\epsilon,m}(t)+\epsilon) - h_{0,m}(\varphi_{\epsilon,m}(t)-\epsilon)\right\}, \\ \left|h'_{\epsilon,m}(t)\right| &\leq \frac{1}{2\epsilon}\left\{h_{0,m}(\varphi_{\epsilon,m}(t)+\epsilon) + h_{0,m}(\varphi_{\epsilon,m}(t)-\epsilon)\right\}. \end{aligned}$$

Since $q_m \leq q$, $\widetilde{q}_m(t) = q_m(t)/\kappa_m$, $q(t)/I(t) \leq C$, and $h_{0,m}(t) = \widetilde{q}_m(t)/I(t)$, we get,

$$0 \leq h_{0,m}(t) = \frac{\widetilde{q}_m(t)}{I(t)} = \frac{q_m(t)}{I(t)\kappa_m} \leq \frac{q(t)}{I(t)\kappa_m} \leq \frac{C}{\kappa_m}.$$

Therefore, there exists C' such that for every ϵ small enough, and every m large enough,

$$\left|h'_{\epsilon,m}(t)\right| \leq \frac{C'}{\epsilon}.$$

The function $h_{\epsilon,m}(t)$ satisfies all conditions of the Proposition 1, so,

$$n\int_{-m}^{m} E_t(\theta^* - t)^2 \widetilde{q}_m(t)dt \geq \frac{\left(\int_{-m}^{m} h_{\epsilon,m}(t)dt\right)^2}{\int_{-m}^{m} I(t)\frac{h^2_{\epsilon,m}(t)}{\widetilde{q}_m(t)}dt + \frac{1}{n}\int_{-m}^{m}\frac{(h'_{\epsilon,m}(t))^2}{\widetilde{q}_m(t)}dt}. \tag{11}$$

2. Let us show that

$$\int_{-m}^{m} h_{\epsilon,m}(t)dt \to \int_{-m}^{m} h_{0,m}(t)dt, \tag{12}$$

and

$$\int_{-m}^{m} I(t)\frac{h^2_{\epsilon,m}(t)}{\widetilde{q}_m(t)}dt \to \int_{-m}^{m}\frac{\widetilde{q}_m(t)}{I(t)}dt, \quad \epsilon \to 0. \tag{13}$$

If we manage to choose ϵ as a function of n so that the term $\frac{1}{n}\int_{-m}^{m}\frac{(h'_{\epsilon,m}(t))^2}{\widetilde{q}_m(t)}dt$ vanishes, the assertions (12)–(13) will imply (11) with $h_{\epsilon,m}$ replaced by h_m, which would provide an opportunity to apply the Lemma 1. The proof of (12)–(13) follows the plan used in [12], so we only show the main points and the main changes.

3. To show (12), we notice that

$$\int_{-m}^{m} h_{\epsilon,m}(t)dt = \int_{-m}^{m} dt \frac{1}{2\epsilon} \int_{\varphi_{\epsilon,m}(t)-\epsilon}^{\varphi_{\epsilon,m}(t)+\epsilon} h_{0,m}(v)dv$$

$$\to \int_{-m+\epsilon}^{m-\epsilon} h_{0,m}(t)dt = \int_{-m}^{m} h_{0,m}(t)dt, \quad \epsilon \to 0.$$

The second convergence here holds true due to the Lebesgue dominated convergence theorem and (9).

4. To show (13), we notice that

$$\int_{-m}^{m} \frac{I(t)}{\widetilde{q}_m(t)} \left(h_{\epsilon,m}^2(t) - h_{0,m}^2(t)\right) dt$$

$$= \int_{-m}^{m} \frac{I(t)}{\widetilde{q}_m(t)} \left(h_{\epsilon,m}(t) - h_{0,m}(t)\right) \left(h_{\epsilon,m}(t) + h_{0,m}(t)\right) dt.$$

Since the terms $I(t)/\widetilde{q}_m(t)$ and $(h_{\epsilon,m}(t) + h_{0,m}(t))$ are uniformly bounded on S_m, it suffices to establish convergence

$$\int_{-m}^{m} |h_{\epsilon,m}(t) - h_{0,m}(t)| \, dt \to 0, \quad \epsilon \to 0. \tag{14}$$

Let $\delta > 0$ be any positive, and let us approximate the function $h_{0,m}(t)$ in $L_1[-m, m]$ by some continuous function $h_{0,m}^{\delta}(t)$ so that

$$\int_{-m}^{m} \left|h_{0,m}(t) - h_{0,m}^{\delta}(t)\right| dt < \delta.$$

Then, denoting

$$h_{\epsilon,m}^{\delta}(t) = \frac{1}{2\epsilon} \int_{\varphi_{\epsilon,m}(t)-\epsilon}^{\varphi_{\epsilon,m}(t)+\epsilon} h_{0,m}^{\delta}(v)dv,$$

we get,

$$\begin{aligned} \int_{-m}^{m} \left|h_{\epsilon,m}(t) - h_{\epsilon,m}^{\delta}(t)\right| dt &= \int_{-m}^{m} \left| \frac{1}{2\epsilon} \int_{\varphi_{\epsilon,m}(t)-\epsilon}^{\varphi_{\epsilon,m}(t)+\epsilon} \left(h_{0,m}(v) - h_{0,m}^{\delta}(v)\right) dv \right| dt \\ &\le \int_{-m}^{m} \frac{1}{2\epsilon} \int_{\varphi_{\epsilon,m}(t)-\epsilon}^{\varphi_{\epsilon,m}(t)+\epsilon} \left|h_{0,m}(v) - h_{0,m}^{\delta}(v)\right| dvdt \end{aligned}$$

$$
\begin{aligned}
&= \int_{-m}^{m} \left| h_{0,m}(v) - h_{0,m}^{\delta}(v) \right| dv \frac{1}{2\epsilon} \int_{\varphi_{\epsilon,m}^{-1}(v-\epsilon)}^{\varphi_{\epsilon,m}^{-1}(v+\epsilon)} dt \\
&\leq 2\delta.
\end{aligned}
$$

Hence,

$$
\int_{-m}^{m} |h_{\epsilon,m}(t) - h_{0,m}(t)|\, dt \leq \int_{-m}^{m} \left| h_{\epsilon,m}(t) - h_{\epsilon,m}^{\delta}(t) \right| dt
$$

$$
+ \int_{-m}^{m} \left| h_{0,m}(t) - h_{0,m}^{\delta}(t) \right| dt + \int_{-m}^{m} \left| h_{\epsilon,m}^{\delta}(t) - h_{0,m}^{\delta}(t) \right| dt
$$

$$
\leq 3\delta + \int_{-m}^{m} \left| h_{\epsilon,m}^{\delta}(t) - h_{0,m}^{\delta}(t) \right| dt.
$$

For every fixed $\delta > 0$, the latter integral tends to zero as $\epsilon \to 0$, because the function $h_{0,m}^{\delta}(t)$ is uniformly continuous, and, hence,

$$
\sup_{x} \left| h_{\epsilon,m}^{\delta}(t) - h_{0,m}^{\delta}(t) \right| dt \to 0, \quad \epsilon \to 0.
$$

Therefore, for every $\delta > 0$,

$$
\limsup_{\epsilon \to 0} \int_{-m}^{m} |h_{\epsilon,m}(t) - h_{0,m}(t)|\, dt \leq 3\delta;
$$

however, the left hand side does not depend on δ, hence, (14) holds true, which implies (13).

5. From (11), (12) and (13) we conclude that

$$
\liminf_{n\to\infty} n \int_{-m}^{m} E_t(\theta^* - t)^2 \widetilde{q}_m(t) dt \geq \frac{\left(\int_{-m}^{m} h_{0,m}(t) dt \right)^2}{\int_{-m}^{m} h_{0,m}(t) dt + \limsup_{n\to\infty} \frac{1}{n} \int_{-m}^{m} \frac{(h'_{\epsilon,m}(t))^2}{\widetilde{q}_m(t)} dt}.
$$

We estimate,

$$
\frac{1}{n} \int_{-m}^{m} \frac{(h'_{\epsilon,m}(t))^2}{\widetilde{q}_m(t)} dt \leq \frac{1}{n} \int_{-m}^{m} \frac{C'^2}{\epsilon^2 \widetilde{q}_m(t)} dt = \frac{C'^2}{\epsilon^2 n} \int_{-m}^{m} \frac{1}{\widetilde{q}_m(t)} dt\,.
$$

Hence, for fixed m we can choose $\epsilon = \epsilon(n) = C'n^{-1/5}$, then $\lim_{n\to\infty} 1/(\epsilon^2 n) = 0$. Hence, we obtain,

$$\liminf_{n\to\infty} n \int_{-m}^{m} E_t(\theta^* - t)^2 \widetilde{q}_m(t)dt \geq \int_{-m}^{m} h_{0,m}(t)dt.$$

Due to the Lemma 1, this implies the desired asymptotic inequality (7). Indeed,

$$h_{0,m}(t) = \frac{\widetilde{q}_m(t)}{I(t)},$$

so,

$$\int_{-m}^{m} h_{0,m}(t)dt = \int_{-m}^{m} \frac{\widetilde{q}_m(t)}{I(t)}\, dt \to J, \quad m \to \infty.$$

The Theorem 1 is proved.

5 Proof of Theorem 2

1. Let us denote,

$$q_-^m := \inf_{-m\leq t\leq m} q(t) > 0,$$

see the assumption (A2'). As in the proof of the Theorem 1, we will approximate q by appropriate $\tilde{q}_m$ and apply the Lemma 1. Except this step, the method is rather close to the calculus in [12, Theorem 1], however, we ought to present it in order to make sure that it works in this new situation, indeed. Let

$$q_m(t) := q(t)1(-m+1 < t < m-1), \quad m > 1,$$

$$\kappa_m = \int_{-m}^{m} q_m(\theta)d\theta \qquad \textit{and} \qquad \widetilde{q}_m(t) = \frac{q_m(t)}{\kappa_m}.$$

To prove the Theorem, it suffices to show that for every m,

$$\liminf_{n\to\infty} n \int_{-m}^{m} E_t(\theta_n^*(X) - t)^2 \widetilde{q}_m(t)dt \geq \int_{-m}^{m} \frac{\widetilde{q}_m(t)}{I(t)}dt. \tag{15}$$

For the function $\int_{-m}^{m} (h_\epsilon'(t))^2/\tilde{q}_m(t)\, dt$, the following notation will be used,

$$H_m(\epsilon) := \int_{-m}^{m} (h_\epsilon'(t))^2/\tilde{q}_m(t)\, dt.$$

Let

$$h_{0,m}(t) := \tilde q_m(t)/I(t),$$

$$\bar h_{\epsilon,m}(t) := \min_{|u|\le\epsilon} \frac{\tilde q_m(t+u)}{I(t+u)}, \quad \tilde h_{\epsilon,m}(t) := \bar h_{\epsilon,m}(t) \wedge \frac{q_-^m}{\epsilon}, \qquad -m \le t \le m,$$

and

$$h_{\epsilon,m}(t) := \frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon} \tilde h_{\epsilon,m}(v)\, dv. \tag{16}$$

With this definition, we clearly have

$$\tilde h_{\epsilon,m}(t) \le h_{0,m}(t), \quad \& \quad 0 \le h_{\epsilon,m}(t) \le h_{0,m}(t). \tag{17}$$

Now, the function $h_{\epsilon,m}$ defined in (16) is absolutely continuous and differentiable almost everywhere, with

$$|h'_{\epsilon,m}(t)| \le \frac{C\tilde q_m(t)}{\epsilon} \wedge \frac{\tilde q_m(t)}{I(t)},$$

and $h_{\epsilon,m}(-m) = h_{\epsilon,m}(m) = 0$, for any $\epsilon > 0$. Due to the assumption (A2'), the function $H_m(\epsilon)$ is finite, and, moreover,

$$H_m(\epsilon) \le \frac{C}{\epsilon^2}\int_{-m}^{m} \frac{\tilde q_m^2(t)}{\tilde q_m(t)}\, dt \le \frac{C}{\epsilon^2}.$$

2. Let us show that

$$\tilde h_{\epsilon,m}(t) \to h_{0,m}(t), \quad \epsilon \downarrow 0 \quad (a.e.). \tag{18}$$

For that, due to the Lebesgue dominated convergence theorem, it suffices to show that

$$\int_{-m}^{m} (h_{0,m}(t) - \tilde h_{\epsilon,m}(t))\, dt \downarrow 0, \quad \epsilon \downarrow 0. \tag{19}$$

This follows similarly to [1, Proof of Theorem 30.5], where this hint is applied to the function q. We have, by virtue of the R.i. condition and the theorem about Darboux' integral sums,

$$\sum_k \bar h_{\delta,m}(2k\delta)\, 2\delta \to \int_{-m}^{m} h_{0,m}(t)\, dt, \quad \delta \to 0,$$

$$\sum_k \bar h_{\delta,m}((2k+1)\delta)\, 2\delta \to \int_{-m}^{m} h_{0,m}(t)\, dt, \quad \delta \to 0.$$

Estimate the difference,

$$0 \le \sum_k (\bar h_{\delta,m}(2k\delta) - \tilde h_{\delta,m}(2k\delta))\, 2\delta$$

$$\le 2\delta \sum_k \bar h_{\delta,m}(2k\delta)\, 1(\bar h_{\delta,m}(2k\delta) > q_-^m/(2\delta)).$$

However, since $h_{0,m}$ is Riemann integrable, it must be bounded on $[-m, m]$, and so is $\bar h_{\delta,m} \le h_{0,m}$. Since $\inf_{t\in[-m,m]} \tilde q_m(t) > 0$, then it follows from (A2') that $\tilde h_{\delta,m} \equiv \bar h_{\delta,m}$ as δ is small enough. Then, of course,

$$1(\bar h_{\delta,m}(2k\delta) > q_-^m/(2\delta)) = 0.$$

Therefore the sum $\sum_k \bar h_{\delta,m}(2k\delta)\, 1(\bar h_{\delta,m}(2k\delta) > q_-^m/(2\delta))$ equals zero if δ is small enough. So,

$$0 \le \sum_k (\bar h_{\delta,m}(2k\delta) - \tilde h_{\delta,m}(2k\delta))\, 2\delta \to 0, \qquad \delta \to 0.$$

Similarly,

$$0 \le \sum_k (\bar h_{\delta,m}((2k+1)\delta) - \tilde h_{\delta,m}((2k+1)\delta))\, 2\delta \to 0, \qquad \delta \to 0.$$

Hence,

$$\int_{-m}^{m} \tilde h_{\epsilon,m}(t)\, dt \ge \left(\sum_k \tilde h_{2\epsilon,m}(4k\epsilon)\, 2\epsilon + \sum_k \tilde h_{2\epsilon,m}((4k+2)\epsilon)\, 2\epsilon \right)$$
$$\to \int_{-m}^{m} h_{0,m}(t) dt,\ \epsilon \to 0. \tag{20}$$

Since $\int_{-m}^{m} \tilde h_{\epsilon,m} \le \int_{-m}^{m} h_{0,m}$, the latter convergence implies (19); strictly speaking, we have shown *just* convergence, not a monotone one; but, by construction, the function $\tilde h_{\epsilon,m}$ increases with ϵ decreasing. Hence, (18) holds true *almost everywhere* for $-m \le t \le m$.

3. Notice that $h_{\epsilon,m}$ satisfies the assumptions of the Proposition 1, being differentiable and since it vanishes at $-m$ and m. So, we get, with $\epsilon = (Cn)^{-1/3}$,

$$n \int_{-m}^{m} E_\theta(\theta^* - \theta)^2\, \tilde q_m(\theta)\, d\theta \ge \frac{\left(\int_{-m}^{m} h_{\epsilon,m}(t)\, dt\right)^2}{\int_{-m}^{m} I(t) h_{\epsilon,m}(t)^2/\tilde q_m(t)\, dt + n^{-1/3}}. \tag{21}$$

Hence, to complete the proof, it suffices to establish

$$\int_{-m}^{m} h_{\epsilon,m}(t)\,dt \to \int_{-m}^{m} h_{0,m}(t)\,dt, \tag{22}$$

and

$$\int_{-m}^{m} I(t) h_{\epsilon,m}(t)^2/\tilde{q}_m(t)\,dt \to \int_{-m}^{m} \tilde{q}_m(t)/I(t)\,dt, \quad \epsilon \to 0. \tag{23}$$

4. We have,

$$0 \le \int_{-m}^{m} \left(h_{0,m}(t) - h_{\epsilon,m}(t)\right)\,dt$$

$$= \int_{-m}^{m} \left(h_{0,m}(t) - \frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon} \tilde{h}_{\epsilon,m}(v)\,dv\right)\,dt$$

$$= \int_{-m}^{m} \frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon} \left(h_{0,m}(t) - \tilde{h}_{\epsilon,m}(v)\right)\,dv\,dt$$

$$= \int_{-m}^{m} \frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon} \left(h_{0,m}(t) - \tilde{h}_{\epsilon,m}(t)\right)\,dv\,dt$$

$$+ \int_{-m}^{m} \frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon} \left(\tilde{h}_{\epsilon,m}(t) - \tilde{h}_{\epsilon,m}(v)\right)\,dv\,dt.$$

Here,

$$\int_{-m}^{m} \frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon} \left(h_{0,m}(t) - \tilde{h}_{\epsilon,m}(t)\right)\,dv\,dt = \int_{-m}^{m} \left(h_{0,m}(t) - \tilde{h}_{\epsilon,m}(t)\right)\,dt \to 0, \quad \epsilon \to 0,$$

due to (19). On the other hand side,

$$\int_{-m}^{m} \frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon} \left(\tilde{h}_{\epsilon,m}(t) - \tilde{h}_{\epsilon,m}(v)\right)\,dv\,dt = \int \tilde{h}_{\epsilon,m}(t)\,dt - \int \tilde{h}_{\epsilon,m}(v)\left(\frac{1}{2\epsilon}\int_{v-\epsilon}^{v+\epsilon} 1\,dt\right)\,dv = 0.$$

Thus, indeed, (22) holds true.

5. Further, by virtue of (17) and (22), we also have,

$$0 \le \int \frac{I(t)}{\tilde{q}_m(t)}(h_{0,m}^2(t) - h_{\epsilon,m}^2(t))\,dt$$

$$= \int h_{0,m}^{-1}(h_{0,m}(t) - h_{\epsilon,m}(t))(h_{0,m}(t) + h_{\epsilon,m}(t))\,dt$$

$$\le \int h_{0,m}^{-1}(h_{0,m}(t) - h_{\epsilon,m}(t)) 2h_{0,m}(t)\,dt$$

$$= 2\int (h_{0,m}(t) - h_{\epsilon,m}(t))\,dt \to 0, \quad \epsilon \to 0.$$

Whence, from (21), (22) and (23) the desired inequality (15) follows. By virtue of the Lemma 1, this finally implies (7). The Theorem 2 is proved.

6 Of optimal choice of h

Theorem 3 *Assume $\int q/I < \infty$. Then the optimal choice of $h(\geq 0)$ in the maximization problem*

$$\sup_h \frac{\left(\int_{-\infty}^{\infty} h(t)dt\right)^2}{\int_{-\infty}^{\infty} I(t)\frac{h^2(t)}{q(t)}dt},$$

is

$$h = c\frac{q}{I}, \qquad \textit{with any} \quad c > 0.$$

Proof. The problem is equivalent to minimization,

$$\min_{h\geq 0:\, \int_\Theta h(t)dt=1} \int_\Theta I(t)\frac{h^2(t)}{q(t)}dt.$$

It is easy to see that only $h_0 = cq/I$ may be a minimizer. Indeed, any other function is not because it cannot satisfy the necessary condition of optimality. More than that, this choice provides, clearly, a *local minimizer,* since for any suitable (admissible) φ, we have the second derivative in any direction positive. We skip the calculus because it suffices to have a *guess* about the minimizer.

The proof of the Theorem follows immediately from the Cauchy–Bouniakovsky–Schwarz inequality. Indeed, for any h,

$$\frac{\left(\int_{-\infty}^{\infty} h(t)dt\right)^2}{\int_{-\infty}^{\infty} I(t)\frac{h^2(t)}{q(t)}dt} \leq \int \frac{q}{I}, \tag{24}$$

since

$$\left(\int_{-\infty}^{\infty} h(t)dt\right)^2 \leq \int \frac{q}{I} \int_{-\infty}^{\infty} I(t)\frac{h^2(t)}{q(t)}\, dt.$$

On the other hand, if we choose $h_0 = cq/I$ (with any c), then

$$\frac{\left(\int_{-\infty}^{\infty} h(t)dt\right)^2}{\int_{-\infty}^{\infty} I(t)\frac{h^2(t)}{q(t)}dt} = \frac{c^2(\int q/I)^2}{\int (I/q)c^2(q/I)^2} = \int q/I.$$

An equality sign in (24) is only possible for the choice of h where $I(t)\frac{h^2(t)}{q(t)} = const\ q/I$, by virtue of the equality part of the Cauchy–Bouniakovsky–Schwarz inequality. The latter equation implies once more that necessarily $h = cq/I$, which confirms that the details about local minimizers omitted above are redundant. The Theorem 3 is proved.

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